Chapter 6

Analysis of $MAP/G_r^{(a,b)}/1/N$ queue with queue length dependent single and multiple vacation

6.1 Introduction

In the literature on classical vacation queueing models, it is mostly assumed that customers are arriving at the system following the Poisson distribution, however, these assumptions of Poisson arrivals fit the model only when the arrival process is uncorrelated. In modern communication systems and ATM networks the arrivals of packetized data to a statistical multiplexer are bursty in nature. These bursty arrivals cannot be modeled well by Poisson processes. The Markovian arrival process (*MAP*) (Neuts (1979, 1981)) is a very good representation for modeling next generation communication networks, viz., 4*G* with bursty and correlated traffic. *MAP* is a rich class of point processes which contain many popular arrival processes such as Poisson process, PH-renewal process, Markov modulated Poisson process (*MMPP*), etc. We refer Neuts (1992) for the applications based on *MAP*. Lucantoni et al. (1990) introduced the correlated arrival process in vacation queueing models. They studied an infinite buffer $MAP/G/1$ queue with general vacation time and obtained steady state queue length and waiting time distribution. In context of *MAP* arrivals various vacation queueing models, either with finite buffer or infinite buffer have been studied, see e.g., Lucantoni et al. (1990), Blondia (1991), Alfa (1995), Choi et al. (1998), Niu and Takahashi (1999), Ho Woo Lee and Park (2001), Gupta et al. (2005), Gupta and Sikdar (2006), Banik et al. (2006b), Gupta et al. (2007), Liu and Wu (2009), Singh et al. (2014b), etc. It seems that Blondia (1991) was the first who analyzed a finite buffer vacation queueing model with *MAP*. Further, Niu and Takahashi (1999) analyzed a $MAP/G/1/N$ queue with single and multiple vacation under exhaustive service discipline along with close-down and/or setup times and obtained the queue length distribution at arbitrary and pre arrival epoch. Gupta and Sikdar (2006) considered $MAP/G/1/N$ queue with single and multiple vacation and using the supplementary variable technique and the embedded Markov chain technique obtained the queue length distribution at various epoch, viz., service completion, vacation termination, departure, arbitrary and pre-arrival epoch.

Although bulk service queues with *MAP* have been studied to a great extent, however, bulk service queues with correlated arrivals under various vacation rules have not been explored much in literature. Bulk service queues with vacation and correlated arrivals are studied in past by few researchers, see e.g., Gupta and Sikdar (2004b), Sikdar and Gupta (2005b), Sikdar (2008), Sikdar and Samanta (2016) and the references therein. Gupta and Sikdar (2004b) studied $MAP/G^{(a,b)}/1/N$ queue with single vacation and then Sikdar (2008) extended their research to multiple vacation. Recently, Sikdar and Samanta (2016) studied $BMAP/G^{Y}/1/N$ queue with single and multiple vacation in an unified way and obtained the queue length distributions at various epochs.

Banerjee et al. (2015) analyzed finite buffer batch size dependent bulk service queue with *MAP* under GBS rule. By using the supplementary variable technique (remaining service time as supplementary variable) and the embedded Markov chain technique they obtained joint distribution of queue content, server content and phase of the arrivals at various epochs. Pradhan and Gupta (2017a) studied infinite buffer $MAP/G_r^{(a,b)}/1$ queue and using bivariate vector generating function method obtained the joint distribution of queue content, server content and phase of arrivals at various epochs. Both the above literature have been studied in continuous time setup. Alfa et al. (1995) considered $DMAP/G^{(1,a,b)}/1/N$ queue in discrete time setup and using matrix analytic method and embedded Markov chain technique they obtained all required joint probabilities at various epochs.

In most of the research on the vacation queueing models it has been considered that the server will go for a vacation of random length which is independent of the queue length at the vacation initiation epoch. The vacation queueing models in which the length of vacation period is modulated depending upon the queue length at vacation initiation epoch is termed as queue length dependent vacation and have been studied by Harris and Marchal (1988), Lee and Srinivasan (1989), Shin and Pearce (1998), Banik (2013a). Banik (2013a) considered $BMAP/G/1/N$ queue with E-limited service and queue length dependent vacation and numerically shown that the queue length dependent vacation policy helps in reducing congestion.

To the best of authors' knowledge $MAP/G_r^{(a,b)}/1/N$ queue with queue length dependent single and multiple vacation has not been addressed by the researchers yet. In view of this, in this chapter we have considered finite buffer bulk service queueing model with single and multiple vacation and MAP. The two vacation policies : Single vacation (SV) and multiple vacation (MV) are discussed in this chapter in an unified way. The arrivals of the customers to the system occur according to the Markovian arrival process (MAP). Service is rendered by a single server following GBS rule. Batch size dependent service and queue length dependent vacation policy both are considered here. The model is analyzed using the embedded Markov chain technique, to obtain the joint distribution of queue content serving batch size and phase of the arrivals; and queue content, vacation type taken by the server and phase of the arrivals at service and vacation completion epoch, respectively. Using the supplementary variable technique we obtain a relation between service/vacation completion epoch and arbitrary epoch joint probabilities.

For use in sequel, let $e_i(r)$, $\mathbf{e}(r)$ are denoting respectively, the column vector of dimension (r) with 1 at i^{th}-position and 0 elsewhere, a column vector of dimension (r) with all entries equal to 1. When there is no need to emphasize the dimension of these vectors we will suppress the suffix. Thus when there is no need to emphasize the dimension of these vectors we will suppress the suffix. Thus, $\mathbf{e}$ will denote a column vector of 1's of appropriate dimension, and e_i will denote a column vector of appropriate dimension consisting 1 at i^{th}-position and

0 elsewhere. The notation ‘T’ appearing in the superscript will stand for the transpose of a matrix.

The outline of the rest of this chapter is as follows: mathematical description along with the use of embedded Markov chain technique, to obtain the joint distributions at service/vacation completion epoch, is explained in Section 6.2 and Section 6.2.1, respectively. Next in Section 6.2.2, a relation between the joint distributions of service/vacation completion epoch and arbitrary epoch is established with the help of supplementary variable technique. Section 6.3 is assigned for the various performance measures. Numerical results and their discussion for several service/vacation time distributions are presented in Section 6.5. Some conclusions of this chapter are drawn in Section 6.6.

6.2 Model description and steady state analysis

Markovian arrival process (MAP)

The Markovian arrival process (MAP) is a generalization of the Poisson process where the arrivals are governed by an underlying m-state Markov chain. With probability c_{ij}, $1 \le i, j \le m$, there is a transition from state i to state j without an arrival, and with probability d_{ij}, $1 \le i, j \le m$, there is a transition from state i to state j with an arrival. The matrix $C = [c_{ij}]$ has nonnegative off-diagonal and negative diagonal elements, and the matrix $D = [d_{ij}]$ has nonnegative elements. Let $\hat{N}(t)$ denote the number of customers arriving in $(0,t]$ and $J(t)$ be the state of the underlying Markov chain at time t. Then $\{\hat{N}(t), J(t)\}$ is a two-dimensional Markov process with state space . The infinitesimal generator of the above Markov process is given by

$$\hat{Q} = \begin{pmatrix} C & D & 0 & 0 & \cdots \\ 0 & C & D & 0 & \cdots \\ 0 & 0 & C & D & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix},$$

$\{\hat{N}(t), J(t)\}$ is called the Markovian arrival process (MAP). Since $\hat{Q}$ is the infinitesimal generator of the MAP, we have

$$(C+D)e = 0,$$

where e is an $m \times 1$ vector with all its elements equal to 1.

Since $C+D(=Q,\ say)$ represents the irreducible infinitesimal generator of the underlying CTMC of the *MAP*, $\{J(t)\}$, there exists a stationary probability vector ϖ satisfying $\varpi Q = 0$ and $\varpi e = 1$. The fundamental arrival rate (average arrival rate) of the above Markov process is given by $\lambda^* = \varpi De$.

Let $\hat{P}(n,t)$ $(n \geq 0, t \geq 0)$ be the square matrices of dimension m whose (i,j)th elements are the conditional probabilities defined as $\hat{p}_{i,j}(n,t) = prob.\{\hat{N}(t) = n, J(t) = j | \hat{N}(0) = 0, J(t) = i\}; n \geq 0, i,j = 1,2,...,m.$

These matrices, associated with the counting process $\{\hat{N}(t), J(t)\}$ $(t \geq 0)$ satisfy the following system of difference-differential equations

$$\begin{aligned} \hat{P}'(0,t) &= \hat{P}(0,t)C, \\ \hat{P}'(n,t) &= \hat{P}(n,t)C + \hat{P}(n-1,t)D, n \geq 1, \end{aligned} \tag{6.1}$$

with $\hat{P}(0,0) = I_m$. For more details see Neuts and Jian-Min (1996).

The matrix generating function $P^*(z,t)$, defined by

$$P^*(z,t) = \sum_{n=0}^{\infty} \hat{P}'(n,t) z^n, \quad |z| \leq 1,$$

satisfies

$$\frac{d}{dt} P^*(z,t) = P^*(z,t)(C+zD), \quad |z| \leq 1,$$

$$P^*(z,0) = I.$$

Solving the above matrix–differential equation, we get

$$P^*(z,t) = e^{(C+zD)t}, \quad |z| \leq 1, \quad t \geq 0.$$

Model description: We consider a finite buffer $MAP/G^{(a,b)}/1$ queue with batch size dependent service and queue length dependent vacation. Customer arrivals obey a Markovian arrival process (*MAP*) which is a more general arrival process. A customer that arrives to find the server busy will wait in the waiting room. We assume that the waiting room capacity is finite. There is a finite waiting room of size N such that any arrival finding the buffer full will be considered lost. Suppose that a and b satisfying $1 \leq a, b \leq N$, are two pre-determined thresholds. The server offers services in batches of varying sizes according to the GBS rule. That is, if the number of customers waiting in the queue is less than a, then the server waits until the number of customers reach a. Specifically, The service discipline is first-come-first-served (FCFS). The service is provided by a single server. The service times are assumed to be generally distributed and dependent on the batch size. Specifically, let $T_r(t)$ denote the service time for a batch of size r with distribution function $S_r(.)$, probability density function (pdf) $s_r(.)$, Laplace–Stieltjes transform (LST) $s_r^*(.)$ and mean service time $\tilde{s_r}$. If at the end of a service the queue length is less than 'a', then the server will go for vacation under certain vacation policy considered (for example, single vacation, multiple vacation).

In this chapter, we have studied the queueing model under consideration with two type of vacation rules: single vacation (SV) and multiple vacation (MV), in an unified way by defining an indicator variable δ_s as follows.

$$\delta_s = \begin{cases} 1, & \text{for SV rule}, \\ 0, & \text{for MV rule}. \end{cases}$$

It should be noted here that, one can obtain the results for $MAP/G_r^{(a,b)}/1/N$ queue with SV by substituting $\delta_s = 1$ and that of MV by substituting $\delta_s = 0$.

We employed here the queue length dependent vacation policy. The vacation rule must be decided at the beginning of the analysis and is not allowed to change in an intermediate stage. We have considered that the vacation time of random length changes dynamically depending on the number of customers remaining in the system at vacation initiation epoch. That is, when the server finishes serving a batch and finds less than 'a' customers in the queue, say 'k' $(0 \leq k \leq a-1)$, then the server leaves for a vacation of random length which is considered to be dependent on the number of customers remaining in the system (i.e., k), at vacation initiation epoch and is termed as $k^{th}-$type of vacation taken by the server throughout the chapter. On returning from a vacation if server finds 'a' or more customers waiting in the queue it resumes its service with maximum of 'b' customers, otherwise it will remain idle or leave for another vacation depending on the vacation rule under consideration, i.e., single vacation or multiple vacation, respectively. The vacation time distribution $(V^{[k]}(\cdot))$ is considered to be generally distributed and dependent on queue length k $(0 \leq k \leq a-1)$ at vacation initiation epoch with pdf $v^{[k]}(\cdot)$, LST $v^{[k]*}(\cdot)$ and mean vacation time $\tilde{v}^{[k]}$.

Note:- It should be noted here that, whenever server leaves for a vacation of random length $V^{[k]}$, leaving k $(0 \leq k \leq a-1)$ in the system, will be termed as *k-th type of vacation* taken by the server or simply *k-th type of vacation*, throughout the chapter.

The steady-state analysis of the model under study will be carried out using the embedded Markov chain approach since the service/vacation times are assumed to be generally distributed. First, we will look at the semi-Markov process embedded at points of departure of customers. Towards this end, we define the following conditional probabilities.

$\left[A_n^{(r)}(x)\right]_{i,j}; x \geq 0,\ 1 \leq i,j \leq m,\ a \leq r \leq b:$ is the conditional probability that, starting with a service/vacation completion which left at least a customers in the queue with the arrival process in state i, the next departure of a batch occurs no later than time x and at that time the phase of the arrival process is j, and exactly n new customers arrive during the service period of the batch of size r $(a \leq r \leq b)$, servicing with service time distribution $S_r(t)$,

$\left[B_{k,n}^{(a)}(x)\right]_{i,j}; x \geq 0,\ 1 \leq i,j \leq m,\ 0 \leq k \leq a-1:$ is the conditional probability that, starting with a vacation completion which left k customers in the queue with the arrival process in state i, the next departure of a batch (of size a) occurs no later than time x and at that time the phase of the arrival process is j, and exactly n new customers arrive during the service period of the batch, servicing with service time distribution $S_a(t)$.

$\left[U_n^{[k]}(x)\right]_{i,j}; x \geq 0,\ 1 \leq i,j \leq m,\ 0 \leq k \leq a-1:$ is the conditional probability that, starting with a service/vacation completion which left k customers in the queue with the arrival process in state i, the (next) vacation completion occurs no later than time x and at that time the phase of the arrival process is j, and exactly n new customers arrive during the vacation period of *k-th type of vacation* time distribution $V^{[k]}(t)$,

Denote by $A_n^{(r)}(x)$, $B_{k,n}^{(a)}(x)$ and $U_n^{[k]}(x)$, the square matrices of order m corresponding to each i,j-th element are given by $\left[A_n^{(r)}(x)\right]_{i,j}$, $\left[B_{k,n}^{(a)}(x)\right]_{i,j}$ and $\left[U_n^{[k]}(x)\right]_{i,j}$ respectively.

