Chapter 7

A general viscous model for tropical cyclonic winds

7.1 Introduction

Cyclone is an extraordinary amazingly huge weather related phenomenon, which originates over tropical oceans due to atmospheric pressure disturbances. It is known as tropical cyclone with wind speed exceeding 33 ms^{-1} in the Atlantic and the eastern North Pacific, typhoon in the western North Pacific and hurricane in the central and eastern Pacific basins. It has enormous impacts on the society.

The most popular model for the azimuthal velocity is the Rankine's (1882) combined vortex model, where the azimuthal velocity depends only on the radial coordinate. This has been extensively used for studying as well as explaining observed tangential wind flow and deduced pressure distribution in dust devils (Sinclair, 1973; Cantor et al., 2006), waterspouts (Leverson and Sinclair, 1977) and tornadoes (Hoecker, 1961; Wakimoto and Wilson, 1989; Winn et al., 1999; Lee and

Samaras, 2004; Wurman and Samaras, 2004; Lee and Wurman, 2005; Tanamachi et al., 2013).

Yet many fundamental questions related to hurricanes continue bewildering the science community. Unexpected variations in the wind direction from the bottom to the top of the hurricane, radial growth in the wind angular momentum in the boundary layer, unpredicted effect of ocean spray, huge rise in the upper boundary layer temperature, etc. are a few of them. Genesis and maturing are still underinvestigated.

Most of the existing theoretical models, idealized in one or the other way, are based on the balanced vortex model in association with Sawyer-Eliassen transverse circulation equation (Eliassen, 1951; Charney and Eliassen, 1964; Ooyama, 1969; Shapiro and Willoughby, 1982; Emanuel, 1986; Schubert and Hack, 1982; Emanuel, 1995; Nolan et al., 2007; Wirth and Dunkerton, 2009). Models based on balanced vortex combine the hydrostatic and gradient wind balances with radial momentum and thermodynamic equations. The balanced vortex model conserves the absolute angular momentum and predicts the azimuthal winds to be maximum at the lower levels and decay upward (Emanuel, 1986; Stern and Zhang, 2016).

Kieu and Zhang (2009) presented an analytical model of tropical cyclones with a purpose to investigate rapid intensification from the perspective of rotational growth and central pressure-falls. They considered a simplified version of the primitive equations with a linear first-order frictional term. In the paper, they separated the entire domain into two regions and called them region 1 of a fixed radius, referred to as the radius of maximum wind, and within which lies the maximum exponential wind growth, and region 2 which lies outside the region 1 and has no vertical component of velocity. They derived velocities for the two regions separately before stepping in any further discussion. If we compare their division of regions with the existing nomenclature of used for hurricane, we find that the region 1 comprises the eye and the eye wall while the region 2 is that lying outside the eyewall. They solved the governing equations for an axisymmetric flow by prescribing a time dependent vertical velocity with exponential growth in the region 1 but no growth in the outer region 2. They finally held the double exponential term, available in the azimuthal velocity derived for the region 1 but absent in that for the region 2, to be responsible for rapid intensification of tropical cyclones. Our curiosity about whether a double exponential term is responsible for intensification of tropical cyclones even when viscosity is of general type led us, unlike them, to consider a much more general viscous term. A perturbation technique is required to include general viscous effects for solution.

It is hence aimed mainly at resolving some of the problems raised above by providing an exact analytical solution of the equations governing hurricane vortex. Analytical solutions for a fully time dependent hurricane vortex have not been found as yet. In this paper we intend to obtain an exact solution to the equations which govern atmospheric vortices, but concentrate only on the hurricane. We derive a class of exact solutions to a simplified version of the governing equations. The vertical velocity has an impact inside the vortex core but zero outside it. The solution is achieved by considering a particular axial variation of the axial velocity in the inner vortex and zero outside of the vortex core. Azimuthal velocity is tried to be derived as much general as possible by duly considering its radial, axial and temporal dependences. To begin with, we assume Rankine type velocity.

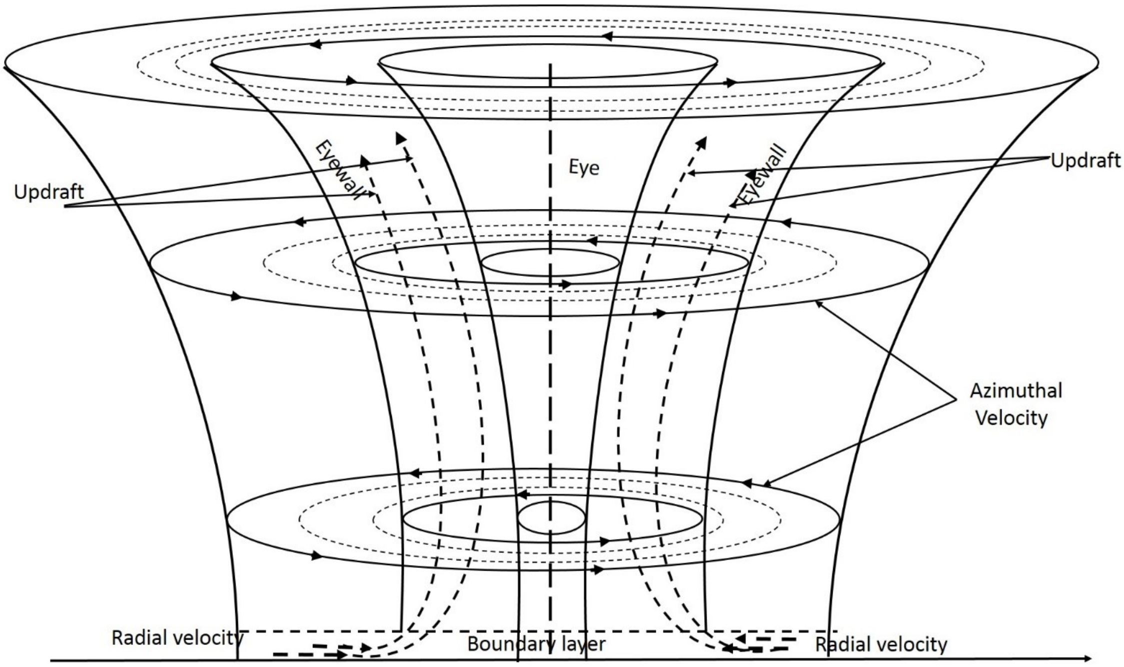

Figure 7.1: Schematic diagram of the physical model of hurricane. The narrow columnar geometry is the relatively motionless warm region called eye placed symmetrically with respect to the vertical axis. The eye is surrounded by the pure updraft zone, within the vertical layer called eye-wall which witnesses heavy downpour. The outermost zone contains violently rotating wind with extremely moist air at the lower altitudes and its radial inflow downdraft supplies moisture for updraft through the boundary layer.

7.2 Mathematical formulation of the problem

7.2.1 The physical model

A hurricane or cyclone is physically a three dimensional natural phenomenon perceived as a solitary vertical air vortex spinning in the cyclonic direction about the axis of rotation having extra radial and vertical winds near the bottom (see Fig. 7.1). In the inner cyclone, there is a calm region of low pressure referred to as eye which is a vertical column of radius 20 km and is wrapped by another region known as eyewall with an external radius $30 - 50\ km$. This wrapping territory will be referred to as inner region. Above the boundary layer of thickness $2 - 3\ km$, the radius of the external eye-wall changes with height (Leonov, 2014). The eye-wall is made up of strongly revolving winds together with radial inflows maximizing at its inner surface. The vertical velocity, which is ideally contained within the eye-wall, is weaker than the radial and azimuthal winds and further weakens outside the eye-wall. In the outermost part of radius $400 - 600\ km$, surrounding the eyewall, the relative rotation of the cyclone declines to zero. The blowing wind along the radial direction is inward at the bottom and outward at the top of the hurricane. The entire vortex is vertically layered into the bottom hurricane boundary layer, and upper adiabatic layer with the total cyclonic height up to $20 - 30\ km$ (Leonov, 2014).

Inflow radial wind blowing from different directions in the hurricane boundary layer is the most significant wind for the genesis of hurricane vortices, and when a hurricane matures, the azimuthal component of the wind velocity is much stronger than the radial and vertical components. These cyclones have been extensively investigated during the last seven decades. But till date no exact analytical model is available for the motion of the real cyclonic vortex and pressure distribution within and outside the vortex. Earlier most of the researchers considered either the linear form of the inflow radial component of velocity or neglected it in comparison with the azimuthal component.

Rankine (1882) was the first to present the radial profile of the azimuthal velocity for incompressible steady inviscid flow with radial and vertical components taken zero. The main problem with the Rankine vortex model is that it has sharp peak at the wall of the core. Later, Burgers (1948) and Rott (1958) independently obtained solution for viscous vortex motion of the steady incompressible flow embedded in a radially inward stagnation point flow over a plane boundary with all non-zero velocity components, which was an improved version of Rankine model. Both models are applicable for single celled vortex flow. Some unrealistic aspects of the Burgers-Rott vortex, however, are that the radial and vertical velocity components increase linearly to infinity.