Hence using the definition of $A_n^{(r)}(x)$, $B_{k,n}^{(a)}(x)$ and $U_n^{[k]}(x)$ we have, for $n \geq 0$,

$$\begin{aligned}
A_n^{(r)}(x) &= \int_0^x \hat{P}(n,t)dS_r(t),\\
B_{k,n}^{(a)}(x) &= \int_0^x \hat{P}(a-1-k,x-t)DA_n^{(a)}(t)dt,\\
U_n^{[k]}(x) &= \int_0^x \hat{P}(n,t)dV^{[k]}(t),\\
\bar{A}_n^{(r)}(x) &\triangleq \sum_{j=n+1}^{\infty} A_j^{(r)}(x),\\
\bar{B}_{k,N}^{(a)}(x) &\triangleq \sum_{j=N+1}^{\infty} B_{k,j}^{(a)}(x),\\
\bar{U}_n^{[k]}(x) &\triangleq \sum_{j=n+1}^{\infty} U_j^{[k]}(x),
\end{aligned} \tag{6.2}$$

for use in sequel, we define

$$\begin{aligned}
\tilde{D} &\triangleq (C)^{-1}D,\\
A_n^{(r)} &\triangleq A_n^{(r)}(\infty),\\
\bar{A}_n^{(r)} &\triangleq \bar{A}_n^{(r)}(\infty),\\
U_n^{[k]} &\triangleq U_n^{[k]}(\infty),\\
\bar{U}_n^{[k]} &\triangleq \bar{U}_n^{[k]}(\infty),\\
B_{k,n}^{(a)} &\triangleq B_{k,n}^{(a)}(\infty) = \tilde{D}^{a-k}A_n^{(a)},\\
\bar{B}_{k,N}^{(a)} &\triangleq \bar{B}_{n,N}^{(a)}(\infty) = \tilde{D}^{a-k}\bar{A}_N^{(a)}.
\end{aligned} \tag{6.3}$$

The last two equations in follow from the fact that $\int_0^{\infty} \hat{P}(n,t)Ddt = \tilde{D}^{n+1}$, $n \geq 0$ (see Appendix A).

6.2.1 Probability distribution at service/vacation completion epoch

In this section, we obtain the joint distribution of the number of customers in the queue and number with the serving batch and the phase of the arrival process at service completion-epoch, by tracking down the number of customers with the server besides observing the number left behind in the queue by the serving batch and change in phase of the arrival during service/vacation completion, also we obtain joint distribution of queue content as well as type of the vacation taken by the server and phase of the arrival at vacation termination epoch by tracking down the number of customers in the queue when $k^{th}-$type vacation taken besides observing the number of customers enters into the queue during vacation period and change in phase of arrival during service/vacation completion. Towards this end consider the system at service completion/vacation termination epochs which are taken as embedded points. Let $t_0, t_1, t_2, ...,$ be the epochs at which either service completion or vacation termination occurs. The state of the system at t_i is defined as $\Omega = \{N_q(t_i), N_s(t_i), J(t_i)\} \cup \{N_q(t_i), \kappa(t_i), J(t_i)\}$ where, $N_q(t_i)$ is the number of customers in the queue at time t_i, $N_s(t_i)$ denotes the number of customers with the serving batch when server is in busy state at time t_i, $\kappa(t_i)$ denotes the vacation type taken by the server which is about to finish at time epoch t_i, $J(t_i)$ denotes phase of the arrival process when server is in busy/vacation state at time t_i. Let us define the following probabilities at the embedded points :

- $\pi_i^+(n,r)$ be the probability that there are n customers are in the queue at the service completion epoch of a batch of customers of size r and the phase of the arrival process is in i, $0 \le n \le N$, $a \le r \le b$, $1 \le i \le m$,
- $\omega_i^+(n,k)$ be the probability that there are $n+k$ customers are present in the queue at k^{th}-type vacation termination epoch of the server and the phase of the arrival process is in i, $0 \le k \le a-1$, $0 \le n \le N-k$, $1 \le i \le m$,
- denote the vectors

$\pi^+(n,r) \equiv \left(\pi_1^+(n,r), \pi_2^+(n,r), ..., \pi_m^+(n,r)\right)$, $\omega^+(n,k) \equiv \left(\omega_1^+(n,k), \omega_2^+(n,k), ..., \omega_m^+(n,k)\right)$,

The joint distribution $\pi_i^+(n,r)$ and $\omega_i^+(n,k)$ can be obtained by solving the system of equations $\Pi\mathscr{P} = \Pi$, where

- $\Pi = (\tilde{\pi}, \tilde{\omega}) = (\pi^+(0), \pi^+(1), ..., \pi^+(N), \omega^+(0), \omega^+(1), ..., \omega^+(N))$, where,

- $\tilde{\pi}$ and $\tilde{\omega}$ are row vectors of dimension $(N+1)$ and is defined by

$$\tilde{\pi} = \left(\pi^+(0), \pi^+(1), ..., \pi^+(N)\right),\ \tilde{\omega} = \left(\omega^+(0), \omega^+(1), ..., \omega^+(N)\right),$$

- each $\pi^+(n)$ $(0 \leq n \leq N)$ is a row vector of dimension '$m(b-a+1)$' and is given by

$$\pi^+(n) = \left(\pi^+(n,a), \pi^+(n,a+1), ..., \pi^+(n,b)\right),$$

- each $\omega^+(n)$ is a row vector of dimension $(n+1)$ for $0 \leq n \leq a-2$, and of dimension a for $a-1 \leq n \leq N$ and is given by

$$\omega^+(n) \equiv \begin{cases} (\omega^+(n,0), \omega^+(n-1,1), ..., \omega^+(0,n)) & \text{for}\, 0 \leq n \leq a-2, \\ (\omega^+(n,0), \omega^+(n-1,1), ..., \omega^+(n-a+1,a-1)) & \text{for}\, a-1 \leq n \leq N, \end{cases}$$

- $\left(\pi^+(n)\mathbf{e}\right)$ be the probability that there are n customers are in the queue at service completion epoch of a batch, $0 \leq n \leq N$, where $\mathbf{e}$ is a column vector of dimension $m(b-a+1)$,

- $(\omega^+(n)\mathbf{e})$ be the probability that there are n customers are in the queue at vacation completion epoch of the server, $0 \leq n \leq N$, where $\mathbf{e}$ is a column vector of appropriate dimension, i.e., of dimension $m(n+1)$ for $0 \leq n \leq a-2$ and of dimension ma for $a-1 \leq n \leq N$.

$\mathscr{P}$ is the one-step transition probability matrix (TPM) of dimension $m\left((N+1)(b-a+1) + \frac{a(a-1)}{2} + a(N-a+1)\right)$, and is given by

$$\mathscr{P} = \begin{pmatrix} \Phi & \Theta \\ \Lambda & \Psi \end{pmatrix},$$

where, Φ, Θ, Λ and Ψ are block matrices of dimension $m(N+1)(b-a+1)\times m(N+1)(b-a+1)$, $m(N+1)(b-a+1)\times m\left(\frac{a(a-1)}{2}+a(N-a+1)\right)$, $m\left(\frac{a(a-1)}{2}+a(N-a+1)\right)\times m(N+1)(b-a+1)$ and $m\left(\frac{a(a-1)}{2}+a(N-a+1)\right)\times m\left(\frac{a(a-1)}{2}+a(N-a+1)\right)$, respectively.

Note : It is to be noted here that $\mathscr{A}_j^{(r)}(l,k)$, $\mathfrak{B}_{n,j}^{(r)}(l,k)$, $U_j^{(r)}(l,k)$ are the block matrices of dimension $lm\times km$. However, in sequel we will denote $\mathscr{A}_j^{(r)}(l,k)$, $\mathfrak{B}_{n,j}^{(r)}(l,k)$ and $U_j^{(r)}(l,k)$ by $\mathscr{A}_j^{(r)}$, $\mathfrak{B}_{n,j}^{(r)}$ and $U_j^{(r)}$ respectively and they will represent the block matrices of appropriate dimension.

Let us now describe the block matrices Φ, Θ, Λ and Ψ in detail for completeness. Each and every elements of the block matrix Φ represents the transition probabilities among the service completion epochs and is given by

$$\Phi=\begin{array}{c} \\ 0\\ \vdots\\ a-1\\ a\\ a+1\\ \vdots\\ b\\ b+1\\ \vdots\\ N\end{array}\begin{array}{c}\begin{array}{cccccccc}0 & 1 & \dots & N-b-1 & N-b & \dots & N-1 & N\end{array}\\ \left(\begin{array}{cccccccc}
0 & 0 & \dots & 0 & 0 & \dots & 0 & 0\\
\vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots\\
0 & 0 & \dots & 0 & 0 & \dots & 0 & 0\\
\mathscr{A}_0^{(1)} & \mathscr{A}_1^{(1)} & \dots & \mathscr{A}_{N-b-1}^{(1)} & \mathscr{A}_{N-b}^{(1)} & \dots & \mathscr{A}_{N-1}^{(1)} & \bar{\mathscr{A}}_N^{(1)}\\
\mathscr{A}_0^{(2)} & \mathscr{A}_1^{(2)} & \dots & \mathscr{A}_{N-b-1}^{(2)} & \mathscr{A}_{N-b}^{(2)} & \dots & \mathscr{A}_{N-1}^{(2)} & \bar{\mathscr{A}}_N^{(2)}\\
\vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots\\
\mathscr{A}_0^{(b-a+1)} & \mathscr{A}_1^{(b-a+1)} & \dots & \mathscr{A}_{N-b-1}^{(b-a+1)} & \mathscr{A}_{N-b}^{(b-a+1)} & \dots & \mathscr{A}_{N-1}^{(b-a+1)} & \bar{\mathscr{A}}_N^{(b-a+1)}\\
0 & \mathscr{A}_0^{(b-a+1)} & \dots & \mathscr{A}_{N-b-2}^{(b-a+1)} & \mathscr{A}_{N-b-1}^{(b-a+1)} & \dots & \mathscr{A}_{N-2}^{(b-a+1)} & \bar{\mathscr{A}}_{N-1}^{(b-a+1)}\\
\vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots\\
0 & 0 & \dots & 0 & \mathscr{A}_0^{(b-a+1)} & \dots & \mathscr{A}_{b-1}^{(b-a+1)} & \bar{\mathscr{A}}_b^{(b-a+1)}
\end{array}\right)\end{array},$$

where each 0, $\mathscr{A}_j^{(i)}$ and $\bar{\mathscr{A}}_j^{(i)}$ are the square matrices of dimension $m(b-a+1)$ and are given as follows.

- $\mathscr{A}_j^{(i)}=e_i^T\otimes\kappa_j^{(i+a-1)}$, $1\le i\le b-a+1,\ 0\le j\le N-1,$
- $\bar{\mathscr{A}}_N^{(i)}=e_i^T\otimes\bar{\kappa}_N^{(i+a-1)}$, $1\le i\le b-a,$
- $\bar{\mathscr{A}}_j^{(b-a+1)}=e_{b-a+1}^T\otimes\bar{\kappa}_{j-1}^{(b)}$, $b\le j\le N,$

with $\kappa_j^{(r)}=\mathbf{e}\otimes A_j^{(r)}$ and $\bar{\kappa}_j^{(r)}=\mathbf{e}\otimes\bar{A}_j^{(r)}$.

The block matrix Λ describes the transition probabilities from vacation completion epoch to the service completion epoch and is given by

$$\Lambda = \begin{array}{c} \\ 0 \\ \vdots \\ a-1 \\ a \\ a+1 \\ \vdots \\ b \\ b+1 \\ \vdots \\ N \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & \dots & N-b-1 & N-b & \dots & N-1 & N \end{array} \\ \left(\begin{array}{cccccccc} \delta_s\mathfrak{B}^{(1)}_{0,0} & \delta_s\mathfrak{B}^{(1)}_{0,1} & \dots & \delta_s\mathfrak{B}^{(1)}_{0,N-b-1} & \delta_s\mathfrak{B}^{(1)}_{0,N-b} & \dots & \delta_s\mathfrak{B}^{(1)}_{0,N-1} & \delta_s\bar{\mathfrak{B}}^{(1)}_{0,N} \\ \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \delta_s\mathfrak{B}^{(1)}_{a-1,0} & \delta_s\mathfrak{B}^{(1)}_{a-1,1} & \dots & \delta_s\mathfrak{B}^{(1)}_{a-1,N-b-1} & \delta_s\mathfrak{B}^{(1)}_{a-1,N-b} & \dots & \delta_s\mathfrak{B}^{(1)}_{a-1,N-1} & \delta_s\bar{\mathfrak{B}}^{(1)}_{a-1,N} \\ \mathscr{B}^{(1)}_0 & \mathscr{B}^{(1)}_1 & \dots & \mathscr{B}^{(1)}_{N-b-1} & \mathscr{B}^{(1)}_{N-b} & \dots & \mathscr{B}^{(1)}_{N-1} & \bar{\mathscr{B}}^{(1)}_N \\ \mathscr{B}^{(2)}_0 & \mathscr{B}^{(2)}_1 & \dots & \mathscr{B}^{(2)}_{N-b-1} & \mathscr{B}^{(2)}_{N-b} & \dots & \mathscr{B}^{(2)}_{N-1} & \bar{\mathscr{B}}^{(2)}_N \\ \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \mathscr{B}^{(b-a+1)}_0 & \mathscr{B}^{(b-a+1)}_1 & \dots & \mathscr{B}^{(b-a+1)}_{N-b-1} & \mathscr{B}^{(b-a+1)}_{N-b} & \dots & \mathscr{B}^{(b-a+1)}_{N-1} & \bar{\mathscr{B}}^{(b-a+1)}_N \\ 0 & \mathscr{B}^{(b-a+1)}_0 & \dots & \mathscr{B}^{(b-a+1)}_{N-b-2} & \mathscr{B}^{(b-a+1)}_{N-b-1} & \dots & \mathscr{B}^{(b-a+1)}_{N-2} & \bar{\mathscr{B}}^{(b-a+1)}_{N-1} \\ \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \mathscr{B}^{(b-a+1)}_0 & \dots & \mathscr{B}^{(b-a+1)}_{b-1} & \bar{\mathscr{B}}^{(b-a+1)}_b \end{array}\right) \end{array},$$

where, the dimension of the each element of the matrix Λ, i.e., $\Lambda_{i,j}$ is given by

$$\Lambda_{i,j} \equiv \begin{cases} \text{matrix of dimension } (i+1)m \times (b-a+1)m, & 0 \le i \le a-1, 0 \le j \le N, \\ \text{matrix of dimension } am \times (b-a+1)m, & a-1 \le i \le N, 0 \le j \le N. \end{cases}$$

Each of the matrices $\mathscr{B}^{(i)}_j$, $\mathfrak{B}^{(1)}_{n,j}$, $\bar{\mathscr{B}}^{(i)}_j$ and $\bar{\mathfrak{B}}^{(1)}_{n,N}$ are described as follows.

- $\mathscr{B}^{(i)}_j = e_i^T \otimes \kappa_j^{(i+a-1)}$, $1 \le i \le b-a+1, 0 \le j \le N-1$,
- $\mathfrak{B}^{(1)}_{n,j} = e_1^T \otimes \eta^{(a)}_{n,j}$, $0 \le j \le N-1, 0 \le n \le a-1$,
- $\bar{\mathscr{B}}^{(i)}_N = e_i^T \otimes \bar{\kappa}^{(i+a-1)}_{N-1}$, $1 \le i \le b-a$,
- $\bar{\mathscr{B}}^{(b-a+1)}_j = e^T_{b-a+1} \otimes \bar{\kappa}^{(b)}_{j-1}$, $b \le j \le N$,
- $\bar{\mathfrak{B}}^{(1)}_{n,N} = e_1^T \otimes \bar{\eta}^{(a)}_{n,N-1}$, $0 \le n \le a-1$,

with $\kappa^{(r)}_j = \mathbf{e} \otimes A^{(r)}_j$, $\bar{\kappa}^{(r)}_j = \mathbf{e} \otimes \bar{A}^{(r)}_j$, $\eta^{(a)}_{n,j} = \mathbf{e} \otimes B^{(a)}_{n,j}$and $\bar{\eta}^{(a)}_{n,N} = \mathbf{e} \otimes B^{(a)}_{n,j}$.

The block matrix Θ describes the transition probabilities from service completion epoch to vacation completion epoch and is given by

$$\Theta = \begin{array}{c} \\ 0 \\ 1 \\ \vdots \\ a-1 \\ a \\ \vdots \\ N \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & \dots & a-2 & a-1 & \dots & N-1 & N \end{array} \\ \begin{pmatrix} U_0^{(1)} & U_1^{(1)} & \dots & U_{a-2}^{(1)} & U_{a-1}^{(1)} & \dots & U_{N-1}^{(1)} & \bar{U}_N^{(1)} \\ 0 & U_0^{(2)} & \dots & U_{a-3}^{(2)} & U_{a-2}^{(2)} & \dots & U_{N-2}^{(2)} & \bar{U}_{N-1}^{(2)} \\ \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & U_0^{(a)} & \dots & U_{N-a}^{(a)} & \bar{U}_{N-a+1}^{(a)} \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \end{array}$$

where each element of the block matrix Θ are again matrices $\Theta_{i,j}$, $0 \le i, j \le N$ of different dimension and is described as follows.

$$\Theta_{i,j} \equiv \begin{cases} \text{matrix of dimension } (b-a+1)m \times (j+1)m, & 0 \le i \le N, 0 \le j \le a-1, \\ \text{matrix of dimension } (b-a+1)m \times am, & 0 \le i \le N, a \le j \le N. \end{cases}$$

The elements $U_j^{(k)}$ are described by

- $U_j^{(k)} = e_k^T \otimes \vartheta_j^{(k-1)}$, $1 \le k \le a, 0 \le j \le N-1$,

- $\bar{U}_j^{(k)} = e_k^T \otimes \bar{\vartheta}_j^{(k-1)}$, $1 \le k \le a-1, N-a+2 \le j \le N, j+k = N+1$,

- $\bar{U}_{N-a+1}^{(a)} = e_a^T \otimes \bar{\vartheta}_{N-a}^{(a-1)}$,

with $\vartheta_j^{(k)} = \mathbf{e} \otimes U_j^{(k)}$ and $\bar{\vartheta}_j^{(k)} = \mathbf{e} \otimes \bar{U}_j^{(k)}$.

The block matrix Ψ describes the transition probabilities among the vacation completion epochs and is given by

$$\Psi = \begin{array}{c} \\ 0 \\ 1 \\ \vdots \\ a-1 \\ a \\ \vdots \\ N \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & \dots & a-2 & a-1 & \dots & N-1 & N \end{array} \\ \begin{pmatrix} (1-\delta_s)W_0^{(1)} & (1-\delta_s)W_1^{(1)} & \dots & (1-\delta_s)W_{a-2}^{(1)} & (1-\delta_s)W_{a-1}^{(1)} & \dots & (1-\delta_s)W_{N-1}^{(1)} & (1-\delta_s)\bar{W}_N^{(1)} \\ 0 & (1-\delta_s)W_0^{(2)} & \dots & (1-\delta_s)W_{a-3}^{(2)} & (1-\delta_s)W_{a-2}^{(2)} & \dots & (1-\delta_s)W_{N-2}^{(2)} & (1-\delta_s)\bar{W}_{N-1}^{(2)} \\ \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & (1-\delta_s)W_0^{(a)} & \dots & (1-\delta_s)W_{N-a}^{(a)} & (1-\delta_s)\bar{W}_{N-a+1}^{(a)} \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \end{array}$$

and the order of each $\Psi_{i,j}$ is described as follows.

$$\Psi_{i,j} \equiv \begin{cases} \text{matrix of dimension } (i+1)m \times (j+1)m, & 0 \le i, j \le a-1, \\ \text{matrix of dimension } am \times am, & a \le i, j \le N, \\ \text{matrix of dimension } (i+1)m \times am, & 0 \le i \le a-1, a \le j \le N, \\ \text{matrix of dimension } am \times (j+1)m, & a \le i \le N, 0 \le j \le a-1. \end{cases}$$

Each element $W_j^{(k)}$ of the matrix Ψ is described as follows.

- $W_j^{(k)} = e_k^T \otimes \vartheta_j^{(k-1)}$, $1 \le k \le a$, $0 \le j \le N-1$,
- $\bar{W}_j^{(k)} = e_k^T \otimes \bar{\vartheta}_{j-1}^{(k-1)}$, $1 \le k \le a-1$, $N-a+2 \le j \le N$, $j+k = N+1$,
- $\bar{W}_{N-a+1}^{(a)} = e_a^T \otimes \bar{\vartheta}_{N-a}^{(a-1)}$,

with $\vartheta_j^{(k)} = \mathbf{e} \otimes U_j^{(k)}$ and $\bar{\vartheta}_j^{(k)} = \mathbf{e} \otimes \bar{U}_j^{(k)}$.

Remark : According to Theorem 3.1 given in Abolnikov and Dukhovny (1991) every Markov chain whose TPM can be represented as a finite positive delta matrix is ergodic. Since the TPM $\mathscr{P}$ of the model considered in this chapter is of finite positive $\triangle_{m,n}$-type matrix, one can conclude that the corresponding Markov chain is ergodic which ensures the existence of steady state distribution.

6.2.2 Probability distribution at arbitrary epoch

In this section, we obtain two type of joint distributions at arbitrary epoch associated with the current model. (i). Joint distribution of the number of customers in the queue and number with the serving batch and the phase of the arrival process at an arbitrary epoch, (ii). Joint distribution of queue content as well as type of the vacation taken by the server and phase of the arrival at arbitrary epoch.

Towards this end let us notify the required stochastic process as follows

- $N_q(t) \equiv$ the number of customers present in the queue at time t,
- $N_s(t) \equiv$ the number of customers in service when server is busy,
- $J(t) \equiv$ the state of the underlying Markov chain.
- $\chi(t) \equiv$ the state of the server, i.e.,

$$\chi(t) = \begin{cases} 0, & \text{if server is in dormancy state}, \\ k, & \text{if server is in k}^{\text{th}} - \text{type of vacation } (0 \le k \le a-1), \\ r, & \text{if server is busy in serving batch of size } r \ (a \le r \le b). \end{cases}$$

- $U(t) \equiv$ the remaining service time of a batch of customers under service, if any.
- $\widetilde{V}(t) \equiv$ the remaining vacation time of the server, if any.

Now we define the following state probabilities, at time t for $1 \le i \le m$, as follows.

- $\hat{p}_i(n,0,t) \equiv prob.\{N_q(t) = n,\ \chi(t) = 0, J(t) = i\}, \quad 0 \le n \le a-1.$
- $\hat{\pi}_i(n,r,x,t)dx \equiv prob.\{N_q(t) = n, N_s(t) = r, J(t) = i, x \le U(t) \le x+dx, \chi(t) = r\}$, $0 \le n \le N,\ a \le r \le b, x \ge 0.$
- $\hat{\omega}_i(n,k,x,t)dx \equiv prob.\{N_q(t) = n+k, J(t) = i, x \le \widetilde{V}(t) \le x+dx, \chi(t) = k\}, \quad 0 \le k \le a-1, 0 \le n \le N-k, x \ge 0.$

More precisely,

- $\hat{p}_i(n,0,t)$ represents the probability that, at time t, there are n $(0 \le n \le a-1)$ customers are present in the queue and server is in dormant state and phase of the arrival process is i.

- $\hat{\pi}_i(n,r,x,t)dx$ represents the probability that, at time t, phase of the arrival process is i, there are n customers are in the queue and server is busy in serving r $(a \le r \le b)$ customers and remaining service time of the server lies between x and $x+dx$.

- $\hat{\omega}_i(n,k,x,t)dx$ represents the probability that, at time t, phase of the arrival process is i, there are $n+k$ customers in the queue and server is in $k^{th}-$type vacation $(0 \le k \le a-1)$, and remaining vacation time of the server lies between x and $x+dx$.

Relating the state of the system at time t and $t+dt$ we obtain the Kolmogrov equations of the model under consideration nas follows, for $1 \le i \le m$ and $x \ge 0$

$$\frac{d}{dt}\hat{p}_i(0,0,t) = \delta_s \sum_{j=1}^{m} \hat{p}_j(0,0,t)c_{ji} + \delta_s \hat{\omega}_i(0,0,0,t), \tag{6.4}$$

$$\frac{d}{dt}\hat{p}_i(n,0,t) = \delta_s \sum_{j=1}^{m} \hat{p}_j(n,0,t)c_{ji} + \delta_s \sum_{j=1}^{m} \hat{p}_j(n-1,0,t)d_{ji} + \delta_s \sum_{k=0}^{n} \hat{\omega}_i(n-k,k,0,t),\ 1 \le n \le a-1, \tag{6.5}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\hat{\pi}_i(0,a,x,t) = \sum_{j=1}^{m} \hat{\pi}_j(0,a,x,t)c_{ji} + \delta_s \sum_{j=1}^{m} \hat{p}_j(a-1,0,t)d_{ji}s_a(x) + \sum_{r=a}^{b} \hat{\pi}_i(a,r,0,t)s_a(x) + \sum_{k=0}^{a-1} \hat{\omega}_i(a-k,k,0,t)s_a(x), \tag{6.6}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\hat{\pi}_i(0,r,x,t) = \sum_{j=1}^{m} \hat{\pi}_j(0,r,x,t)c_{ji} + \sum_{k=a}^{b} \hat{\pi}_i(r,k,0,t)s_r(x) + \sum_{k=0}^{a-1} \hat{\omega}_i(r-k,k,0)s_r(x),\ a+1 \le r \le b, \tag{6.7}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\hat{\pi}_i(n,r,x,t) = \sum_{j=1}^{m} \hat{\pi}_j(n,r,x,t)c_{ji} + \sum_{j=1}^{m} \hat{\pi}_j(n-1,r,x,t)d_{ji},\quad a \le r \le b-1,\ 1 \le n \le N-1, \tag{6.8}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\hat{\pi}_i(n,b,x,t) = \sum_{j=1}^{m} \hat{\pi}_j(n,b,x,t)c_{ji} + \sum_{j=1}^{m} \hat{\pi}_j(n-1,b,x,t)d_{ji} + \sum_{r=a}^{b} \hat{\pi}_i(n+b,r,0,t)s_b(x) +$$

$$\sum_{k=0}^{a-1} \hat{\omega}_i(n+b-k,k,0,t)s_b(x),\ 1 \le n \le N-b, \tag{6.9}$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)\hat{\pi}_i(n,b,x,t) = \sum_{j=1}^{m} \hat{\pi}_j(n,b,x,t)c_{ji} + \sum_{j=1}^{m} \hat{\pi}_j(n-1,b,x,t)d_{ji}, \quad N-b+1 \le n \le N-1, \tag{6.10}$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)\hat{\pi}_i(N,r,x,t) = \sum_{j=1}^{m} \hat{\pi}_j(N,r,x,t)c_{ji} + \sum_{j=1}^{m} \hat{\pi}_j(N,r,x,t)d_{ji} + \sum_{j=1}^{m} \hat{\pi}_j(N-1,r,x,t)d_{ji},\ a \le r \le b, \tag{6.11}$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)\hat{\omega}_i(0,k,x,t) = \sum_{j=1}^{m} \hat{\omega}_j(0,k,x,t)c_{ji} + \left[\sum_{r=a}^{b} \hat{\pi}_i(k,r,0,t) + (1-\delta_s)\sum_{j=0}^{k} \hat{\omega}_i(k-j,j,0,t)\right] v^{[k]}(x),\ 0 \le k \le a-1, \tag{6.12}$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)\hat{\omega}_i(n,k,x,t) = \sum_{j=1}^{m} \hat{\omega}_j(n,k,x,t)c_{ji} + \sum_{j=1}^{m} \hat{\omega}_j(n-1,k,x,t)d_{ji}, \quad 1 \le n \le N-1,\ 0 \le k \le min(a-1,N-n-1), \tag{6.13}$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)\hat{\omega}_i(N-k,k,x,t) = \sum_{j=1}^{m} \hat{\omega}_j(N-k,k,x,t)c_{ji} + \sum_{j=1}^{m} \hat{\omega}_j(N-k,k,x,t)d_{ji} + \sum_{j=1}^{m} \hat{\omega}_j(N-k-1,k,x,t)d_{ji},\ 0 \le k \le a-1. \tag{6.14}$$

Since, we are interested in obtaining steady state joint probabilities, let us define the steady state joint probabilities, as $t \to \infty$, as follows, for $1 \le i \le m$

$$\lim_{t\to\infty} \hat{p}_i(n,0,t) = p_i(n,0);\ 0 \le n \le a-1,$$

$$\lim_{t\to\infty} \hat{\pi}_i(n,r,x,t) = \tilde{\pi}_i(n,r,x);\ 0 \le n \le N,\ a \le r \le b,$$

$$\lim_{t\to\infty} \hat{\omega}_i(n,k,x,t) = \tilde{\omega}_i(n,k,x);\ 0 \le k \le a-1,\ 0 \le n \le N-k.$$