As an attempt to understand the complete dynamics of mature hurricanes, we present some new analytical solutions of general viscous incompressible equations governing cyclones. Analytical solution of the equations governing atmospheric vortices like tornado, hurricane, typhoon, cyclone etc. have always been a challenging task due to their complex formulations. In this paper we intend to model only cyclonic vortex.

7.2.2 Mathematical model of cyclonic vortex

We consider the cylindrical polar coordinates (r, θ, z) for the problem undertaken, where r, θ, z respectively stand for the radial, angular and vertical coordinates. The vertical coordinate is log-pressure coordinate defined as $z = -H \log(p/p_s)$ with respect to the reference pressure p_s with p being pressure and H the scale height of the hurricane. It is observed that practically, a rotating fluid mass in the form of a mature vortex does not seem to differ much at different angles during its rotation about the vertical axis. Thus, it is reasonable to consider the flow as symmetric about the axis. This removes all terms where the angular coordinate θ is involved. Hence, the three-dimensional model of atmospheric flows under the elastic and axisymmetric approximation (Wilhelmson and Ogure, 1972; Willoughby, 1979) of an incompressible Newtonian viscous fluid may be given by

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial r} + F_u, \tag{7.1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + w\frac{\partial v}{\partial z} + \frac{uv}{r} + fu = F_u, \tag{7.2}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + b + F_w, \tag{7.3}$$

$$\frac{1}{r}\frac{\partial (ru)}{\partial r} + \frac{\partial w}{\partial z} - \frac{w}{H} = 0. \tag{7.4}$$

$$\frac{\partial b}{\partial t} + u\frac{\partial b}{\partial r} + N^2 w = Q, \tag{7.5}$$

where u, v and w are the three wind components of velocity in the r, θ, z directions, p is the pressure, $b = g(T - T_{ref})/T_{ref}$ is buoyancy with $T_{ref}(z)$ being the reference temperature of the undisturbed atmosphere, F_u, F_v, F_w are frictional forces, f is the Coriolis parameter; and N^2 is the Brunt-Vaisala frequency.

Characteristic quantities are required to make various parameters dimensionless. The radius, symbolized as a, of the region 1 which consists of the eye and the eye wall could be considered an appropriate characteristic length. Further, the azimuthal wind velocity, symbolized as v_a, at the periphery of region 1 seems to be the only probable characteristic velocity for non-dimensionalisation. Therefore, the system of Eqs. (7.1)−(7.4), along with the boundary conditions are non-dimensionalised in terms of the following dimensionless parameters:

$$\bar{t} = \frac{t}{T}, \bar{r} = \frac{r}{r_a}, \bar{z} = \frac{z}{r_a}, \bar{u} = \frac{u}{v_a}, \bar{v} = \frac{v}{v_a}, \bar{w} = \frac{w}{v_a}, \bar{p} = \frac{p}{P}. \tag{7.6}$$

where T and P will be defined later. Further we consider b constant for analytical solutions.

The dimensionless form of Eqs. (7.1)−(7.4), by dropping the bar, are transformed to

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} - Sv = -\frac{\partial p}{\partial r} + \bar{F}_u, \tag{7.7}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial r}+w\frac{\partial v}{\partial z}+\frac{uv}{r}+Su=\bar{F}_v, \tag{7.8}$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}=-\frac{\partial p}{\partial z}+\bar{b}+\bar{F}_w, \tag{7.9}$$

$$\frac{1}{r}\frac{\partial (ru)}{\partial r}+\frac{\partial w}{\partial z}-\frac{w}{H_1}=0. \tag{7.10}$$

here $S = af/v_a$, $H_1 = H/a$, $P = \rho v_a^2$ and $T = a/v_a$ and quantities with bar denotes non-dimensional parameters.

The classical Rankine-combined vortex is the solution of steady the two-dimensional Euler equation governing an ideal inviscid fluid. The velocity field in this is purely azimuthal and is given, in cylindrical polar coordinates, as $\boldsymbol{q} = [0, v(r), 0]$, where $v(r) = \zeta r/2,\ r \leq a$ and $v(r) = \zeta a^2/2r,\ r > a$. This flow consists of the circular inner region ($r \leq a$) of radius a moving with constant vorticity ζ surrounded by irrotational flow everywhere outside the inner region.

We seek to apply here the method of separation of variables and hence assume the azimuthal velocity as the product of $F(r)$, a function exclusively of r, and $G(z,t)$, a function exclusively of z and t, i.e. $v(r,z,t) = F(r)\times G(z,t)$. Following the Rankine-combined vortex model (Rankine, 1882), we further assume the radial variation $F(r)$ of the azimuthal velocity in the non-dimensional form $F(r) = \zeta r/2$, for $0 \leq r \leq a$, and $F(r) = \zeta a^2/2r$, for $r > a$. In order to obtain a more general z-dependent solution, we consider a piecewise solution in r with the azimuthal velocity taken as

$$v(r,z,t)=\begin{cases} rG(z,t),\ for\ 0\leq r\leq 1 \\ G_1(z,t)/r,\ for\ r>1 \end{cases} ,and\ 0\leq z\leq H, \tag{7.11}$$

In view of the quasi-balanced constraints, Charney and Elliasen (1964), Yanai (1964) and Ooyama (1969) found it theoretically correct to describe the secondary circulation growth in terms of an instability mode. Therefore, following Kein and Zhang

(2009), we assume the diabatically induced ascending velocity as

$$w(r,z,t) = W_0 \sin(\lambda z) e^{\beta t},\ for\ 0 \le r \le 1\ ,\ w(r,z,t) = 0,\ for\ r > 1. \tag{7.12}$$

where W_0, β, λ are constants non-dimensionlised respectively by v_a, a/v_a, a. This is to be noted that there cannot be any sort of discontinuity at $r = 1$. Therefore, since $w(r,z,t) = 0$ at $r > 1$, even $w(1,z,t) = 0$ which simply implies that z is an integral multiple of π/λ at $r = 1$. Kieu and Zhang (2009) argue that β, the growth rate of the vertical flow, is affected by friction and surface heat fluxes, hence is a function of the axial coordinate z and the buoyancy frequency. However, it typically being dimensionally of the order of $10^{-6} - 10^{-5}\ s^{-1}$ (Ooyama, 1969) may be approximated to a constant.

The dimensionless governing equations (7.7)−(7.9) are now constrained by the following boundary conditions:

$$\left.\begin{aligned} u(r,z,t)_{r=0} = 0,\ u(r,z,t)_{r=R_m/a};\ v(r,z,t)_{r=0} = 0, \\ v(r,z,t)_{r=R_m/a} = 0;\ w(r,z,t)_{z=0} = 0,\ w(r,z,t)_{z=H_0/a}. \end{aligned}\right\} \tag{7.13}$$

7.2.3 Analytical solutions

In this section, we present an analytical solution for time dependent viscous incompressible flows in the cyclonic vortex governed by the azimuthal momentum Eq. (7.8). In Eq. (7.8), $\bar{F_v}$ represents the nondimensional viscous term in the azimuthal direction. Most of the earlier researchers either considered, $\bar{F_v}$ negligibly small or of linear form (Kieu and Zhang, 2009). Here, we assume

$$\bar{F_v} = \frac{1}{Re}\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

Now, we solve the following equation for azimuthal velocity separately for the two regions (1) $0 \leq r \leq 1$ and (2) $r > 1$ separately by supplying radial and vertical velocities

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + w\frac{\partial v}{\partial z} + \frac{uv}{r} + Su = \frac{1}{Re}\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2}\right), \tag{7.14}$$

7.2.3.1 Solution for the region 1

Assuming $w_1(r,z,t) = W(z)e^{\beta t}$, where $W(z) = W_0 \sin(\lambda z)$ as in Eq. (7.12), the radial wind in the region 1 with vanishing radial velocity at $r = 0$, may be obtained, from the continuity equation (7.10), as

$$u_1(r,z,t) = \frac{rW_0}{2}\left[\frac{1}{H_1}\sin(\lambda z) - \lambda\cos(\lambda z)\right]e^{\beta t}, \tag{7.15}$$

Using Eqs. (7.12) and (7.15) into Eq. (7.14), we obtain the following equation for the tangential wind in the region 1:

$$\frac{\partial v_1}{\partial t} + W_0\sin(\lambda z)\frac{\partial v_1}{\partial z}e^{\beta t} + \frac{rW_0}{2}\left\{\frac{1}{H_1}\sin(\lambda z) - \lambda\cos(\lambda z)\right\}\left(\frac{\partial v_1}{\partial r} + \frac{v_1}{r} + S\right)e^{\beta t} \tag{7.16}$$
$$= \frac{1}{Re}\left(\frac{\partial^2 v_1}{\partial r^2} + \frac{1}{r}\frac{\partial v_1}{\partial r} - \frac{v_1}{r^2} + \frac{\partial^2 v_1}{\partial z^2}\right),$$

The only solution separable in the radial and axial-temporal coordinates of Eq. (7.16) can be of the form $v_1(r,z,t) = rF_1(z,t)$. Hence, using this form in Eq. (7.16), we get

$$\frac{\partial F_1}{\partial t} + \left[W_0\sin(\lambda z)\frac{\partial F_1}{\partial z} + \frac{W_0}{2}\left\{\frac{1}{H_1}\sin(\lambda z) - \lambda\cos(\lambda z)\right\}(2F_1 + S)\right]e^{\beta t} = \frac{1}{Re}\frac{\partial^2 F_1}{\partial z^2}, \tag{7.17}$$