Hence the corresponding steady state equations of the equations (6.4)-(6.14) are obtained as follows

$$0 = \delta_s \sum_{j=1}^{m} p_j(0,0)c_{ji} + \delta_s \tilde{\omega}_i(0,0,0), \tag{6.15}$$

$$0 = \delta_s \sum_{j=1}^{m} p_j(n,0)c_{ji} + \delta_s \sum_{j=1}^{m} p_j(n-1,0)d_{ji}$$

$$+\delta_s \sum_{k=0}^{n} \tilde{\omega}_i(n-k,k,0),\ 1 \le n \le a-1, \tag{6.16}$$

$$-\frac{\partial}{\partial x}\tilde{\pi}_i(0,a,x) = \sum_{j=1}^{m} \tilde{\pi}_j(0,a,x)c_{ji} + \delta_s \sum_{j=1}^{m} p_j(a-1,0)d_{ji}s_a(x) + \sum_{r=a}^{b} \tilde{\pi}_i(a,r,0)s_a(x) + \sum_{k=0}^{a-1} \tilde{\omega}_i(a-k,k,0)s_a(x), \tag{6.17}$$

$$-\frac{\partial}{\partial x}\tilde{\pi}_i(0,r,x) = \sum_{j=1}^{m} \tilde{\pi}_j(0,r,x)c_{ji} + \sum_{k=a}^{b} \tilde{\pi}_i(r,k,0)s_r(x) + \sum_{k=0}^{a-1} \tilde{\omega}_i(r-k,k,0)s_r(x),,\ a+1 \le r \le b, \tag{6.18}$$

$$-\frac{\partial}{\partial x}\tilde{\pi}_i(n,r,x) = \sum_{j=1}^{m} \tilde{\pi}_j(n,r,x)c_{ji} + \sum_{j=1}^{m} \tilde{\pi}_j(n-1,r,x)d_{ji},\ a \le r \le b-1,\ 1 \le n \le N-1, \tag{6.19}$$

$$-\frac{\partial}{\partial x}\tilde{\pi}_i(n,b,x) = \sum_{j=1}^{m} \tilde{\pi}_j(n,b,x)c_{ji} + \sum_{j=1}^{m} \tilde{\pi}_j(n-1,b,x)d_{ji} + \sum_{r=a}^{b} \tilde{\pi}_i(n+b,r,0)s_b(x) + \sum_{k=0}^{a-1} \tilde{\omega}_i(n+b-k,k,0)s_b(x),\ 1 \le n \le N-b, \tag{6.20}$$

$$-\frac{\partial}{\partial x}\tilde{\pi}_i(n,b,x) = \sum_{j=1}^{m} \tilde{\pi}_j(n,b,x)c_{ji} + \sum_{j=1}^{m} \tilde{\pi}_j(n-1,b,x)d_{ji},\ N-b+1 \le n \le N-1, \tag{6.21}$$

$$-\frac{\partial}{\partial x}\tilde{\pi}_i(N,r,x) = \sum_{j=1}^{m} \tilde{\pi}_j(N,r,x)c_{ji} + \sum_{j=1}^{m} \tilde{\pi}_j(N,r,x)d_{ji} + \sum_{j=1}^{m} \tilde{\pi}_j(N-1,r,x)d_{ji},\ a \le r \le b, \tag{6.22}$$

$$-\frac{\partial}{\partial x}\tilde{\omega}_i(0,k,x) = \sum_{j=1}^{m} \tilde{\omega}_j(0,k,x)c_{ji} + \left[\sum_{r=a}^{b} \tilde{\pi}_i(k,r,0) + (1-\delta_s)\sum_{j=0}^{k} \tilde{\omega}_i(k-j,j,0)\right] v^{[k]}(x),\ 0 \le k \le a-1, \tag{6.23}$$

$$-\frac{\partial}{\partial x}\tilde{\omega}_i(n,k,x) = \sum_{j=1}^{m} \tilde{\omega}_j(n,k,x)c_{ji} + \sum_{j=1}^{m} \tilde{\omega}_j(n-1,k,x)d_{ji},\ 1 \le n \le N-1,\ 0 \le k \le min(a-1,N-n-1), \tag{6.24}$$

$$-\frac{\partial}{\partial x}\tilde{\omega}_i(N-k,k,x) = \sum_{j=1}^{m} \tilde{\omega}_j(N-k,k,x)c_{ji} + \sum_{j=1}^{m} \tilde{\omega}_j(N-k,k,x)d_{ji} + \sum_{j=1}^{m} \tilde{\omega}_j(N-k-1,k,x)d_{ji},\ 0 \le k \le a-1. \tag{6.25}$$

For use in sequel, we define the vectors $\mathbf{p}(n,0)$, $\tilde{\pi}(n,r,x)$ and $\tilde{\omega}(n,k,x)$ of dimension m as follows $\mathbf{p}(n,0)=(p_1(n,0),p_2(n,0),...,p_m(n,0))$, $\tilde{\pi}(n,r,x)=(\tilde{\pi}_1(n,r,x),\tilde{\pi}_2(n,r,x),...,\tilde{\pi}_m(n,r,x))$ and $\tilde{\omega}(n,k,x)=(\tilde{\omega}_1(n,k,x),\tilde{\omega}_2(n,k,x),...,\tilde{\omega}_m(n,k,x))$. Using these vector notations we can write the above equations in the following matrix form

$$0 = \delta_s\mathbf{p}(0,0)C+\delta_s\tilde{\omega}(0,0,0), \tag{6.26}$$

$$0 = \delta_s\mathbf{p}(n,0)C+\delta_s\mathbf{p}(n-1,0)D+\delta_s\sum_{k=0}^{n}\tilde{\omega}(n-k,k,0), \quad 1\le n\le a-1, \tag{6.27}$$

$$-\frac{\partial\tilde{\pi}(0,a,x)}{\partial x} = \tilde{\pi}(0,a,x)C+\delta_s\mathbf{p}(a-1,0)Ds_a(x)+\sum_{r=a}^{b}\tilde{\pi}(a,r,0)s_a(x) +\sum_{k=0}^{a-1}\tilde{\omega}(a-k,k,0)s_a(x), \tag{6.28}$$

$$-\frac{\partial\tilde{\pi}(0,r,x)}{\partial x} = \tilde{\pi}(0,r,x)C+\sum_{k=a}^{b}\tilde{\pi}(r,k,0)s_r(x)+\sum_{k=0}^{a-1}\tilde{\omega}(r-k,k,0)s_r(x), \quad a+1\le r\le b, \tag{6.29}$$

$$-\frac{\partial\tilde{\pi}(n,r,x)}{\partial x} = \tilde{\pi}(n,r,x)C+\tilde{\pi}(n-1,r,x)D, \quad a\le r\le b-1,\ 1\le n\le N-1, \tag{6.30}$$

$$-\frac{\partial\tilde{\pi}(n,b,x)}{\partial x} = \tilde{\pi}(n,b,x)C+\tilde{\pi}(n-1,b,x)D+\sum_{r=a}^{b}\tilde{\pi}(n+b,r,0)s_b(x) +\sum_{k=0}^{a-1}\tilde{\omega}(n+b-k,k,0)s_b(x),\ 1\le n\le N-b, \tag{6.31}$$

$$-\frac{\partial\tilde{\pi}(n,b,x)}{\partial x} = \tilde{\pi}(n,b,x)C+\tilde{\pi}(n-1,b,x)D,\ N-b+1\le n\le N-1, \tag{6.32}$$

$$-\frac{\partial\tilde{\pi}(N,r,x)}{\partial x} = \tilde{\pi}(N,r,x)(C+D)+\tilde{\pi}(N-1,r,x)D,\ a\le r\le b, \tag{6.33}$$

$$-\frac{\partial\tilde{\omega}(0,k,x)}{\partial x} = \tilde{\omega}(0,k,x)C+\left[\sum_{r=a}^{b}\tilde{\pi}(k,r,0)+ (1-\delta_s)\sum_{j=0}^{k}\tilde{\omega}(k-j,j,0)\right]v^{[k]}(x),\ 0\le k\le a-1, \tag{6.34}$$

$$-\frac{\partial\tilde{\omega}(n,k,x)}{\partial x} = \tilde{\omega}(n,k,x)C+\tilde{\omega}(n-1,k,x)D, \quad 1\le n\le N-1,\ 0\le k\le min(a-1,N-n-1), \tag{6.35}$$

$$-\frac{\partial\tilde{\omega}(N-k,k,x)}{\partial x} = \tilde{\omega}(N-k,k,x)\,(C+D)+\tilde{\omega}(N-k-1,k,x)D, \quad 0\le k\le a-1. \tag{6.36}$$

Multiplying (6.28)-(6.36) by $e^{-\theta x}$ and integrating with respect to x over 0 to ∞ and using above defined vector notations we find

$$
\begin{aligned}
-\theta\pi^*(0,a,\theta)+\tilde{\pi}(0,a,0) &= \pi^*(0,a,\theta)C+\delta_s\mathbf{p}(a-1,0)Ds_a^*(\theta) \\
&\quad +\sum_{r=a}^{b}\tilde{\pi}(a,r,0)s_a^*(\theta)+\sum_{k=0}^{a-1}\tilde{\omega}(a-k,k,0)s_a^*(\theta), \qquad (6.37)\\
-\theta\pi^*(0,r,\theta)+\tilde{\pi}(0,r,0) &= \pi^*(0,r,\theta)C+\sum_{k=a}^{b}\tilde{\pi}(r,k,0)s_r^*(\theta) \\
&\quad +\sum_{k=0}^{a-1}\tilde{\omega}(r-k,k,0)s_r^*(\theta),\ a+1\le r\le b, \qquad (6.38)\\
-\theta\pi^*(n,r,\theta)+\tilde{\pi}(n,r,0) &= \pi^*(n,r,\theta)C+\pi^*(n-1,r,\theta)D, \\
&\quad a\le r\le b-1,\ 1\le n\le N-1, \qquad (6.39)\\
-\theta\pi^*(n,b,\theta)+\tilde{\pi}(n,b,0) &= \pi^*(n,b,\theta)C+\pi^*(n-1,b,\theta)D+\sum_{r=a}^{b}\tilde{\pi}(n+b,r,0)s_b^*(\theta), \\
&\quad +\sum_{k=0}^{a-1}\tilde{\omega}(n+b-k,k,0)s_b^*(\theta),\ 1\le n\le N-b, \qquad (6.40)\\
-\theta\pi^*(n,b,\theta)+\tilde{\pi}(n,b,0) &= \pi^*(n,b,\theta)C+\pi^*(n-1,b,\theta)D, \\
&\quad N-b+1\le n\le N-1, \qquad (6.41)\\
-\theta\pi^*(N,r,\theta)+\tilde{\pi}(N,r,0) &= \pi^*(N,r,\theta)(C+D)+\pi^*(N-1,r,\theta)D, \\
&\quad a\le r\le b, \qquad (6.42)\\
-\theta\omega^*(0,k,\theta)+\tilde{\omega}(0,k,0) &= \omega^*(0,k,\theta)C+\sum_{r=a}^{b}\tilde{\pi}(k,r,0)v^{[k]*}(\theta)+ \\
&\quad (1-\delta_s)\sum_{j=0}^{k}\tilde{\omega}(k-j,j,0)v^{[k]*}(\theta),\ 0\le k\le a-1, \qquad (6.43)\\
-\theta\omega^*(n,k,\theta)+\tilde{\omega}(n,k,0) &= \omega^*(n,k,\theta)C+\omega^*(n-1,k,\theta)D, \\
&\quad 1\le n\le N-1,\ 0\le k\le min(a-1,N-n-1), \qquad (6.44)\\
-\theta\omega^*(N-k,k,\theta)+\tilde{\omega}(N-k,k,0) &= \omega^*(N-k,k,\theta)(C+D)+\omega^*(N-k-1,k,\theta)D, \\
&\quad 0\le k\le a-1. \qquad (6.45)
\end{aligned}
$$

where,

$$
\begin{aligned}
\int_0^\infty e^{-\theta x}\tilde{\pi}_i(n,r,x)dx &= \pi_i^*(n,r,\theta),\ 0\le n\le N,\ a\le r\le b,\theta\ge 0,\\
\int_0^\infty e^{-\theta x}\tilde{\omega}_i(n,k,x)dx &= \omega_i^*(n,k,\theta),\ 0\le k\le a-1,\ 0\le n\le N-k,\ \theta\ge 0,\\
\int_0^\infty e^{-\theta x}s_r(x)dx &= s_r^*(\theta),\ a\le r\le b,\ \theta\ge 0,
\end{aligned}
$$

$$\int_0^\infty e^{-\theta x} v^{[k]}(x)dx \quad = \quad v^{[k]*}(\theta),\ 0 \le k \le a-1,\ \theta \ge 0.$$

Using above, $\pi_i(n,r) \equiv \pi_i^*(n,r,0)$ and $\omega_i(n,k) \equiv \omega_i^*(n,k,0)$, we find $\pi_i(n,r)$ is the arbitrary epoch probability that n customers in the queue and server is busy with r customers and at that time the phase of the arrival process is in i and $\omega_i(n,k)$ is the arbitrary epoch probability that $n+k$ customers in the queue and server is in $k^{th}-$type vacation and at that time the phase of the arrival process is i.

Using the above, we define row vectors of order m as :

$\pi(n,r) \equiv \pi^*(n,r,0) = (\pi_1(n,r), \pi_2(n,r), ..., \pi_m(n,r))$ and

$\omega(n,k) \equiv \omega^*(n,k,0) = (\omega_1(n,k), \omega_2(n,k), ..., \omega_m(n,k))$.

Now we first obtain following results in form of Lemma which will be used to develop relations between service/vacation completion epoch and arbitrary epoch probabilities.

Lemma 6.1. The service completion epoch probabilities $\pi^+(n,r)$ and vacation termination epoch probabilities $\omega^+(n,k)$ are proportional to the probabilities $\tilde{\pi}(n,r,0)$, $\tilde{\omega}(n,k,0)$ respectively, and are given by

$$\pi^+(n,r) \quad = \quad \sigma\tilde{\pi}(n,r,0),\ 0 \le n \le N,\ a \le r \le b, \tag{6.46}$$

$$\omega^+(n,k) \quad = \quad \sigma\tilde{\omega}(n,k,0),\ 0 \le k \le a-1,\ 0 \le n \le N-k, \tag{6.47}$$

where, $\sigma^{-1} = \sum_{n=0}^{N}\sum_{r=a}^{b}\tilde{\pi}(n,r,0)\mathbf{e} + \sum_{k=0}^{a-1}\sum_{n=0}^{N-k}\tilde{\omega}(n,k,0)\mathbf{e}$.