In terms of $G(z,t)\,(=F_1(z,t)+S/2)$, Eq. (7.17) is transformed to

$$\frac{\partial G}{\partial t}+W_0\left[\sin(\lambda z)\frac{\partial G}{\partial z}+\left\{\frac{1}{H_1}\sin(\lambda z)-\lambda\cos(\lambda z)\right\}G\right]e^{\beta t}=\frac{1}{Re}\frac{\partial^2 G}{\partial z^2}, \tag{7.18}$$

We suppose that the vortex Reynolds number Re is very large; or equivalently that $\epsilon = Re^{-1} \ll 1$. In view of this, we seek an asymptotic solution of Eq. (7.18) of the form

$$G(z,t)=G_0(z,t)+\epsilon G_1(z,t)+\epsilon^2 G_2(z,t)+..... \tag{7.19}$$

Assuming that this series expansion converges for higher orders, we can obtain a solution for $G(z,t)$ for various orders of ϵ by substituting Eq. (7.19) into Eq. (7.18). However, in order to avoid unnecessary derivations in view of $\epsilon \ll 1$, we present the solution only up to the first order of ϵ. Thus, equations of the zeroth and the first order of ϵ are

$$\epsilon^0:\frac{\partial G_0}{\partial t}+W_0\left[\sin(\lambda z)\frac{\partial G_0}{\partial z}+\left\{\frac{1}{H_1}\sin(\lambda z)-\lambda\cos(\lambda z)\right\}G_0\right]e^{\beta t}=0, \tag{7.20}$$

$$\epsilon^1:\frac{\partial G_1}{\partial t}+W_0\left[\sin(\lambda z)\frac{\partial G_1}{\partial z}+\left\{\frac{1}{H_1}\sin(\lambda z)-\lambda\cos(\lambda z)\right\}G_1\right]e^{\beta t}=\frac{\partial^2 G_0}{\partial z^2}. \tag{7.21}$$

A possible solution of Eq. (7.20) is of the form

$$G_0(z,t)=G_0'(z)\exp(\mu e^{\beta t}), \tag{7.22}$$

where μ is an arbitrary positive dimensionless number. Substituting Eq. (7.22) into Eq. (7.20), we get

$$\frac{dG_0'}{dz}=-\left[\left\{\frac{1}{H_1}-\lambda\cot(\lambda z)\right\}+\frac{\beta\mu}{W_0}\csc(\lambda z)\right]G_0', \tag{7.23}$$

which on integration, yields $G_0(z,t)$ in the explicit form as

$$G_0(z,t) = Ke^{(-z/H_1)}\left\{\sin(\lambda z)\right\}^{(1-\beta\mu/\lambda W_0)}\left\{2\cos\left(\frac{\lambda z}{2}\right)\right\}^{2(\beta\mu/\lambda W_0)}\exp\left(\mu e^{\beta t}\right), \quad (7.24)$$

where K is an integration constant that determines the initial strength of the vortex (See **Appendix A** for detailed analysis). The zeroth order tangential wind velocity is given by

$$v_{10}(r,z,t) = r\left[Ke^{(-z/H_1)}\left\{\sin(\lambda z)\right\}^{(1-\beta\mu/\lambda W_0)}\left\{2\cos\left(\frac{\lambda z}{2}\right)\right\}^{2(\beta\mu/\lambda W_0)}\exp\left(\mu e^{\beta t}\right) - \frac{S}{2}\right], \quad (7.25)$$

which has an infinite number of possible solutions depending on the values of μ. However, the requirements for the regularity of Eq. (7.25) at $z=0$ impose a strong restriction on the range of μ. Thus for a regular solution of Eq. (7.25), we have $\beta\mu/\lambda W_0 \le 1$. Following Kieu and Zhang (2009), we take $\beta\mu/\lambda W_0 = 1-\delta$, where $0 \le \delta \le 1$. This substitution transforms Eq. (7.25) to

$$G_0(z,t) = 2Ke^{(-z/H_1)}\left\{\sin(\lambda z)\right\}^{\delta}\left\{\cos\left(\frac{\lambda z}{2}\right)\right\}^{2-\delta}\exp\left(\frac{\lambda W_0}{\beta}(1-\delta)e^{\beta t}\right), \quad (7.26)$$

and the corresponding zeroth order azimuthal wind velocity, in terms of δ, may be given by

$$v_{10}(r,z,t) = r\left[2Ke^{(-z/H_1)}\left\{\sin\left(\frac{\lambda z}{2}\right)\right\}^{\delta}\left\{\cos\left(\frac{\lambda z}{2}\right)\right\}^{(2-\delta)}\exp\left(\frac{\lambda W_0}{\beta}(1-\delta)e^{\beta t}\right) - \frac{S}{2}\right], \quad (7.27)$$

Now we seek to solve Eq. (7.21) by presenting it in the form

$$e^{-\beta t}\frac{\partial G_1}{\partial t} + W_0\sin(\lambda z)\frac{\partial G_1}{\partial z} + W_0\left\{\frac{1}{H_1}\sin(\lambda z) - \lambda\cos(\lambda z)\right\}G_1 = e^{-\beta t}\frac{\partial^2 G_0}{\partial z^2}, \quad (7.28)$$

Assuming $G_1(z,t) = \Gamma(z,t)G_0(z,t)$ in Eq. (7.28) and using Eq. (7.20), we obtain

$$e^{-\beta t}\frac{\partial \Gamma}{\partial t} + W_0 \sin(\lambda z)\frac{\partial \Gamma}{\partial z} = F(z,t), \tag{7.29}$$

where

$$\begin{aligned} F(z,t) = e^{-\beta t}\Bigg[&\frac{\lambda^2\delta(\delta-1)}{4}\cot^2\left(\frac{\lambda z}{2}\right) + \frac{\lambda^2(2-\delta)(1-\delta)}{4}\tan^2\left(\frac{\lambda z}{2}\right) \\ &- \frac{\delta\lambda}{H_1}\cot\left(\frac{\lambda z}{2}\right) + \frac{(2-\delta)\lambda}{H_1}\tan\left(\frac{\lambda z}{2}\right) + \left(\frac{1}{H_1^2} - \frac{(2\delta-\delta^2+1)\lambda^2}{2}\right)\Bigg], \end{aligned} \tag{7.30}$$

The solution of Eq. (7.30) is given by (the detailed analysis is given in **Appendix B**),

$$\begin{aligned} \Gamma(z,t) = \frac{1}{\beta}\Bigg[&\frac{\lambda^2\delta(\delta-1)}{4}\exp\left(2\lambda W_0\frac{e^{\beta t}}{\beta}\right)\cot^2\left(\frac{\lambda z}{2}\right) I_{11} \\ &+ \frac{\lambda^2(2-\delta)(1-\delta)}{4}\exp\left(-2\lambda W_0\frac{e^{\beta t}}{\beta}\right)\tan^2\left(\frac{\lambda z}{2}\right) I_{22} \\ &- \frac{\delta\lambda}{H_1}\exp\left(\lambda W_0\frac{e^{\beta t}}{\beta}\right)\cot\left(\frac{\lambda z}{2}\right) I_{33} \\ &+ \frac{(2-\delta)\lambda}{H_1}\exp\left(-\lambda W_0\frac{e^{\beta t}}{\beta}\right)\tan\left(\frac{\lambda z}{2}\right) I_{44} \\ &+ \left(\frac{1}{H_1^2} - \frac{(2\delta-\delta^2+1)\lambda^2}{2}\right)\log\left(\frac{e^{\beta t}}{\beta}\right)\Bigg] + \phi(A), \end{aligned} \tag{7.31}$$

where $\phi(A)$ is an arbitrary function of $A = e^{\beta t}/\beta - (1/(\lambda W_0))\log(\lambda z/2)$, which is itself a constant and is given by $(B4)$ in **Appendix B**. This may be noted that A has a singularity at $z = 0$, and hence $\phi(A)$ too, which may be eliminated by considering $\phi(A) = \gamma$ a constant. Thus, the first order solution of $G_1(z,t)$ may be given by

$$G_1(z,t) \tag{7.32}$$
$$= 2Ke^{(-z/H_1)} \left\{\sin\left(\frac{\lambda z}{2}\right)\right\}^{\delta} \left\{2\cos\left(\frac{\lambda z}{2}\right)\right\}^{(2-\delta)} \left[\frac{1}{\beta}\left\{\frac{\lambda^2\delta(\delta-1)}{4}\exp\left(2\lambda W_0\frac{e^{\beta t}}{\beta}\right)\times\right.\right.$$
$$\cot^2\left(\frac{\lambda z}{2}\right) I_{11} + \frac{\lambda^2(2-\delta)(1-\delta)}{4}\exp\left(-2\lambda W_0\frac{e^{\beta t}}{\beta}\right)\tan^2\left(\frac{\lambda z}{2}\right) I_{22} - \frac{\delta\lambda}{H_1}\exp\left(\lambda W_0\frac{e^{\beta t}}{\beta}\right)\times$$
$$\cot\left(\frac{\lambda z}{2}\right) I_{33} + \frac{(2-\delta)\lambda}{H_1}\exp\left(-\lambda W_0\frac{e^{\beta t}}{\beta}\right)\tan\left(\frac{\lambda z}{2}\right) I_{44}$$
$$\left.\left. + \left(\frac{1}{H_1^2} - \frac{(2\delta-\delta^2+1)\lambda^2}{2}\right)\log\left(\frac{e^{\beta t}}{\beta}\right)\right\} + \gamma\right]\exp\left(\frac{\lambda W_0}{\beta}(1-\delta)e^{\beta t}\right),$$

With the first order viscous correction, the tangential wind velocity in region 1 is now given by

$$v_1(r,z,t) = r\left[G_0(z,t) + \epsilon G_1(z,t) - S/2\right] = r\psi(z,t). \tag{7.33}$$

where

$$\psi(z,t) = 2Ke^{(-z/H_1)} \left\{\sin\left(\frac{\lambda z}{2}\right)\right\}^{\delta} \left\{2\cos\left(\frac{\lambda z}{2}\right)\right\}^{(2-\delta)} \left[1\right. \tag{7.34}$$
$$+\frac{1}{Re}\left\{\frac{1}{\beta}\left\{\frac{\lambda^2\delta(\delta-1)}{4}\exp\left(2\lambda W_0\frac{e^{\beta t}}{\beta}\right)\cot^2\left(\frac{\lambda z}{2}\right) I_{11} + \frac{\lambda^2(2-\delta)(1-\delta)}{4}\times\right.\right.$$
$$\exp\left(-2\lambda W_0\frac{e^{\beta t}}{\beta}\right)\tan^2\left(\frac{\lambda z}{2}\right) I_{22} - \frac{\delta\lambda}{H_1}\exp\left(\lambda W_0\frac{e^{\beta t}}{\beta}\right)\cot\left(\frac{\lambda z}{2}\right) I_{33}$$
$$\left.\left.\left. + \frac{(2-\delta)\lambda}{H_1}\exp\left(-\lambda W_0\frac{e^{\beta t}}{\beta}\right)\tan\left(\frac{\lambda z}{2}\right) I_{44} + \left(\frac{1}{H_1^2} - \frac{(2\delta-\delta^2+1)\lambda^2}{2}\right)\log\left(\frac{e^{\beta t}}{\beta}\right)\right\} + \gamma\right\}\right]$$
$$\times$$
$$\exp\left(\frac{\lambda W_0}{\beta}(1-\delta)e^{\beta t}\right) - \frac{S}{2}.$$

The radial pressure gradient is obtained, by using Eqs. (7.12), (7.15) and (7.34) into Eq. (7.7), as

$$\frac{\partial p_1}{\partial r} = -r\left[\frac{W_0 e^{\beta t}}{4}\left\{\frac{1}{H_1}\sin(\lambda z) - \lambda\cos(\lambda z)\right\}\left\{2\beta + W_0\left(\frac{1}{H_1}\sin(\lambda z) - \lambda\cos(\lambda z)\right)e^{\beta t}\right.\right.$$
$$\left.\left. + \frac{2\lambda^2}{Re}\right\} + \frac{W_0^2\lambda}{2}\sin(\lambda z)\left\{\frac{1}{H_1}\cos(\lambda z) + \lambda\sin(\lambda z)\right\}e^{2\beta t} - \psi^2 - S\psi\right], \tag{7.35}$$

Integration of Eq. (7.35), with respect to r, gives

$$p_1(r,z,t) - p_1(1,z,t) = \frac{(1-r^2)}{2}\left[\frac{W_0 e^{\beta t}}{4}\left\{\frac{1}{H_1}\sin(\lambda z) - \lambda\cos(\lambda z)\right\}\left\{2\beta + W_0\left(\frac{1}{H_1}\sin(\lambda z) - \lambda\cos(\lambda z)\right)e^{\beta t} + \frac{2\lambda^2}{Re}\right\} + \frac{W_0^2\lambda}{2}\sin(\lambda z)\left\{\frac{1}{H_1}\cos(\lambda z) + \lambda\sin(\lambda z)\right\}e^{2\beta t} - \psi^2 - S\psi\right], \tag{7.36}$$

The axial pressure gradient is obtained, by using Eqs. (7.12), (7.15) into Eq. (7.9), as

$$\frac{\partial p_1}{\partial z} = -\frac{W_0^2\lambda}{2}\sin(2\lambda z)e^{2\beta t} + \bar{b} - W_0\left(\beta + \frac{\lambda^2}{Re}\right)\sin(\lambda z)e^{\beta t}, \tag{7.37}$$

Integrating Eq. (7.37) from the initial level z_0 to z, we get

$$p_1(r,z,t) - p_1(r,z_0,t) = \frac{W_0^2}{4}\left(\cos(2\lambda z) - \cos(2\lambda z_0)\right)e^{2\beta t} + \frac{W_0}{\lambda}\left(\beta + \frac{\lambda^2}{Re}\right) \times \left(\cos(2\lambda z) - \cos(2\lambda z_0)\right)e^{\beta t} + \int_{z_0}^{z}\bar{b}dz, \tag{7.38}$$

Then substituting $r = 1$ into Eq. (7.38) and substituting in Eq. (7.36), we get

$$p_1(r,z,t) - p_1(1,z,t) = \frac{(1-r^2)}{2}\left[\frac{W_0 e^{\beta t}}{4}\left\{\frac{1}{H_1}\sin(\lambda z) - \lambda\cos(\lambda z)\right\}\left\{2\beta + W_0\left(\frac{1}{H_1}\sin(\lambda z) - \lambda\cos(\lambda z)\right)e^{\beta t} + \frac{2\lambda^2}{Re}\right\} + \frac{W_0^2\lambda}{2}\sin(\lambda z)\left\{\frac{1}{H_1}\cos(\lambda z) + \lambda\sin(\lambda z)\right\}e^{2\beta t} - \psi^2 - S\psi\right] + \frac{W_0^2}{4}\left(\cos(2\lambda z) - \cos(2\lambda z_0)\right)e^{2\beta t} + \frac{W_0}{\lambda}\left(\beta + \frac{\lambda^2}{Re}\right) \times \left(\cos(2\lambda z) - \cos(2\lambda z_0)\right)e^{\beta t} + \int_{z_0}^{z}\bar{b}dz, \tag{7.39}$$

7.2.3.2 Solution for the region 2

From the continuity equation (7.10) and assuming $w_2(r, z, t) = 0$, the radial wind in region 2 may be given by

$$u_2(r, z, t) = \frac{c_1(z, t)}{r}, \tag{7.40}$$

where $c_1(z, t)$, an integral function, is obtained by matching the radial velocities at the outer boundary of the core, i.e., at $r = 1$. Thus the radial velocity may be given by

$$u_2(r, z, t) = \frac{W_0}{2r}\left[\frac{1}{H_1}\sin(\lambda z) - \lambda\cos(\lambda z)\right]e^{\beta t}. \tag{7.41}$$

Substitution of u_2 into Eq. (7.14), followed by some manipulations, yields

$$\begin{aligned}&\frac{\partial v_2}{\partial t} + \frac{W_0}{2r}\left\{\frac{1}{H_1}\sin(\lambda z) - \lambda\cos(\lambda z)\right\}\left(\frac{\partial v_2}{\partial r} + \frac{v_2}{r} + S\right)e^{\beta t} \\ &= \epsilon\left(\frac{\partial^2 v_2}{\partial r^2} + \frac{1}{r}\frac{\partial v_2}{\partial r} - \frac{v_2}{r^2} + \frac{\partial^2 v_2}{\partial z^2}\right),\end{aligned} \tag{7.42}$$

A possible separable solution of Eq. (7.42) is of the form $v_2(r, z, t) = F_2(z, t)/r$ so that we have

$$\frac{\partial F_2}{\partial t} - \epsilon\frac{\partial^2 F_2}{\partial z^2} = \frac{W_0}{2}\left\{\frac{1}{H_1}\sin(\lambda z) - \lambda\cos(\lambda z)\right\}Se^{\beta t}, \tag{7.43}$$

which itself is separable in the form $F_2(z, t) = M(z)e^{\beta t}$, where $M(z)$ satisfies the following equation:

$$\frac{d^2 M}{dz^2} - \frac{\beta}{\epsilon}M = \frac{W_0}{2\epsilon}\left\{\frac{1}{H_1}\sin(\lambda z) - \lambda\cos(\lambda z)\right\}S, \tag{7.44}$$

whose solution is obtained as

$$M(z) = c_1 e^{-\sqrt{\beta Re}z} + c_2 e^{\sqrt{\beta Re}z} - \frac{W_0 S}{2(\lambda Re^{-1} + \beta)}\left\{\frac{1}{H_1}\sin(\lambda z) - \lambda\cos(\lambda z)\right\}., \tag{7.45}$$