Proof. Using Bayes' theorem, for $0 \le n \le N$, $a \le r \le b$, $1 \le i \le m$ we have

$\pi_i^+(n,r) =$ prob.$\{n$ customers are in the queue at the service completion epoch of a batch of size $r\}$

$=$ prob.$\{n$ customers are in the queue and phase of the arrival is i just prior to the service completion epoch of a batch of size r $|\le N$ customers are in the queue just prior to the service completion epoch of a batch of size $a \le r \le b$ or vacation completion epoch of k-th type vacation with $0 \le k \le a-1$.$\}$

$$= \frac{\tilde{\pi}_i(n,r,0)}{\sum_{n=0}^{N}\sum_{r=a}^{b}\tilde{\pi}(n,r,0)\mathbf{e} + \sum_{k=0}^{a-1}\sum_{n=0}^{N-k}\tilde{\omega}(n,k,0)\mathbf{e}},$$

With the similar argument one can write, for $0 \leq k \leq a-1$, $0 \leq n \leq N-k$, $1 \leq i \leq m$

$$\omega_i^+(n,k) = \frac{\tilde{\omega}_i(n,k,0)}{\sum\limits_{n=0}^{N}\sum\limits_{r=a}^{b}\tilde{\pi}(n,r,0)\mathbf{e}+\sum\limits_{k=0}^{a-1}\sum\limits_{n=0}^{N-k}\tilde{\omega}(n,k,0)\mathbf{e}}.$$

Writing these expressions in vector form we get the required result.

Lemma 6.2. In case of single vacation, the dormant steady state probability vectors $\mathbf{p}(n,0)$ $(0 \leq n \leq a-1)$, are given by

$$\mathbf{p}(n,0) = \left(\sum_{i=0}^{n}\sum_{k=0}^{i}\tilde{\omega}(n-k,k,0)\tilde{D}^{n-i}\right)(-C)^{-1},\ 0 \leq n \leq a-1, \tag{6.48}$$

Proof. Using (6.26) in (6.27), we get our desired result (6.48).

Lemma 6.3. The value of σ^{-1} as appeared in Lemma 6.1 is given by

$$\sigma^{-1} = \sum_{n=0}^{N}\sum_{r=a}^{b}\tilde{\pi}(n,r,0)\mathbf{e}+\sum_{k=0}^{a-1}\sum_{n=0}^{N-k}\tilde{\omega}(n,k,0)\mathbf{e} = \frac{1-\delta_s\sum\limits_{n=0}^{a-1}\mathbf{p}(n,0)\mathbf{e}}{g}, \tag{6.49}$$

where, $g = \sum\limits_{n=0}^{a-1}\left(\tilde{v}^{[n]}\sum\limits_{r=a}^{b}\pi^+(n,r)\mathbf{e}+\left(\delta_s\tilde{s_a}+(1-\delta_s)\tilde{v}^{[n]}\right)\sum\limits_{k=0}^{n}\omega^+(n-k,k)\mathbf{e}\right)$
$+\sum\limits_{n=a}^{b}\left(\sum\limits_{k=0}^{a-1}\omega^+(n-k,k)\mathbf{e}+\sum\limits_{r=a}^{b}\pi^+(n,r)\mathbf{e}\right)\tilde{s_n}+\sum\limits_{n=b+1}^{N}\left(\sum\limits_{k=0}^{a-1}\omega^+(n-k,k)\mathbf{e}+\sum\limits_{r=a}^{b}\pi^+(n,r)\mathbf{e}\right)\tilde{s_b}.$

Proof. Post-multiplying (6.26) and (6.27) by $\mathbf{e}$ and adding the resulting equations,and using the fact that $(C+D)\mathbf{e} = 0$, we get

$$\mathbf{p}(a-1,0)D\mathbf{e} = \sum_{k=0}^{a-1}\sum_{j=0}^{k}\tilde{\omega}(k-j,j,0)\mathbf{e}, \tag{6.50}$$

Similarly, post-multiplying (6.37) by $\mathbf{e}$ and using (6.50), we get

$$\begin{aligned} -\theta\pi^*(0,a,\theta)\mathbf{e}+\tilde{\pi}(0,a,0)\mathbf{e} &= \pi^*(0,a,\theta)C\mathbf{e}+\delta_s\sum_{k=0}^{a-1}\sum_{j=0}^{k}\tilde{\omega}(k-j,j,0)\mathbf{e}s_a^*(\theta) \\ &\quad +\sum_{r=a}^{b}\tilde{\pi}(a,r,0)\mathbf{e}s_a^*(\theta)+\sum_{k=0}^{a-1}\tilde{\omega}(a-k,k,0)\mathbf{e}s_a^*(\theta), \end{aligned} \tag{6.51}$$

again post-multiplying (6.38)-(6.45) by $\mathbf{e}$ and summing the resulting equations and (6.51). we get

$$\sum_{n=0}^{N}\sum_{r=a}^{b}\pi^*(n,r,\theta)\mathbf{e}+\sum_{k=0}^{a-1}\sum_{n=0}^{N-k}\omega^*(n,k,\theta)\mathbf{e} = \sum_{n=0}^{a-1}\Bigg(\frac{1-v^{[n]*}(\theta)}{\theta}\sum_{r=a}^{b}\tilde{\pi}(n,r,0)\mathbf{e}+ \frac{1-\delta_s s_a^*(\theta)-(1-\delta_s)v^{[n]*}(\theta)}{\theta}\sum_{k=0}^{n}\tilde{\omega}(n-k,k,0)\mathbf{e}\Bigg)+ \sum_{n=a}^{b}\frac{1-s_n^*(\theta)}{\theta}\left(\sum_{r=a}^{b}\tilde{\pi}(n,r,0)\mathbf{e}+\sum_{k=0}^{a-1}\tilde{\omega}(n-k,k,0)\mathbf{e}\right) + \sum_{n=b+1}^{N}\frac{1-s_b^*(\theta)}{\theta}\left(\sum_{r=a}^{b}\tilde{\pi}(n,r,0)\mathbf{e}+\sum_{k=0}^{a-1}\tilde{\omega}(n-k,k,0)\mathbf{e}\right). \tag{6.52}$$

Dividing by θ in above expression and taking limit as $\theta \to 0$ using L'Hôpital's rule, and the fact that

$$\sum_{n=0}^{a-1}\mathbf{p}(n,0)\mathbf{e}+\sum_{n=0}^{N}\sum_{r=a}^{b}\pi(n,r)\mathbf{e}+\sum_{k=0}^{a-1}\sum_{n=0}^{N-k}\omega(n,k)\mathbf{e}=\varpi\mathbf{e}=1, \tag{6.53}$$

we get

$$1-\delta_s\sum_{n=0}^{a-1}\mathbf{p}(n,0)\mathbf{e} = \sum_{n=0}^{a-1}\left(\tilde{v}^{[n]}\sum_{r=a}^{b}\tilde{\pi}(n,r,0)\mathbf{e}+\left(\delta_s\tilde{s}_a+(1-\delta_s)\tilde{v}^{[n]}\right)\tilde{\omega}(n-k,k,0)\mathbf{e}\right) +\sum_{n=a}^{b}\tilde{s}_n\left(\sum_{r=a}^{b}\tilde{\pi}(n,r,0)\mathbf{e}+\sum_{k=0}^{a-1}\tilde{\omega}(n-k,k,0)\mathbf{e}\right) +\sum_{n=b+1}^{N}\tilde{s}_b\left(\sum_{r=a}^{b}\tilde{\pi}(n,r,0)\mathbf{e}+\sum_{k=0}^{a-1}\tilde{\omega}(n-k,k,0)\mathbf{e}\right), \tag{6.54}$$

Now using Lemma 6.1 in (6.54) and manipulating, we obtain the desired result (6.49).

Now we are ready to state the main result of this section.

Theorem 6.1. The steady state probability vectors $\{\mathbf{p}(n,0),\, \pi(n,r),\, \omega(n,r)\}$ and $\{\pi^+(n,r),\, \omega^+(n,r)\}$ are related by

$$\mathbf{p}(n,0) = E^{*-1}\left(\sum_{i=0}^{n}\sum_{k=0}^{i}\omega^+(n-k,k)\tilde{D}^{n-i}\right)(-C)^{-1},\ 0\le n\le a-1, \tag{6.55}$$

$$\pi(0,a) = \Bigg[E^{*-1}\left\{\pi^+(0,a)-\sum_{r=a}^{b}\pi^+(a,r)-\sum_{k=0}^{a-1}\omega^+(a-k,k)\right\} \tag{6.56}$$

$$-\mathbf{p}(a-1,0)D\Bigg](C)^{-1}, \tag{6.57}$$

$$\pi(0,r) = E^{*-1}\left\{\pi^+(0,r)-\sum_{k=a}^{b}\pi^+(r,k)-\sum_{k=0}^{a-1}\omega^+(r-k,k)\right\}(C)^{-1},$$

$$a+1 \le r \le b, \tag{6.58}$$

$$\pi(n,r) = \left\{E^{*-1}\pi^{+}(n,r) - \pi(n-1,r)D\right\}(C)^{-1}, 1 \le n \le N-1, a \le r \le b-1, \tag{6.59}$$

$$\pi(n,b) = \left[\left(E^{*-1}\left\{\pi^{+}(n,b) - \sum_{r=a}^{b}\pi^{+}(n+b,r) - \sum_{k=0}^{a-1}\omega^{+}(n+b-k,k)\right\}\right.\right.$$
$$\left.-\pi(n-1,b)D\right](C)^{-1}, \quad 1 \le n \le N-b, \tag{6.60}$$

$$\pi(n,b) = \left[E^{*-1}\pi^{+}(n,b) - \pi(n-1,b)D\right](C)^{-1}, N-b+1 \le n \le N-1, \tag{6.61}$$

$$\omega(0,k) = E^{*-1}\left\{\omega^{+}(0,k) - \sum_{r=a}^{b}\pi^{+}(k,r) - (1-\delta_s)\sum_{j=0}^{k}\omega^{+}(k-j,j)\right\}(C)^{-1},$$
$$0 \le k \le a-1, \tag{6.62}$$

$$\omega(n,k) = \left[E^{*-1}\omega^{+}(n,k) - \omega(n-1,k)D\right](C)^{-1},$$
$$1 \le n \le N-1, 0 \le k \le min(N-n-1, a-1), \tag{6.63}$$

where, $E^{*} = \left[g + \sum_{n=0}^{a-1}\sum_{j=0}^{n}\sum_{k=0}^{j}\omega^{+}(j-k,k)\tilde{D}^{n-j}(-C)^{-1}\mathbf{e}\right]$, and g is given in Lemma 6.3.

Proof. On dividing (6.26)-(6.27) by σ^{-1} and with the help of Lemma 6.3, after some algebraic manipulations we get

$$\left(g + \sum_{n=0}^{a-1}\sum_{j=0}^{n}\sum_{k=0}^{j}\omega^{+}(j-k,k)\tilde{D}^{n-j}(-C)^{-1}\mathbf{e}\right)\mathbf{p}(0,0)\mathbf{e} = \omega^{+}(0,0)(-C)^{-1}\mathbf{e}, \tag{6.64}$$

Using recursively equation (6.27) and with the help of (6.64) we get the desired result (6.55). Now putting $\theta = 0$ in (6.37)-(6.41) and (6.43)-(6.45), solving it recursively after some algebraic manipulations, we get the desired results (6.56)-(6.63).

It may here note that Theorem 6.1 does not have any expression for the probability vectors $\pi(N,r)\,(a \le r \le b)$ and $\omega(N-k,k)\,(0 \le k \le a-1)$. However, one can obtain the probabilities useful in defining key system performance measures, in the following section.

6.2.2.1 Evaluation of $\pi(N,r)\mathbf{e}\,(a \le r \le b)$

As mentioned earlier that the probability vectors $\pi(N,r)$, cannot be obtained using the relations given in Theorem 6.1. However, some probabilities useful in defining key system performance measures can be obtained. In the following we will denote $\pi^{*'}(n,r,0)$ the derivative of $\pi^{*}(n,r,\theta)$ with respect to θ evaluated at $\theta = 0$. Differentiating the functions in equations

(6.37)-(6.42) with respect to θ, setting $\theta = 0$ and post-multiplying the resulting ones by $\mathbf{e}$ and using $(C+D)\,\mathbf{e} = 0$ we get

$$\begin{aligned}
\pi^{*'}(0,a,0)D\mathbf{e} &= \pi(0,a)\mathbf{e} - \delta_s\tilde{s_a}\mathbf{p}(a-1,0)D\mathbf{e} - \sum_{r=a}^{b}\tilde{s_a}\tilde{\pi}(a,r,0)\mathbf{e} \\
&\quad - \sum_{k=0}^{a-1}\tilde{s_a}\tilde{\omega}(a-k,k,0)\mathbf{e}, \qquad (6.65)\\
\pi^{*'}(0,r,0)D\mathbf{e} &= \pi(0,r)\mathbf{e} - \sum_{k=a}^{b}\tilde{s_r}\tilde{\pi}(r,k,0)\mathbf{e} - \sum_{k=0}^{a-1}\tilde{s_r}\tilde{\omega}(r-k,k,0)\mathbf{e}, \\
&\quad a+1 \le r \le b, \qquad (6.66)\\
\pi^{*'}(n,r,0)D\mathbf{e} &= \pi(n,r)\mathbf{e} + \pi^{*'}(n-1,r,0)D\mathbf{e}, \\
&\quad a \le r \le b-1\,,\, 1 \le n \le N-1, \qquad (6.67)\\
\pi^{*'}(n,b,0)D\mathbf{e} &= \pi(n,b)\mathbf{e} + \pi^{*'}(n-1,b,0)D\mathbf{e} - \sum_{r=a}^{b}\tilde{s_b}\tilde{\pi}(n+b,r,0)\mathbf{e} \\
&\quad - \sum_{k=0}^{a-1}\tilde{s_b}\tilde{\omega}(n+b-k,k,0)\mathbf{e},\;\; 1 \le n \le N-b, \qquad (6.68)\\
\pi^{*'}(n,b,0)D\mathbf{e} &= \pi(n,b)\mathbf{e} + \pi^{*'}(n-1,b,0)D\mathbf{e},\; N-b+1 \le n \le N-1, \qquad (6.69)\\
\pi(N,r)\mathbf{e} &= -\pi^{*'}(N-1,r,0)D\mathbf{e},\; a \le r \le b, \qquad (6.70)
\end{aligned}$$

Using the facts of Lemma 6.1 and Lemma 6.3 in the equations (6.65)-(6.69) yield a recursive procedure to solve $\pi^{*'}(N-1,r,0)D\mathbf{e}$, and hence from (6.70) we can calculate $\pi(N,r)\mathbf{e}$.