For a finite solution of $v_2(r,z,t)$, $F_2(z,t)$ and hence $M(z)$ must be finite, which is possible only when $c_2 = 0$, which increases it infinitely. This constraint reduces Eq. (7.45) to

$$M(z) = c_1 e^{-\sqrt{\beta Re} z} - \frac{W_0 S}{2(\lambda Re^{-1} + \beta)} \left\{ \frac{1}{H_1} \sin(\lambda z) - \lambda \cos(\lambda z) \right\}. \tag{7.46}$$

Thus the azimuthal velocity for this region may be given by

$$v_2(r,z,t) = \frac{e^{\beta t}}{r} M(z), \tag{7.47}$$

We obtain c_1 by using the second condition, i.e., the azimuthal velocities v_1 and v_2 of the two regions are the same when $t = 0$, $z = 0$ and $r = 1$. Thus, we have

$$v_2(r,z,t) = -\frac{S}{2r} \left[\left\{ 1 + \frac{\lambda W_0}{(\lambda Re^{-1} + \beta)} \right\} e^{-\sqrt{\beta Re} z} + \frac{W_0 S}{(\lambda Re^{-1} + \beta)} \left\{ \frac{1}{H_1} \sin(\lambda z) - \lambda \cos(\lambda z) \right\} \right] e^{\beta t}. \tag{7.48}$$

Corresponding radial pressure gradient is obtained, by applying Eqs. (7.12), (7.41) and Eq. (7.48) into Eq. (7.7), as

$$\begin{aligned} \frac{\partial p_2}{\partial r} = &-\frac{W_0}{2} \left\{ \frac{1}{H_1} \sin(\lambda z) - \lambda \cos(\lambda z) \right\} \left[\frac{1}{r} \left(\beta + \lambda^2 Re^{-1} \right) \right. \\ &\left. - \frac{W_0}{2r^3} \left\{ \frac{1}{H_1} \sin(\lambda z) - \lambda \cos(\lambda z) \right\} e^{\beta t} \right] e^{\beta t} + \frac{M^2}{r^3} e^{2\beta t} + \frac{SM}{r} e^{\beta t}. \end{aligned} \tag{7.49}$$

Further, integrating Eq. (7.49) with respect to r, we get

$$\begin{aligned} p_2(r,z,t) - P(z,t) = &-\frac{W_0}{2} \left\{ \frac{1}{H_1} \sin(\lambda z) - \lambda \cos(\lambda z) \right\} \left[\left(\beta + \lambda^2 Re^{-1} \right) \log(r) \right. \\ &\left. + \frac{W_0}{4r^2} \left\{ \frac{1}{H_1} \sin(\lambda z) - \lambda \cos(\lambda z) \right\} e^{\beta t} \right] e^{\beta t} \\ &- \frac{M^2}{2r^2} e^{2\beta t} + SM \log(r) e^{\beta t}. \end{aligned} \tag{7.50}$$

$P(z,t)$ may be obtained by using the second condition $p_2(R_m,z,t)=0$, at $r=R_m$. Thus, we have

$$\begin{aligned} p_2(r,z,t) = &-\frac{W_0}{2}\left\{\frac{1}{H_1}\sin(\lambda z)-\lambda\cos(\lambda z)\right\}\Bigg[\left(\beta+\lambda^2 Re^{-1}\right)\log\left(\frac{r}{R_m}\right) \\ &+\frac{W_0}{4}\left(\frac{1}{r^2}-\frac{1}{R_m^2}\right)\left\{\frac{1}{H_1}\sin(\lambda z)-\lambda\cos(\lambda z)\right\}e^{\beta t}\Bigg]e^{\beta t} \\ &-\frac{M^2}{2}\left(\frac{1}{r^2}-\frac{1}{R_m^2}\right)e^{2\beta t}+SMe^{\beta t}\log\left(\frac{r}{R_m}\right). \end{aligned} \tag{7.51}$$

7.3 Results and discussion

This is an analytical model of an intense hurricane vortex by considering a diabatically induced ascending motion proposed by Kieu and Zhang (2009). The particular form of the vertical velocity assumed by Kieu and Zhang (2009) is used to solve the governing equations analytically. We have further assumed that the vertical velocity is independent of the radial coordinate r.

In most the investigations the vortex motion has been considered inviscid. However, Kieu and Zhang (2009) considered viscous flow but took a linear form of viscosity. Unlike them, we have considered the general form and used a perturbation technique to analyze the contribution of viscosity to hurricane dynamics despite the fact that the Reynolds number is very large in such a rotational motion. Due to the large Reynolds number and highly complicated expressions, we confined the entire solution to the first order of $\epsilon = Re^{-1} (\ll 1)$. Besides the azimuthal velocity, pressure too has been considered worth discussing.

7.3.1 Analysis of the solution in the region 1

As per our assumptions made in Section (7.2.2), we have only azimuthal velocity derived for the two regions viz., the inner region and the outer region. In the eye-wall, updraft and rotational wind motion about the vertical axis are witnessed. Since the vertical velocity is the same as that assumed by Kieu and Zhang (2009), we shall confine the discussion around the azimuthal velocity.

7.3.1.1 Azimuthal velocity

The role of δ which is a parameter in the formulation of the azimuthal velocity, the contribution due to the perturbation term and the edge of general viscosity consideration over the linear form assumed by Kieu and Zhang (2009) is required for discussion.

In order to examine the impact of δ, where $\beta\mu/(\lambda W_0) = 1 - \delta$, $0 \leq \delta \leq 1$ on the unperturbed azimuthal velocity v_{10}, we plot v_{10} versus z, displayed in Fig. 7.2, against a wide range of δ at $t = 0$ and for $\lambda = 2$, $\beta = 0.5$, $W_0 = 0.12$ and $r = 1$ which is the interface of the two regions. It is observed that the azimuthal velocity increases while ascending along the vertical axis up to a certain height and then begins to fall in magnitude. An interesting observation is that up to that height v_{10}, the zeroth order azimuthal velocity, increases with δ but coincides at non-dimensional $z = 1$ $\forall\delta$. Trends are exactly reverse above that. This is almost the same as that Kieu and Zhang (2009) discovered. This is to be noted that they used dimensional parameters.

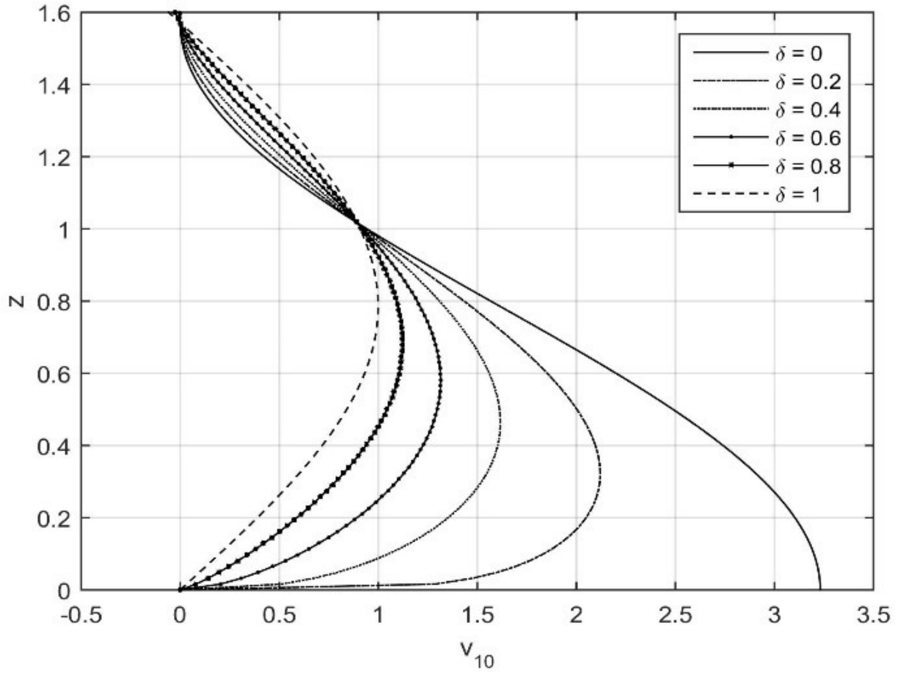

Figure 7.2: The diagram, based on Eq. (7.27), represents vertical profile of v_{10}, the zeroth order azimuthal velocity, for $t = 0$. Here $\lambda = 2$, $\beta = 0.5$, $W_0 = 0.12$, are the parameters used for the plot.

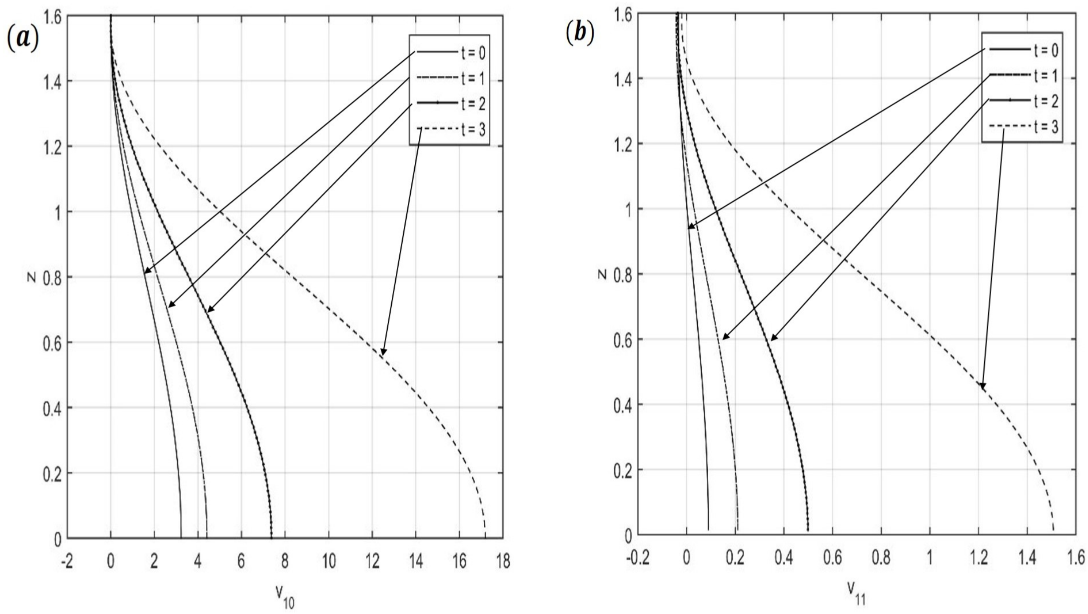

Figure 7.3: The diagram displays (a) the zeroth order and (b) the first order perturbed azimuthal velocity along the vertical axis for $\delta = 0$ at different instants mentioned in the legend.

The contribution of the first perturbation term is another vital aspect of this investigation. Accordingly, we plot v_{10} and v_{11} versus z by varying t in the range $0 - 3$ in Figs. 7.3. The two have similar patterns with v_1 exceeding a little bit in magnitude but both of them increase with time t. However, the real contribution of the perturbation term will be much less as it is multiplied by $\epsilon = Re^{-1}$ which is of the order of 10^{-4} for a real hurricane. The combined effect has been displayed in Figs. 7.4(a) and 7.4(b) respectively for $Re = 10000$ and 100. For comparatively small Reynolds number, the contribution of the perturbation term, in terms of magnitude, is quite significant. Moreover, the pattern we get here is quite similar to what Kieu and Zhang (2009) observed. Scales are distinct for the reason that they used dimensional parameters; however, they claim the figures to use non-dimensional units. If so, then probably they used different characteristic parameters which are nowhere mentioned in the article.

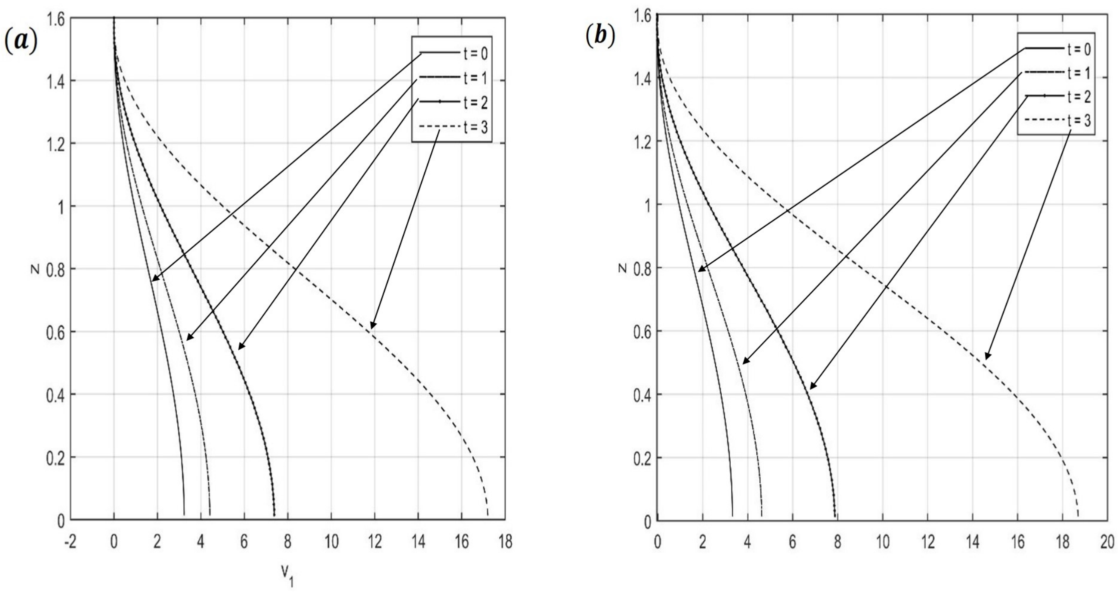

Figure 7.4: The diagram displays the azimuthal velocity, (a) for Reynolds number $Re = 10000$ (b) Reynolds number $Re = 100$, for $\delta = 0$ along the vertical axis at different instants.

7.3.1.2 Vertical pressure distribution

Pressure for region 1 is given by Eq. (7.39). Apart from the radial and axial coordinates, it depends also on time, viscosity and the radius of maximum wind. Therefore, in this subsection, we would discuss temporal, viscous and the radius of maximum wind impacts on radial and vertical pressure distribution. Vertical pressure distribution is displayed in Figs. 7.6, in which the parameters viz., time, the Reynolds number and the radial distance are varied respectively in Fig. $7.6(a)-(c)$ in order to examine their effects when other parameters are kept constant.

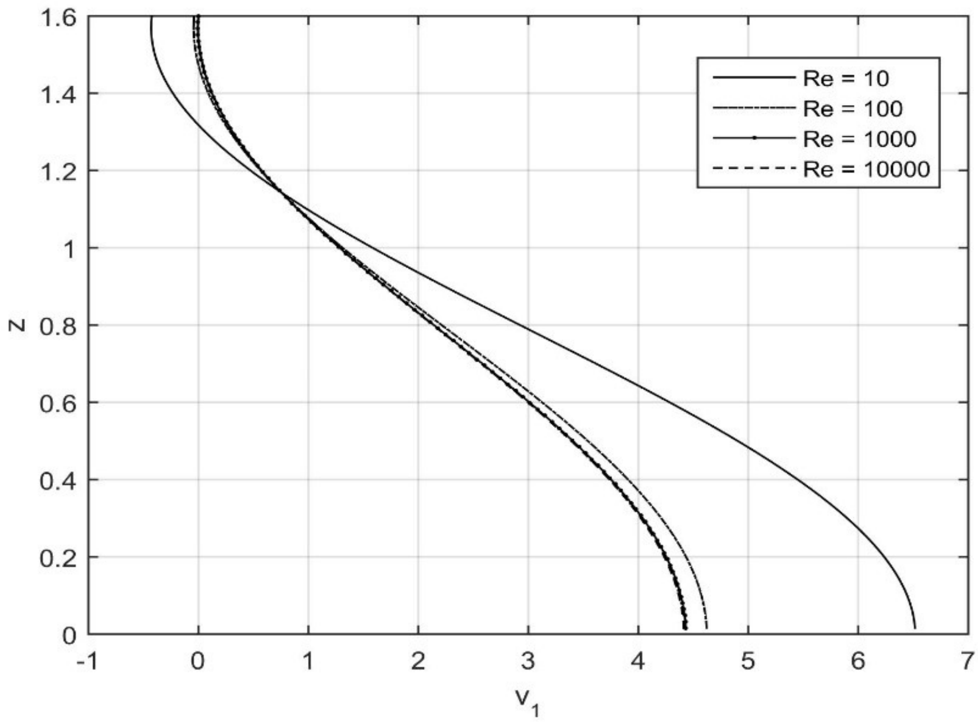

Figure 7.5: The diagram represents the azimuthal velocity v vs. z for different Reynolds number at time $t = 1$, and $\delta = 0$.

Keeping the reference pressure $p_1(1, 0, t)$, i.e., that at the radius of maximum wind on the ground we observe that with all variations in the three parts of the figure, pressure ascends with height. Setting $Re = 10000$ and the radius of maximum wind $a = 100$, temporal variation reveals in Fig. 7.6(a) that pressure drops with increasing time. Fixing $r = 20$, $a = 100$, $t = 1$, we find in Fig. 7.6(b) that pressure drops when Re is increased; whereas when $Re = 10000$, $a = 100$, $t = 1$, are kept unaltered, pressure ascends with the radial distance (see Fig. 7.6(c)).