6.2.2.2 Evaluation of $\omega(N-k,k)\mathbf{e}$ $\quad(0 \le k \le a-1)$

In the following we will denote $\omega^{*'}(n,r,0)$ as the derivative of $\omega^{*}(n,r,\theta)$ with respect to θ evaluated at $\theta = 0$. Differentiating the functions in (6.43)-(6.45) with respect to θ, setting $\theta = 0$ and post-multiplying the resulting ones by $\mathbf{e}$ and using $(C+D)\,\mathbf{e} = 0$ we get

$$\begin{aligned}
\omega^{*'}(0,k,0)D\mathbf{e} &= \omega(0,k)\mathbf{e} - \tilde{v}^{[k]}\sum_{r=a}^{b}\tilde{\pi}(k,r,0)\mathbf{e} - (1-\delta_s)\tilde{v}^{[k]}\sum_{j=0}^{k}\tilde{\omega}(k-j,j,0)\mathbf{e}, \\
&\quad 0 \le k \le a-1, \qquad (6.71)\\
\omega^{*'}(n,k,0)D\mathbf{e} &= \omega(n,k)\mathbf{e} + \omega^{*'}(n-1,k,0)D\mathbf{e}, \\
&\quad 1 \le n \le N-1,\, 0 \le k \le min(a-1,N-n-1), \qquad (6.72)\\
\omega(N-k,k)\mathbf{e} &= -\omega^{*}(N-k-1,k,0)D\mathbf{e},\, 0 \le k \le a-1. \qquad (6.73)
\end{aligned}$$

Using the facts of Lemma 6.1 and Lemma 6.3 in the equations (6.71)-(6.72) yield a recursive procedure to solve $\omega^{*'}(N-k-1,k,0)D\mathbf{e}$, and hence from (6.73) we can calculate $\omega(N-k,k)\mathbf{e}$.

Henceforth, we have obtained important joint distribution of queue and server content, joint distribution of queue content as well as type of the vacation taken by the server. Now the other significant distribution, which are useful in computing various performance measures, can be obtained as follows.

- the distribution of queue content, π_n^{queue} $(0 \leq n \leq N)$, is given by

$$\pi_n^{queue} = \begin{cases} \delta_s \mathbf{p}(n,0)\mathbf{e} + \sum_{r=a}^{b} \pi(n,r)\mathbf{e} + \sum_{k=0}^{n} \omega(n-k,k)\mathbf{e}, & 0 \leq n \leq a-1, \\ \sum_{r=a}^{b} \pi(n,r)\mathbf{e} + \sum_{k=0}^{a-1} \omega(n-k,k)\mathbf{e}. & a \leq n \leq N. \end{cases}$$

- the distribution of the system content (including number of customers with the server), π_n^{sys} $(0 \leq n \leq N+b)$, is given by

$$\pi_n^{sys} = \begin{cases} \delta_s \mathbf{p}(n,0)\mathbf{e} + \sum_{k=0}^{n} \omega(n-k,k)\mathbf{e}, & 0 \leq n \leq a-1, \\ \sum_{r=a}^{min(b,n)} \pi(n-r,r)\mathbf{e} + \sum_{k=0}^{a-1} \omega(n-k,k)\mathbf{e}, & a \leq n \leq N, \\ \sum_{r=a}^{b} \pi(n-r,r)\mathbf{e}, & N+1 \leq n \leq N+a, \\ \sum_{r=n-N}^{b} \pi(n-r,r)\mathbf{e}. & N+a+1 \leq n \leq N+b. \end{cases}$$

- the queue length distribution when server is busy is given by $\pi_n^{busy} = \sum_{r=a}^{b} \pi(n,r)\mathbf{e}$, $(0 \leq n \leq N)$.

- the queue length distribution when server is in vacation $\omega_n^{[vac]}$ $(0 \leq n \leq N)$, is given by $\omega_n^{[vac]} = \sum_{k=0}^{min(n,a-1)} \omega(n-k,k)\mathbf{e}$.

- the probability that the server is in dormant state $(\mathbf{p}_{dor})$, busy (π_{busy}) and in vacation (ω_{vac}) are given by $\mathbf{p}_{dor} = \sum_{n=0}^{a-1} \mathbf{p}(n,0)\mathbf{e}$, $\pi_{busy} = \sum_{r=a}^{b}\sum_{n=0}^{N} \pi(n,r)\mathbf{e}$ and $\omega_{vac} = \sum_{k=0}^{a-1}\sum_{n=0}^{N-k} \omega(n,k)\mathbf{e}$, respectively.

- the distribution of the server content given that the server is busy, π_r^{ser} $(a \leq r \leq b)$, is given by

$$\pi_r^{ser} = \sum_{n=0}^{N} \pi(n,r)\mathbf{e}/\pi_{busy}.$$

- the distribution that the server is in $k^{th}-$type vacation given that the server is in vacation state, $\zeta_k\,(0 \le k \le a-1)$, is given by $\zeta_k = \sum_{n=0}^{N-k} \omega(n,k)\mathbf{e}/\omega_{vac}$.

6.3 Performance measure

The performance measures of the present model are evaluated and presented as follows.

1. Average queue length $(L_q) = \sum_{n=0}^{N} n\pi_n^{queue}$.
2. Average system length $(L) = \sum_{n=0}^{N+b} n\pi_n^{sys}$.
3. Average number of customers with the server when server is busy $(L_s) = \sum_{r=a}^{b} r\pi_r^{ser}$.
4. Average vacation type (average number of customers in the queue at vacation initiation epoch) $(\zeta) = \sum_{k=0}^{a-1} k\zeta_k$.
5. Average queue length when server is in dormancy (L_q^{dor}), busy (L_q^{busy}) and in vacation (L_q^{vac}) are obtained as $L_q^{dor} = \sum_{n=0}^{a-1} n\mathbf{p}(n,0)\mathbf{e}/\mathbf{p}_{dor}$, $L_q^{busy} = \sum_{n=0}^{N} n.\pi_n^{busy}/\pi_{busy}$ and $L_q^{vac} = \sum_{n=0}^{N} n.\omega_n^{[vac]}/\omega_{vac}$ respectively.
6. Blocking Probability : The probability that an arriving customer will be lost is given by

$$\pi_{Block} = \frac{1}{\lambda^*}\left(\sum_{r=a}^{b} \pi(N,r)D\mathbf{e} + \sum_{k=0}^{a-1} \omega(N-k,k)D\mathbf{e}\right).$$

7. Finally, using Little's law, the average waiting time of an arbitrary customer in the queue $(W_q) = L_q/\bar{\lambda}$ as well as the system $(W) = L/\bar{\lambda}$,

where, $\bar{\lambda}$ is the effective arrival rate of the system and is given by $\bar{\lambda} = \lambda^*(1-\pi_{Block})$.

6.4 Phase type services and vacations

A PH-distribution is the time until absorption in a finite state Markov chain with one absorption state. It is characterized by an initial probability vector and a square matrix governing the transitions among the transient states. For details on PH-distributions and their properties, we refer the reader to Neuts (1981). The following theorem shows the computational aspects in the case of PH-services and PH-vacations.

Theorem 6.2. Suppose that $S_r(t)$ $(a \leq r \leq b)$ follows a PH-distribution with an irreducible representation (β_r, h_r) of dimension τ_1 and $V^{[k]}(t)$ $(0 \leq k \leq a-1)$ follows PH-distribution with irreducible representation (α_k, u_k) of dimension τ_2. Then the matrices appearing in (6.3) are given by

$$\begin{aligned}
A_k^{(r)} &= \left[(I_m \otimes \beta_r) M_k^{(r)} \left(I_m \otimes h_r^0\right)\right], 0 \leq k \leq N, a \leq r \leq b, \\
\bar{A}_k^{(r)} &= \left[(I_m \otimes \beta_r) \tilde{M}_k^{(r)} \left(I_m \otimes h_r^0\right)\right], 0 \leq k \leq N, a \leq r \leq b, \\
U_k^{[n]} &= \left[(I_m \otimes \alpha_n) L_k^{(n)} \left(I_m \otimes u_n^0\right)\right], 0 \leq k \leq N, 0 \leq n \leq a-1, \\
\bar{U}_k^{[n]} &= \left[(I_m \otimes \alpha_n) \tilde{L}_k^{(n)} \left(I_m \otimes u_n^0\right)\right], 0 \leq k \leq N, 0 \leq n \leq a-1, \\
B_{n,k}^{(a)} &= \tilde{D}^{a-n} A_k^{(a)}, 0 \leq k \leq N, 0 \leq n \leq a-1, \qquad (6.74)\\
\bar{B}_{n,N}^{(a)} &= \tilde{D}^{a-n} \bar{A}_N^{(a)}, 0 \leq n \leq a-1,
\end{aligned}$$

where,

$h_r^0 = -h_r e$, $M_k^{(r)} = -M_{k-1}^{(r)} (D \otimes I_{\tau_1})(C \oplus h_r)^{-1}$, $M_0^{(r)} = -(C \oplus h_r)^{-1}$,

$\tilde{M}_k^{(r)} = -M_k^{(r)} (D \otimes I_{\tau 1}) [(C \oplus h_r) + (D \otimes I_{\tau_1})]^{-1}$,

$u_n^0 = -u_n e$, $L_k^{(n)} = -L_{k-1}^{(n)} (D \otimes I_{\tau_2})(C \oplus u_n)^{-1}$, $L_0^{(n)} = -(C \oplus u_n)^{-1}$,

$\tilde{L}_k^{(n)} = -M_k^{(n)} (D \otimes I_{\tau_2}) [(C \oplus u_n) + (D \otimes I_{\tau_2})]^{-1}$.

Proof. For $0 \leq k \leq N$, $a \leq r \leq b$, we have

$$\begin{aligned}
A_k^{(r)} &= \int_0^\infty \hat{P}(k,t)\, dS_r(t) \\
&= \int_0^\infty \hat{P}(k,t) \otimes \beta_r e^{(h_r t)} h_r^0 dt
\end{aligned}$$

$$= \int_0^\infty I_m \hat{P}(k,t) I_m \otimes \beta_r e^{(h_r t)} h_r^0 dt$$

$$= (I_m \otimes \beta_r) \left(\int_0^\infty \hat{P}(k,t) \otimes e^{(h_r t)} dt \right) \left(I_m \otimes h_r^0\right),$$

Therefore we obtain

$$A_k^{(r)} = (I_m \otimes \beta_r) M_k^{(r)} \left(I_m \otimes h_r^0\right), \tag{6.75}$$

Where,

$$M_k^{(r)} = \int_0^\infty \hat{P}(k,t) \otimes e^{(h_r t)} dt, \quad 0 \le k \le N, a \le r \le b \tag{6.76}$$

Now in order to calculate $M_k^{(r)}$, we integrate RHS of (6.76) by parts, and obtain

$$M_k^{(r)} = -\left(\hat{P}(k,0) \otimes h_r^{-1}\right) - \int_0^\infty \hat{P}'(k,t) \otimes e^{(h_r t)} h_r^{-1} dt. \tag{6.77}$$

$$[\text{as}\, t \to \infty\, \text{and}\, \hat{P}(k,t) \to 0]$$

Now setting $k = 0$ in (6.77), we obtain for $a \le r \le b$

$$M_0^{(r)} = -\left(\hat{P}(0,0) \otimes h_r^{-1}\right) - \int_0^\infty \hat{P}'(0,t) \otimes e^{(h_r t)} h_r^{-1} dt$$

$$= -\left(I_m \otimes h_r^{-1}\right) - \int_0^\infty \hat{P}(0,t) C \otimes e^{(h_r t)} h_r^{-1} dt$$

$$[\text{as}\, \hat{P}'(0,t) = \hat{P}(0,t)C]$$

$$= -\left(I_m \otimes h_r^{-1}\right) - \int_0^\infty \left(\hat{P}(0,t) \otimes e^{(h_r t)} dt\right) \left(C \otimes h_r^{-1}\right)$$

$$= -\left(I_m \otimes h_r^{-1}\right) - M_0^{(r)} \left(C \otimes h_r^{-1}\right).$$

Therefore,

$$M_0^{(r)} + M_0^{(r)}\left(C \otimes h_r^{-1}\right) = -\left(I_m \otimes h_r\right)^{-1}, \ \left(as\ (A \otimes B)^{-1} = A^{-1} \otimes B^{-1}\right)$$

So the above expression can be now written in the following form:

$$M_0^{(r)}\left(I_{m\tau_1} + \left(C \otimes h_r^{-1}\right)\right) = \left(I_m \otimes h_r\right)^{-1}$$

Since both C and h_r are invertible, the term $\left(I_{m\tau_1} + \left(C \otimes h_r^{-1}\right)\right)$ is invertible. Hence, post multiplying both side of the above expression by $\left(I_{m\tau_1} + \left(C \otimes h_r^{-1}\right)\right)^{-1}$, we have for $a \leq r \leq b$

$$\begin{aligned}
M_0^{(r)} &= -\left(I_m \otimes h_r\right)^{-1}\left(I_{m\tau_1} + \left(C \otimes h_r^{-1}\right)\right)^{-1} \\
&= -\left[\left(I_{m\tau_1} + \left(C \otimes h_r^{-1}\right)\right)\left(I_m \otimes h_r\right)\right]^{-1} \\
&= -\left[I_{m\tau_1}\left(I_m \otimes h_r\right) + \left(C \otimes h_r^{-1}\right)\left(I_m \otimes h_r\right)\right]^{-1} \\
&= -\left[\left(I_m \otimes h_r\right) + CI_m \otimes h_r^{-1}h_r\right]^{-1} \\
&= -\left[\left(I_m \otimes h_r\right) + C \otimes I_{\tau_1}\right]^{-1} \\
&= -\left[C \otimes I_{\tau_1} + I_m \otimes h_r\right]^{-1},
\end{aligned}$$

Therefore,

$$M_0^{(r)} = -\left(C \oplus h_r\right)^{-1}, \ \left[as A \oplus B = A \otimes I_m + I_n \otimes B\right] \tag{6.78}$$

So (6.78) gives the value of $M_0^{(r)}$ $(a \leq r \leq b)$. Now from (6.77) for $1 \leq k \leq N$, $a \leq r \leq b$, we have

$$\begin{aligned}
M_k^{(r)} &= -\left(\hat{P}(k,0) \otimes h_r^{-1}\right) - \int_0^\infty \hat{P}'(k,t) \otimes e^{(h_r t)} h_r^{-1} dt \\
&= -\left(0_m \otimes h_r^{-1}\right) - \int_0^\infty \left(\hat{P}(k,t)C + \hat{P}(k-1,t)D\right) \otimes e^{(h_r t)} h_r^{-1} dt, \\
&\left[\because \hat{P}(k,0) = 0_m \,\&\, \hat{P}'(k,t) = \hat{P}(k,t)C + \hat{P}(k-1,t)D, \forall k \geq 1\right]
\end{aligned}$$

$$\begin{aligned}
&= -0_{m\tau_1} - \int_0^\infty \hat{P}(k,t)C \otimes e^{(h_r t)} h_r^{-1} dt - \int_0^\infty \hat{P}(k-1,t)D \otimes e^{(h_r t)} h_r^{-1} dt, \\
&= -\left(\int_0^\infty \hat{P}(k,t) \otimes e^{(h_r t)} dt\right)\left(C \otimes h_r^{-1}\right) - \left(\int_0^\infty \hat{P}(k-1,t) \otimes e^{(h_r t)} dt\right)\left(D \otimes h_r^{-1}\right) \\
&= -M_k^{(r)}\left(C \otimes h_r^{-1}\right) - M_{k-1}^{(r)}\left(D \otimes h_r^{-1}\right) \\
&= \left[-M_k^{(r)}\left(C \otimes I_{\tau_1}\right) - M_{k-1}^{(r)}\left(D \otimes I_{\tau_1}\right)\right]\left(I_m \otimes h_r^{-1}\right).
\end{aligned}$$

Now post-multiplying both sides of the above expression by $(I_m \otimes h_r)$, we have

$$M_{k,r}\left(I_m \otimes h_r\right) = -M_k^{(r)}\left(C \otimes I_{\tau_1}\right) - M_{k-1,r}\left(D \otimes I_{\tau_1}\right),$$

The above expression can be written in the following form

$$M_{k,r}\left(C \oplus h_r\right) = -M_{k-1,r}\left(D \otimes I_{\tau_1}\right).$$

$$\therefore M_{k,r} = -M_{k-1,r}\left(D \otimes I_{\tau_1}\right) M_{0,r}, \ [\text{as } M_{0,r} = -\left(C \oplus h_r\right)^{-1}]. \tag{6.79}$$

Since $M_{0,r}$ $(a \le r \le b)$ are known from (6.78), hence the value of $M_{k,r}$ $(0 \le k \le N, a \le r \le b)$ can be calculated recursively from (6.79).