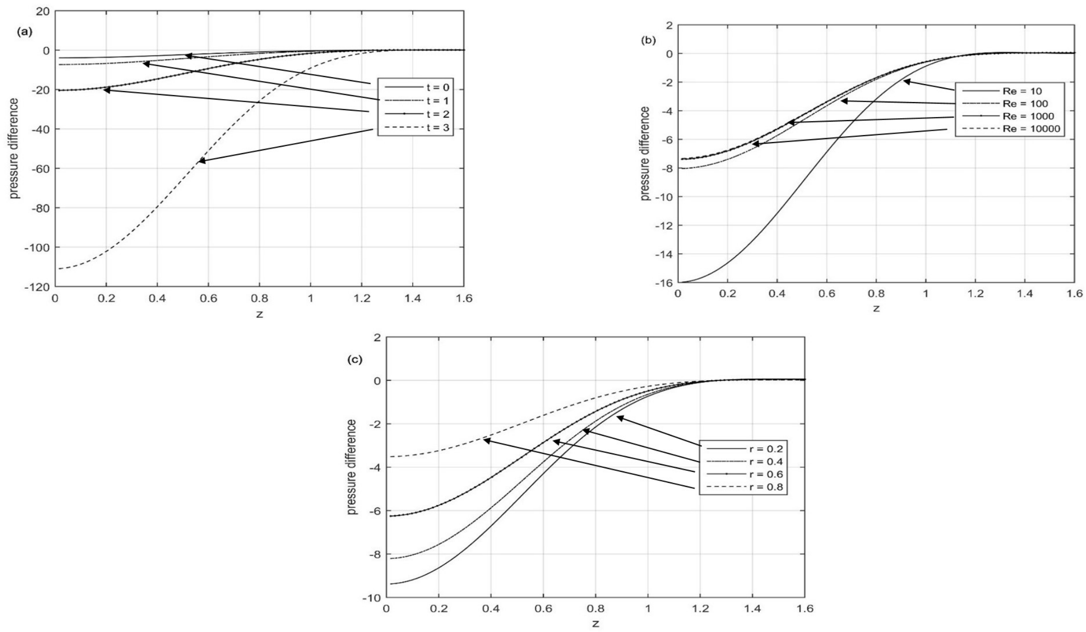

Figure 7.6: The diagrams display vertical pressure distribution for $\delta = 0$ and the impact of (a) time t ($Re = 10000$, $a = 100$) (b) the Reynolds number Re ($r = 20$, $a = 100$, $t = 1$) and (c) the radial distance ($Re = 10000$, $a = 100$, $t = 1$).

7.3.2 Analysis of the solution in the region 2

The region 2 which contains violently rotating wind with extremely moist air at the lower altitudes and its radial inflow downdraft supplies moisture for updraft through the boundary layer has an extremely important role to play for the updraft in the eye wall. We study the azimuthal velocity with the vertical profile of the azimuthal velocity plotted in Fig. 7.7 in a temporal range $0 - 3$. Unlike within the region 1 for which perturbation technique had to be used, exact solution has been obtained in region 2.

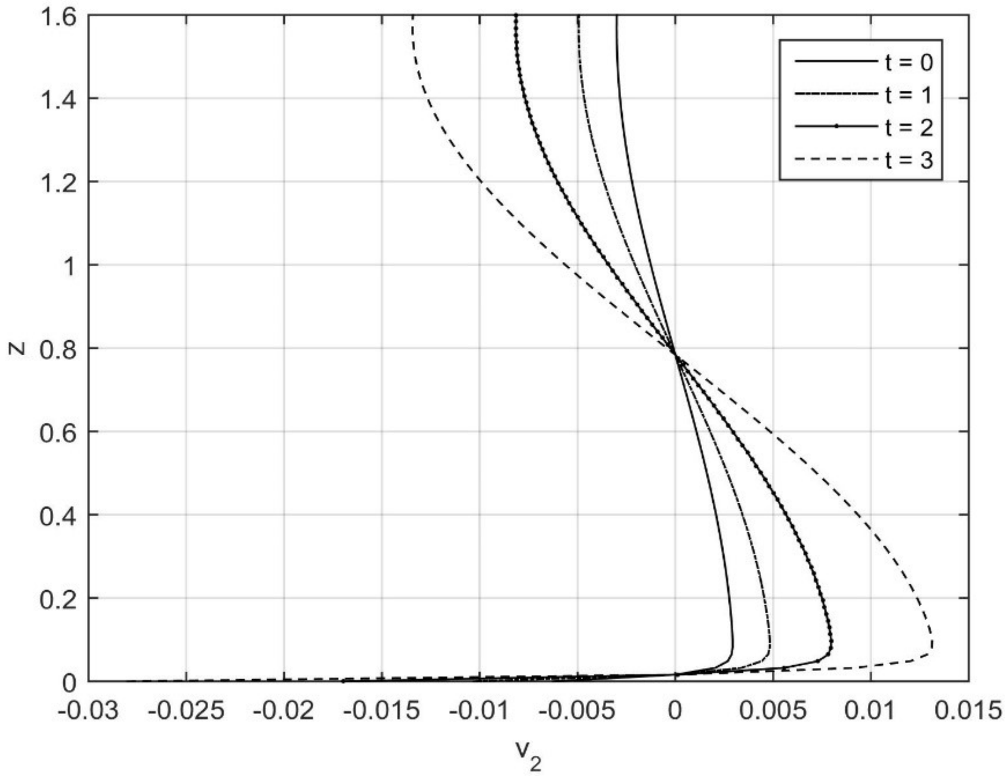

Figure 7.7: The diagram represents azimuthal velocity along the vertical axis for $\delta = 0$ at different instants mentioned in the legend for the second region.

Near the ground the azimuthal velocity is found to rise very fast with time, but reverse is the trend at a little bit height and becomes independent of time at high altitudes. It drops further with time at even higher altitudes. The trends are qualitatively similar to that in the region 1 but quantitative difference is there. In fact the two regions conform to the Rankines' model as assumed in the beginning. The presence of sine and cosine terms seem to periodically change the trends with height. This region has no vertical velocity.

7.3.3 Pressure deficit

The relationship between the central pressure deficit and the peak wind speed near the ground surface in a tropical cyclone has important consequences in meteorology from physical point of view. It is also related to risk of damage and loss of life (Chavas et al., 2017). The central pressure deficit in a tropical cyclone is defined as the difference in pressure between the centre of the storm and outside it. We denote it by $\Delta\tilde{p} = 1 - p_m/p_0$, where p_m is the minimum central pressure near the surface and p_0 is the environmental pressure at the outer edge of the storm.

The minimum central pressure near the surface may be obtained with the help of Eq. (7.39) and Eq. (7.51), which is

$$p_m(t) = -\frac{\lambda W_0 e^{\beta t}}{2}\left[\left(\beta + \frac{\lambda^2}{Re}\right)\log(R_m) + \frac{\lambda W_0}{4}\left(1 - \frac{1}{R_m^2}\right)e^{\beta t}\right] + \frac{S^2}{8}\left(1 - \frac{1}{R_m^2}\right)e^{2\beta t} + \frac{S^2}{2}e^{\beta t}\log(R_m) - \frac{1}{8}\left[\lambda W_0 e^{\beta t}\left\{2\left(\beta + \frac{\lambda^2}{Re}\right) - \lambda W_0 e^{\beta t}\right\} - S^2\right]. \tag{7.52}$$

It is observed that the central presure drop decreases with time (Fig. 7.8). The observation is similar to that Kieu and Zhang (2009) who found it to conform experimental dara.

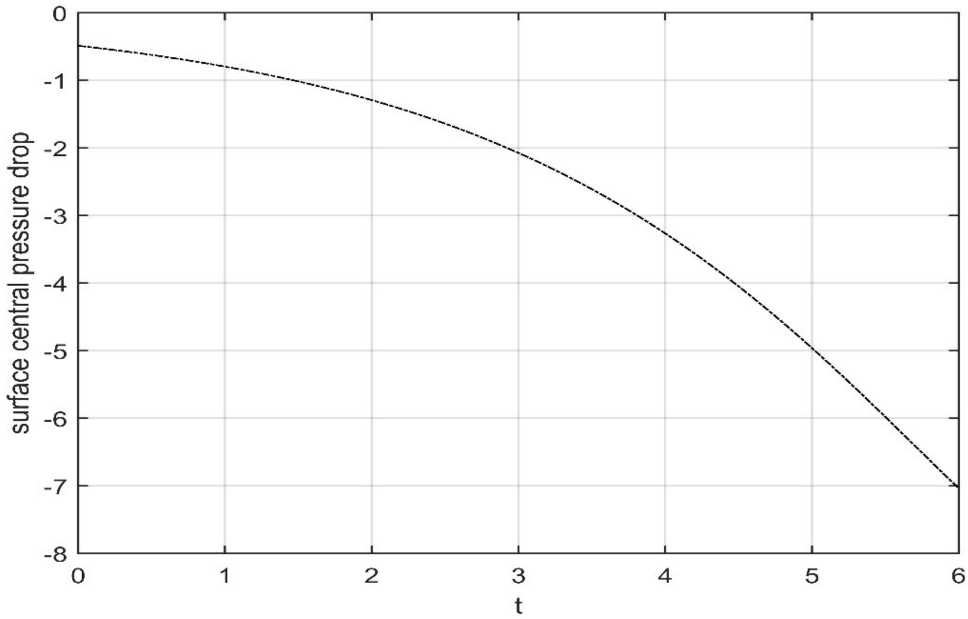

Figure 7.8: The diagram represents the variation of surface central pressure drop with time based on Eq. (7.52).

7.4 Conclusions

Unlike a special linear form of viscosity, we assumed a general type of viscosity for investigating the reason behind rapid intensification of cyclonic wind. The existence of double exponential terms was discovered as the reason for linear viscosity. Similar terms are observed even for the general form of viscosity. Hence, it is concluded that double exponential terms accelerate the rotational motion irrespective of the form of the viscosity.

The domain of analysis is split into two regions, one which is distinct in the sense that all the updraft in confined in this region only which entirely lies within the radius of maximum velocity and the other beyond it, which is without updraft, has only azimuthal velocity.