Now for $0 \le k \le N$, $a \le r \le b$,

$$\begin{aligned}
\bar{A}_k^{(r)} &= \sum_{j=k+1}^{\infty} A_j^{(r)} = \sum_{j=k+1}^{\infty} \left(I_m \otimes \beta_r\right) M_j^{(r)}\left(I_m \otimes h_r^0\right) \\
&= \left(I_m \otimes \beta_r\right)\left(\sum_{j=k+1}^{\infty} M_j^{(r)}\right)\left(I_m \otimes h_r^0\right) \\
&= \left(I_m \otimes \beta_r\right) \tilde{M}_k^{(r)}\left(I_m \otimes h_r^0\right).
\end{aligned} \tag{6.80}$$

Where, $\tilde{M}_k^{(r)} = \left(\sum_{j=k+1}^{\infty} M_j^{(r)}\right)$ which we have to evaluate. Hence, from (6.80) we have

$$\begin{aligned}
\tilde{M}_k^{(r)} &= \sum_{j=k+1}^{\infty} M_j^{(r)}, \\
&= \sum_{j=k+1}^{\infty} M_{j-1}^{(r)} (D \otimes I_{\tau_1}) M_0^{(r)}, \\
&= -\left(\sum_{j=k+1}^{\infty} M_{j-1}^{(r)}\right) (D \otimes I_{\tau_1}) (C \oplus h_r)^{-1}, \\
&= -\left(M_k^{(r)} + \sum_{j=k+2}^{\infty} M_{j-1}^{(r)}\right) (D \otimes I_{\tau_1}) (C \oplus h_r)^{-1} \\
&= -\left(M_k^{(r)} + \tilde{M}_{k+1}^{(r)}\right) (D \otimes I_{\tau_1}) (C \oplus h_r)^{-1}
\end{aligned}$$

$$\therefore \tilde{M}_k^{(r)} \left(I_{m\tau_1} + (D \otimes I_{\tau_1}) (C \oplus h_r)^{-1}\right) = -M_k^{(r)} (D \otimes I_{\tau_1}) (C \oplus h_r)^{-1}$$

Now post multiplying both side of the above expression by $\left(I_{m\tau_1} + (D \otimes I_{\tau_1}) (C \oplus h_r)^{-1}\right)^{-1}$, we obtain

$$\begin{aligned}
\tilde{M}_k^{(r)} &= -M_k^{(r)} (D \otimes I_{\tau_1}) (C \oplus h_r)^{-1} \left(I_{m\tau_1} + (D \otimes I_{\tau_1}) (C \oplus h_r)^{-1}\right)^{-1}, \\
&= -M_k^{(r)} (D \otimes I_{\tau_1}) \left[\left(I_{m\tau_1} + (D \otimes I_{\tau_1}) (C \oplus h_r)^{-1}\right) (C \oplus h_r)\right]^{-1}, \\
&= -M_k^{(r)} (D \otimes I_{\tau_1}) [(C \oplus h_r) + (D \otimes I_{\tau_1})]^{-1}.
\end{aligned}$$

Therefore from (6.80) we obtain

$$\bar{A}_k^{(r)} = (I_m \otimes \beta_r) \left[-M_k^{(r)} (D \otimes I_{\tau_1}) [(C \oplus h_r) + (D \otimes I_{\tau_1})]^{-1}\right] (I_m \otimes h_r^0).$$

Since $A_k^{(r)}$ and $\bar{A}_k^{(r)}$ are known now, $B_{n,k}^{(a)}$ and $\bar{B}_{n,N}^{(a)}$ can be obtained as

$$B_{n,k}^{(a)} = \tilde{D}^{a-n} A_k^{(a)}, \quad 0 \le n \le a-1, 0 \le k \le N$$

$$\bar{B}^{(a)}_{n,N} = \tilde{D}^{a-n}\bar{A}^{(a)}_{N}, \quad 0 \leq n \leq a-1.$$

Similar steps will follow for the computation of $U_k^{[n]}$ and $\bar{U}_k^{[n]}$.

6.5 Numerical results

This section discusses the implementation of the analytical results, derived in this chapter, in the form of tabular representation and presents a comparative analysis of our model with the existing ones in terms of congestion. To bring out the implementation of analytical result we present the tabular representation of joint probability distribution of queue content, server content and phase of arrival process at service completion epoch and at arbitrary epoch in Table 6.1 and Table 6.2, respectively, for SV. Also joint probability distribution of queue content, vacation type taken by the server and phase of the arrival process at vacation termination epoch and at arbitrary epoch are presented in Table 6.3 and Table 6.4, respectively, for SV. The similar tabular representation of joint probability distribution of queue content, server content and phase of the arrival process at service completion epoch and at arbitrary epoch are shown in Table 6.5 and Table 6.6, respectively, for MV, and the joint probability distribution of queue content, vacation type taken by the server and phase of the arrival process at vacation termination epoch and at arbitrary epoch in Table 6.7 and Table 6.8, respectively, for MV. The input parameters for Tables 6.1-6.8 are taken as $a = 4$, $b = 7$, $N = 15$ and the matrix corresponding to MAP are taken as $C = \begin{pmatrix} -1.00222 & 1.00222 & 0 \\ 0 & -1.00222 & 0 \\ 0 & 0 & -225.75 \end{pmatrix}$, $D = \begin{pmatrix} 0 & 0 & 0 \\ 0.99220 & 0 & 0.01002 \\ 2.2575 & 0 & 223.4925 \end{pmatrix}$ so that $\varpi = \begin{bmatrix} 0.4989 & 0.4989 & 0.00221 \end{bmatrix}$ and $\lambda^* = 1.0$. The service time distribution is considered as PHD (E_3) with irreducible representation (β_r, h_r), with $\beta_r = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$, $h_r = \begin{pmatrix} -3\mu_r & 3\mu_r & 0 \\ 0 & -3\mu_r & 3\mu_r \\ 0 & 0 & -3\mu_r \end{pmatrix}$, $\mu_r = 1/\tilde{s}_r = 1/2r$ $(a \leq r \leq b)$ and the vacation time distribution is again a PHD with irreducible representation (α_k, u_k) where, $\alpha_k = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}$, $u_k = \begin{pmatrix} -v_{1,k} & 0.25v_k & 0 & 0 \\ 0 & -v_{2,k} & 0.3v_{2,k} & 0 \\ 0 & 0 & -v_{3,k} & 0.5v_{3,k} \\ 0 & 0 & 0 & -v_{4,k} \end{pmatrix}$, $v_{i,k} = v_i + \frac{1}{a-k}$ $(0 \le k \le a-1,\ 1 \le i \le 4)$ with $v_1 = 2.0$, $v_2 = 2.1$, $v_3 = 2.3$, $v_4 = 2.5$. These results are presented here to show the numerical compatibility of our analytical results. The important performance measures of the queueing model under consideration are also presented at the bottom of the Table 6.2 and Table 6.6.

After demonstrating the tabular representation of implementation of the analytical result, we now turn our attention to specify the advantage of our current model with the existing ones in terms of congestion (performance measures). For this purpose we have considered $MAP/G_r^{(6,9)}/1/N$ queue with SV and MV and presented a comparison between queue length dependent vacation policy with queue length independent vacation for both SV and MV. The MAP parameters are taken as $C = \begin{pmatrix} -1.00222 & 1.00222 & 0 \\ 0 & -1.00222 & 0 \\ 0 & 0 & -225.75 \end{pmatrix}$, $D = \begin{pmatrix} 0 & 0 & 0 \\ 0.01002 & 0 & 0.99220 \\ 223.4925 & 0 & 2.2575 \end{pmatrix}$, so that $\varpi = \begin{bmatrix} 0.4989 & 0.4989 & 0.00221 \end{bmatrix}$ and $\lambda^* = 1.0$. The service time distribution (STD) is taken as PHD with irreducible representation (β_r, h_r), where $\beta_r = \begin{pmatrix} 0.7 & 0.3 \end{pmatrix}$, $h_r = \begin{pmatrix} -1.5\mu_r & 1.5\mu_r \\ 0.2\mu_r & -1.5\mu_r \end{pmatrix}$ with $\mu_r = 1.5/r$ $(a \le r \le b)$ and the vacation time distribution is again a PHD with irreducible representation (α_k, u_k) where, $\alpha_k = \begin{pmatrix} 0.4 & 0.6 \end{pmatrix}$, $u_k = \begin{pmatrix} -2.0v_k & 1.5v_k \\ 0.5v_k & -2.5v_k \end{pmatrix}$ with $v_k = 0.25(k/4+1)$ $(0 \le k \le a-1)$. These PHD representation of service time distribution are chosen in such a way that mean service time of a batch is increasing with the increase in batch size. On the other hand the PHD representation of vacation time distribution ensures that the mean vacation time is decreasing when more customers $(< a)$ are waiting in the queue at vacation initiation epoch.

For the comparison purpose the following inputs are considered for two different cases,

Case 1. The queue length dependent vacation rates are taken as $\frac{1}{\alpha_k u_k^{-1}\mathbf{e}}$,

Case 2. The queue length independent vacation rate is considered as $\frac{1}{\alpha_0 u_0^{-1}\mathbf{e}}$.

The corresponding mean vacation time and mean service time is shown in the Table 6.9.

Table 6.1: Service completion epoch joint distributions for SV

n	$r=4$			$r=5$			$r=6$			$r=7$			$\sum_{r=a}^{b}\pi^+(n,r)\mathbf{e}$
	$\pi_1^+(n,4)$	$\pi_2^+(n,4)$	$\pi_3^+(n,4)$	$\pi_1^+(n,5)$	$\pi_2^+(n,5)$	$\pi_3^+(n,5)$	$\pi_1^+(n,6)$	$\pi_2^+(n,6)$	$\pi_3^+(n,6)$	$\pi_1^+(n,7)$	$\pi_2^+(n,7)$	$\pi_3^+(n,7)$	
0	**0.00365251**	**0.00857214**	**0.00000000**	**0.00032662**	**0.00106358**	**0.00000000**	**0.00019171**	**0.00064556**	**0.00000000**	**0.00012219**	**0.00042067**	**0.00000000**	**0.01499498**
1	**0.01278292**	**0.01581800**	**0.00000038**	**0.00185653**	**0.00256285**	**0.00000005**	**0.00116980**	**0.00167790**	**0.00000003**	**0.00089990**	**0.00156351**	**0.00000002**	**0.03833190**
2	**0.01732208**	**0.01781342**	**0.00000108**	**0.00306752**	**0.00341164**	**0.00000016**	**0.00208744**	**0.00241351**	**0.00000010**	**0.00236482**	**0.00331722**	**0.00000009**	**0.05179909**
3	**0.01723441**	**0.01621335**	**0.00000186**	**0.00354753**	**0.00357289**	**0.00000031**	**0.00260919**	**0.00273228**	**0.00000021**	**0.00433421**	**0.00546220**	**0.00000023**	**0.05570868**
4	**0.01467005**	**0.01309604**	**0.00000257**	**0.00345139**	**0.00328340**	**0.00000047**	**0.00274427**	**0.00271462**	**0.00000033**	**0.00657317**	**0.00775743**	**0.00000048**	**0.05429421**
5	**0.01136356**	**0.00980025**	**0.00000312**	**0.00303135**	**0.00277902**	**0.00000061**	**0.00260589**	**0.00248411**	**0.00000045**	**0.00884521**	**0.00997525**	**0.00000082**	**0.05088962**
6	**0.00826510**	**0.00695661**	**0.00000353**	**0.00248846**	**0.00222289**	**0.00000073**	**0.00231275**	**0.00214820**	**0.00000055**	**0.01093970**	**0.01192510**	**0.00000125**	**0.04726486**
7	**0.00574881**	**0.00475256**	**0.00000380**	**0.00194735**	**0.00170652**	**0.00000082**	**0.00195649**	**0.00178276**	**0.00000064**	**0.01269209**	**0.01346969**	**0.00000177**	**0.04406331**
8	**0.00386961**	**0.00315587**	**0.00000397**	**0.00147079**	**0.00127012**	**0.00000089**	**0.00159711**	**0.00143402**	**0.00000072**	**0.01444396**	**0.01604439**	**0.00000235**	**0.04329380**
9	**0.00254236**	**0.00205220**	**0.00000407**	**0.00108116**	**0.00092296**	**0.00000093**	**0.00126846**	**0.00112576**	**0.00000077**	**0.01753177**	**0.01897556**	**0.00000304**	**0.04550905**
10	**0.00164174**	**0.00131528**	**0.00000413**	**0.00077823**	**0.00065830**	**0.00000096**	**0.00098593**	**0.00086687**	**0.00000081**	**0.01989286**	**0.02060936**	**0.00000386**	**0.04675833**
11	**0.00104882**	**0.00083617**	**0.00000414**	**0.00055116**	**0.00046283**	**0.00000098**	**0.00075329**	**0.00065729**	**0.00000084**	**0.02074015**	**0.02071428**	**0.00000473**	**0.04577470**
12	**0.00066760**	**0.00053120**	**0.00000414**	**0.00038563**	**0.00032197**	**0.00000099**	**0.00056773**	**0.00049230**	**0.00000087**	**0.02018731**	**0.01960248**	**0.00000561**	**0.04276783**
13	**0.00042708**	**0.00034035**	**0.00000412**	**0.00026758**	**0.00022246**	**0.00000100**	**0.00042333**	**0.00036523**	**0.00000088**	**0.01864186**	**0.01771510**	**0.00000642**	**0.03841542**
14	**0.00027764**	**0.00022257**	**0.00000409**	**0.00018487**	**0.00015327**	**0.00000100**	**0.00031315**	**0.00026909**	**0.00000089**	**0.01653261**	**0.01544842**	**0.00000715**	**0.03341475**
15	**0.00527645**	**0.00470279**	**0.00043144**	**0.00152352**	**0.00134247**	**0.00010513**	**0.00205186**	**0.00182810**	**0.00009673**	**0.08783503**	**0.07956321**	**0.00112811**	**0.18588484**