Vertical pressure depends on time, viscosity and the radius of maximum wind. Within the radius of maximum wind, we observe that pressure ascends with height and the radial distance but drops with time and also when the Reynolds number is increased.

The azimuthal velocity rises fast with time close to the ground, but this dependence diminishes at a height little above the ground. At a considerable height, time sieges to be a factor and further, above that, trends reverse.

Near the ground the azimuthal velocity is found to rise very fast with time, but reverse is the trend at a little above and becomes independent of time at high altitudes. It drops further with time at even higher altitudes.

Mathematical method for the solution is perturbation technique. It is found that the perturbation terms behave almost identically with the terms without perturbation. The significance of their contribution definitely depends on the magnitude of the Reynolds number. Unlike within the region 1 for which perturbation technique had to be used, exact solution is obtained for region 1. The trends are qualitatively similar to that in the region 1 but quantitative difference is definitely marked. It is also observed that the central pressure drop decreases with time.

Appendices

Appendix A

Analytical solution for the eye wall

Plunging Eq. (7.22) into Eq. (7.20), we obtain

$$\frac{dG_0'}{dz} = -\left[\left(\frac{1}{H_1} - \lambda\cot(\lambda z)\right) + \frac{\beta\mu}{W_0}\csc(\lambda z)\right] G_0', \tag{A1}$$

On integration, we have

$$\log\left(\frac{G_0'}{K}\right) = -\left[\frac{z}{H_1} - \log(\sin(\lambda z)) - \frac{\beta\mu}{\lambda W_0}\log(\cot(\lambda z) + \csc(\lambda z))\right],$$

$$G_0'(z) = Ke^{-(z/H_1)}\left\{\sin(\lambda z)\right\}^(1 - \beta\mu/\lambda W_0)\left\{2\cos\left(\frac{\lambda z}{2}\right)\right\}^{2(\beta\mu/\lambda W_0)}, \tag{A2}$$

where K is an integration constant (with unit per second) that determines the initial strength of the vortex.

Eq. $(A2)$ contains an infinite number of possible solution depending on the values of μ. However, the requirements for the regularity of Eq. $(A2)$ at $z = 0$ impose a strong restriction on the range of μ. Using L' Hospital rule for regular solution we have $1 \geq \beta\mu/\lambda W_0$. Taking $\beta\mu/\lambda W_0 = 1 - \delta$, where $0 \leq \delta \leq 1$. Thus, Eq. $(A2)$ reduces to

$$G_0'(z) = Ke^{-(z/H_1)}\left\{\sin(\lambda z)\right\}^{\delta}\left\{\cos\left(\frac{\lambda z}{2}\right)\right\}^{2-\delta}, \tag{A3}$$

Appnendix B

To solve Eq. (7.28), we let $G_1(z,t) = \Gamma(z,t)G_0(z,t)$, and use Eq. (7.20) we obtain

$$e^{-\beta t}\frac{\partial \Gamma}{\partial t} + W_0 \sin(\lambda z)\frac{\partial \Gamma}{\partial z} = F(z,t). \tag{B1}$$

where

$$e^{\beta t}F(z,t) = \frac{(2\delta-1)\delta\lambda^2}{4}\cot^2\left(\frac{\lambda z}{2}\right) + \frac{(1-3\delta)\lambda}{2H_1}\cot\left(\frac{\lambda z}{2}\right) + \left\{\frac{1}{H_1^2} - \frac{(5-2\delta)\delta\lambda^2}{4}\right\} + \frac{(3-\delta)\lambda}{2H_1}\tan\left(\frac{\lambda z}{2}\right) - \frac{\delta\lambda^2}{4}\csc^2\left(\frac{\lambda z}{2}\right), \tag{B2}$$

Now we solve Eq. $(B2)$ by applying Lagrange subsidiary equation

$$\frac{dt}{e^{-\beta t}} = \frac{dz}{W_0\sin(\lambda z)} = \frac{d\Gamma}{F(z,t)}, \tag{B3}$$

Solution of the first equality is

$$\frac{e^{\beta t}}{\beta} - \frac{1}{\lambda W_0}\log\left(\frac{\lambda z}{2}\right) = A, \tag{B4}$$

where A is an integration constant.

The second integral is obtained by last two equality

$$\frac{d\Gamma}{dz} = \frac{F(z,t)}{W_0\sin(\lambda z)}, \tag{B5}$$

Integrating Eq. ($B5$), with respect to z, we obtain

$$\beta^{-1}\Gamma(z,t) = \frac{\lambda^2\delta(\delta-1)I_1}{4} + \frac{\lambda^2(2-\delta)(1-\delta)I_2}{4} - \frac{\delta\lambda I_3}{H_1} + \frac{(2-\delta)\lambda I_4}{H_1} + \left(\frac{1}{H_1^2} - \frac{(2\delta-\delta^2+1)\lambda^2}{2}\right) I_5 + \beta^{-1}\phi(A), \tag{B6}$$

where

$$\left.\begin{aligned}
I_1 &= \int \frac{\cot^2\left(\frac{\lambda z}{2}\right)}{W_0\sin(\lambda z)\left(A + \frac{1}{\lambda W_0}\log\left\{\tan\left(\frac{\lambda z}{2}\right)\right\}\right)} dz,\\
I_2 &= \int \frac{\tan^2\left(\frac{\lambda z}{2}\right)}{W_0\sin(\lambda z)\left(A + \frac{1}{\lambda W_0}\log\left\{\tan\left(\frac{\lambda z}{2}\right)\right\}\right)} dz,\\
I_3 &= \int \frac{\cot\left(\frac{\lambda z}{2}\right)}{W_0\sin(\lambda z)\left(A + \frac{1}{\lambda W_0}\log\left\{\tan\left(\frac{\lambda z}{2}\right)\right\}\right)} dz,\\
I_4 &= \int \frac{\tan\left(\frac{\lambda z}{2}\right)}{W_0\sin(\lambda z)\left(A + \frac{1}{\lambda W_0}\log\left\{\tan\left(\frac{\lambda z}{2}\right)\right\}\right)} dz,\\
I_5 &= \int \frac{dz}{W_0\sin(\lambda z)\left(A + \frac{1}{\lambda W_0}\log\left\{\tan\left(\frac{\lambda z}{2}\right)\right\}\right)} dz.
\end{aligned}\right\} \tag{B7}$$

Substitutng $A + \frac{1}{\lambda W_0}\log\left\{\tan\left(\frac{\lambda z}{2}\right)\right\} = P$, we get $dz = W_0\sin(\lambda z)dP$, $\tan\left(\frac{\lambda z}{2}\right) = e^{(P-A)\lambda W_0}$ and $\cot\left(\frac{\lambda z}{2}\right) = e^{-(P-A)\lambda W_0}$, thus integrals ($B7$) reduces to

$$I_1 = e^{2\lambda W_0 A}I_{11},\ I_2 = e^{-2\lambda W_0 A}I_{22},\ I_3 = e^{\lambda W_0 A}I_{33},\ I_4 = e^{-\lambda W_0 A}I_{44},\ I_5 = \int \frac{e^P}{P}dP, \tag{B8}$$

where

$$I_1 = \int \frac{e^{-2\lambda W_0 P}}{P}dP,\ I_{22} = \int \frac{e^{2\lambda W_0 P}}{P}dP,\ I_{33} = \int \frac{e^{-\lambda W_0 P}}{P}dP,\ I_{44} = \int \frac{e^{\lambda W_0 P}}{P}dP, \tag{B9}$$

Thus, we have

$$\beta^{-1}\Gamma(z,t) = \frac{\lambda^2\delta(\delta-1)}{4}\exp\left(2\lambda W_0\frac{e^{\beta t}}{\beta}\right)\cot^2\left(\frac{\lambda z}{2}\right)I_{11} + \frac{\lambda^2(2-\delta)(1-\delta)}{4}$$
$$\times \exp\left(-2\lambda W_0\frac{e^{\beta t}}{\beta}\right)\tan^2\left(\frac{\lambda z}{2}\right)I_{22} - \frac{\delta\lambda}{H_1}\exp\left(\lambda W_0\frac{e^{\beta t}}{\beta}\right)\cot\left(\frac{\lambda z}{2}\right)I_{33} +$$
$$\frac{(2-\delta)\lambda}{H_1}\exp\left(-\lambda W_0\frac{e^{\beta t}}{\beta}\right)\tan\left(\frac{\lambda z}{2}\right)I_{44} + \left(\frac{1}{H_1^2} - \frac{(2\delta-\delta^2+1)\lambda^2}{2}\right)\log\left(\frac{e^{\beta t}}{\beta}\right)$$
$$+ \beta^{-1}\phi(A), \tag{B10}$$

Thus, we get the first order solution as

$$G_1(z,t) = 2Ke^{-(z/H_1)}\left\{\sin\left(\frac{\lambda z}{2}\right)\right\}^{\delta}\left\{\cos\left(\frac{\lambda z}{2}\right)\right\}^{2-\delta}\Gamma(z,t)\exp\left(\frac{\lambda W_0}{\beta}(1-\delta)e^{\beta t}\right). \tag{B11}$$

****