Table 6.2: Arbitrary epoch joint distributions for SV when server is busy

n	$r=4$			$r=5$			$r=6$			$r=7$			π_n^{busy}
	$\pi_1(n,4)$	$\pi_2(n,4)$	$\pi_3(n,4)$	$\pi_1(n,5)$	$\pi_2(n,5)$	$\pi_3(n,5)$	$\pi_1(n,6)$	$\pi_2(n,6)$	$\pi_3(n,6)$	$\pi_1(n,7)$	$\pi_2(n,7)$	$\pi_3(n,7)$	
0	**0.01646077**	**0.01841628**	**0.00000119**	**0.00245013**	**0.00470147**	**0.00000000**	**0.00221923**	**0.00432085**	**0.00000000**	**0.00206444**	**0.00404169**	**0.00000000**	**0.05467607**
1	**0.01704789**	**0.01557913**	**0.00000199**	**0.00448209**	**0.00424412**	**0.00000021**	**0.00416904**	**0.00401324**	**0.00000020**	**0.00590343**	**0.00779227**	**0.00000019**	**0.06323381**
2	**0.01381945**	**0.01216541**	**0.00000267**	**0.00391734**	**0.00360055**	**0.00000040**	**0.00377973**	**0.00355563**	**0.00000037**	**0.00957762**	**0.01141243**	**0.00000054**	**0.06183213**
3	**0.01044950**	**0.00894403**	**0.00000318**	**0.00323605**	**0.00290429**	**0.00000055**	**0.00327865**	**0.00302495**	**0.00000053**	**0.01306074**	**0.01473105**	**0.00000104**	**0.05963457**
4	**0.00749960**	**0.00628359**	**0.00000354**	**0.00255603**	**0.00225116**	**0.00000068**	**0.00274107**	**0.00248901**	**0.00000065**	**0.01612036**	**0.01750519**	**0.00000169**	**0.05745257**
5	**0.00517360**	**0.00426361**	**0.00000378**	**0.00194870**	**0.00169066**	**0.00000077**	**0.00222363**	**0.00199297**	**0.00000076**	**0.01853792**	**0.01955686**	**0.00000246**	**0.05539572**
6	**0.00346206**	**0.00281611**	**0.00000393**	**0.00144443**	**0.00123803**	**0.00000084**	**0.00176001**	**0.00156054**	**0.00000084**	**0.02018611**	**0.02081000**	**0.00000331**	**0.05328621**
7	**0.00226302**	**0.00182173**	**0.00000402**	**0.00104671**	**0.00088826**	**0.00000088**	**0.00136516**	**0.00119963**	**0.00000090**	**0.02103827**	**0.02128198**	**0.00000420**	**0.05091477**
8	**0.00145326**	**0.00116023**	**0.00000406**	**0.00074480**	**0.00062686**	**0.00000091**	**0.00104136**	**0.00090821**	**0.00000094**	**0.02872870**	**0.03536236**	**0.00000590**	**0.07003758**
9	**0.00092170**	**0.00073114**	**0.00000407**	**0.00052226**	**0.00043656**	**0.00000093**	**0.00078347**	**0.00067894**	**0.00000097**	**0.03339421**	**0.03163226**	**0.00000741**	**0.06911393**
10	**0.00058055**	**0.00045843**	**0.00000406**	**0.00036204**	**0.00030091**	**0.00000094**	**0.00058280**	**0.00050231**	**0.00000099**	**0.02948557**	**0.02757191**	**0.00000874**	**0.05985924**
11	**0.00036559**	**0.00028795**	**0.00000404**	**0.00024885**	**0.00020587**	**0.00000095**	**0.00042958**	**0.00036855**	**0.00000101**	**0.02539013**	**0.02346673**	**0.00000987**	**0.05077911**
12	**0.00023217**	**0.00018285**	**0.00000401**	**0.00017014**	**0.00014024**	**0.00000094**	**0.00031441**	**0.00026870**	**0.00000101**	**0.02137989**	**0.01955972**	**0.00001081**	**0.04226489**
13	**0.00015039**	**0.00011878**	**0.00000397**	**0.00011612**	**0.00009546**	**0.00000094**	**0.00022899**	**0.00019507**	**0.00000101**	**0.01765755**	**0.01601264**	**0.00001157**	**0.03459250**
14	**0.00010076**	**0.00008010**	**0.00000394**	**0.00007946**	**0.00006523**	**0.00000094**	**0.00016633**	**0.00014134**	**0.00000101**	**0.01434350**	**0.01290905**	**0.00001216**	**0.02790382**
15		**0.005563878**			**0.001695069**			**0.002596445**			**0.125097656**		**0.134953048**

p$(0,0)$ =(**0.00024969 0.00096544 0.000004**) **p**$(1,0)$ =(**0.00220551 0.00393882 0.000004**), **p**$(2,0)$ = (**0.00602776 0.00841008 0.000002**), **p**$(3,0)$ =(**0.01093193 0.013566570 0.0000005**).

$L = 13.38011$, $W = 28.28986$, $L_q = 7.385073$, $W_q = 15.61442$, $L_s = 6.33772$, $\zeta = 1.822217$, $L_q^{vac} = 2.126077$, $\mathbf{p}_{dor} = 0.04629667$, $\pi_{busy} = 0.94593$, $\pi_{Block} = 0.5269834$, $\pi_{vac} = 0.007773$,

Table 6.3: Vacation completion epoch joint distributions for SV

n	$k=0$			$k=1$			$k=2$			$k=3$			$\sum_{k=0}^{min(n,a-1)} \omega^+(n-k,k)e$
	$\omega_1^+(n,0)$	$\omega_2^+(n,0)$	$\omega_3^+(n,0)$	$\omega_1^+(n-1,1)$	$\omega_2^+(n-1,1)$	$\omega_3^+(n-1,1)$	$\omega_1^+(n-2,2)$	$\omega_2^+(n-2,2)$	$\omega_3^+(n-2,2)$	$\omega_1^+(n-3,3)$	$\omega_2^+(n-3,3)$	$\omega_3^+(n-3,3)$	
0	**0.0026890**	**0.0077084**	**0.000000**										**0.010397**
1	**0.0028527**	**0.0010674**	**0.0000003**	**0.0106062**	**0.0175995**	**0.0000000**							**0.029273**
2	**0.0003946**	**0.0001463**	**0.000000**	**0.006367**	**0.0023289**	**0.000000**	**0.0161579**	**0.0231814**	**0.0000000**				**0.048576**
3	**0.0000536**	**0.0000195**	**0.000000**	**0.0008412**	**0.000304**	**0.000000**	**0.0080266**	**0.0028099**	**0.000001**	**0.019138**	**0.025239**	**0.000000**	**0.0564318**
4	**0.0000073**	**0.0000026**	**0.0000004**	**0.0001091**	**0.0000388**	**0.0000009**	**0.0009716**	**0.0003368**	**0.0000011**	**0.0077415**	**0.0024024**	**0.0000011**	**0.0116138**
5	**0.0000012**	**0.0000004**	**0.0000004**	**0.0000143**	**0.0000050**	**0.0000009**	**0.0001155**	**0.0000393**	**0.0000011**	**0.0007366**	**0.0002263**	**0.0000012**	**0.0011423**
6	**0.0000005**	**0.0000002**	**0.0000004**	**0.0000024**	**0.0000008**	**0.0000008**	**0.0000140**	**0.0000047**	**0.0000011**	**0.0000691**	**0.0000208**	**0.0000012**	**0.0001160**
7	**0.0000004**	**0.0000001**	**0.0000004**	**0.0000010**	**0.0000004**	**0.0000008**	**0.0000024**	**0.0000008**	**0.0000011**	**0.0000070**	**0.0000021**	**0.0000012**	**0.0000176**
8	**0.0000003**	**0.0000001**	**0.0000003**	**0.0000008**	**0.0000003**	**0.0000008**	**0.0000011**	**0.0000004**	**0.0000011**	**0.0000014**	**0.0000004**	**0.0000012**	**0.0000083**
9	**0.0000003**	**0.0000001**	**0.0000003**	**0.0000008**	**0.0000003**	**0.0000008**	**0.0000010**	**0.0000003**	**0.0000011**	**0.0000009**	**0.0000003**	**0.0000011**	**0.0000074**
10	**0.0000003**	**0.0000001**	**0.0000003**	**0.0000008**	**0.0000003**	**0.0000008**	**0.0000009**	**0.0000003**	**0.0000010**	**0.0000009**	**0.0000003**	**0.0000011**	**0.0000072**
11	**0.0000003**	**0.0000001**	**0.0000003**	**0.0000007**	**0.0000003**	**0.0000008**	**0.0000009**	**0.0000003**	**0.0000010**	**0.0000009**	**0.0000003**	**0.0000011**	**0.0000070**
12	**0.0000003**	**0.0000001**	**0.0000003**	**0.0000007**	**0.0000003**	**0.0000008**	**0.0000009**	**0.0000003**	**0.0000010**	**0.0000008**	**0.0000003**	**0.0000011**	**0.0000069**
13	**0.0000003**	**0.0000001**	**0.0000003**	**0.0000007**	**0.0000003**	**0.0000007**	**0.0000009**	**0.0000003**	**0.0000010**	**0.0000008**	**0.0000003**	**0.0000010**	**0.0000068**
14	**0.0000003**	**0.0000001**	**0.0000003**	**0.0000007**	**0.0000003**	**0.0000007**	**0.0000009**	**0.0000003**	**0.0000010**	**0.0000008**	**0.0000003**	**0.0000010**	**0.0000067**
15	**0.0000178**	**0.0000066**	**0.0000179**	**0.0000393**	**0.0000143**	**0.0000406**	**0.0000476**	**0.0000166**	**0.0000522**	**0.0000403**	**0.0000124**	**0.0000511**	**0.0003567**

Table 6.4: Arbitrary epoch joint distributions for SV when server is in vacation

n	$k=0$			$k=1$			$k=2$			$k=3$			$\omega_n^{[vac]}$
	$\omega_1(n,0)$	$\omega_2(n,0)$	$\omega_3(n,0)$	$\omega_1(n-1,1)$	$\omega_2(n-1,1)$	$\omega_3(n-1,1)$	$\omega_1(n-2,2)$	$\omega_2(n-2,2)$	$\omega_3(n-2,2)$	$\omega_1(n-3,3)$	$\omega_2(n-3,3)$	$\omega_3(n-3,3)$	
0	**0.00014894**	**0.00042689**	**0.00000000**										**0.00057583**
1	**0.00015773**	**0.00005861**	**0.00000002**	**0.00056668**	**0.00094020**	**0.00000000**							**0.00172324**
2	**0.00002143**	**0.00000784**	**0.00000002**	**0.00033954**	**0.00012329**	**0.00000004**	**0.00080633**	**0.00115680**	**0.00000000**				**0.00245530**
3	**0.00000283**	**0.00000102**	**0.00000002**	**0.00004404**	**0.00001574**	**0.00000005**	**0.00039993**	**0.00013902**	**0.00000005**	**0.00079735**	**0.00105186**	**0.00000000**	**0.00245190**
4	**0.00000038**	**0.00000013**	**0.00000002**	**0.00000555**	**0.00000195**	**0.00000005**	**0.00004753**	**0.00001626**	**0.00000006**	**0.00032251**	**0.00009944**	**0.00000005**	**0.00049392**
5	**0.00000006**	**0.00000002**	**0.00000002**	**0.00000071**	**0.00000024**	**0.00000005**	**0.00000549**	**0.00000184**	**0.00000006**	**0.00003015**	**0.00000914**	**0.00000005**	**0.00004784**
6	**0.00000002**	**0.00000001**	**0.00000002**	**0.00000012**	**0.00000004**	**0.00000004**	**0.00000065**	**0.00000021**	**0.00000006**	**0.00000275**	**0.00000082**	**0.00000005**	**0.00000479**
7	**0.00000002**	**0.00000001**	**0.00000002**	**0.00000005**	**0.00000002**	**0.00000004**	**0.00000011**	**0.00000004**	**0.00000005**	**0.00000027**	**0.00000008**	**0.00000005**	**0.00000077**
8	**0.00000002**	**0.00000001**	**0.00000002**	**0.00000004**	**0.00000002**	**0.00000004**	**0.00000006**	**0.00000002**	**0.00000005**	**0.00000006**	**0.00000002**	**0.00000005**	**0.00000040**
9	**0.00000002**	**0.00000001**	**0.00000002**	**0.00000004**	**0.00000001**	**0.00000004**	**0.00000005**	**0.00000002**	**0.00000005**	**0.00000004**	**0.00000001**	**0.00000005**	**0.00000036**
10	**0.00000002**	**0.00000001**	**0.00000002**	**0.00000004**	**0.00000001**	**0.00000004**	**0.00000005**	**0.00000002**	**0.00000005**	**0.00000004**	**0.00000001**	**0.00000005**	**0.00000035**
11	**0.00000002**	**0.00000001**	**0.00000002**	**0.00000004**	**0.00000001**	**0.00000004**	**0.00000005**	**0.00000002**	**0.00000005**	**0.00000004**	**0.00000001**	**0.00000005**	**0.00000034**
12	**0.00000002**	**0.00000001**	**0.00000002**	**0.00000004**	**0.00000001**	**0.00000004**	**0.00000005**	**0.00000002**	**0.00000005**	**0.00000003**	**0.00000001**	**0.00000004**	**0.00000034**
13	**0.00000002**	**0.00000001**	**0.00000002**	**0.00000004**	**0.00000001**	**0.00000004**	**0.00000004**	**0.00000002**	**0.00000005**	**0.00000003**	**0.00000001**	**0.00000004**	**0.00000033**
14	**0.00000002**	**0.00000001**	**0.00000002**	**0.00000004**	**0.00000001**	**0.00000004**	**0.00000004**	**0.00000002**	**0.00000005**	**0.00000003**	**0.00000001**	**0.00000004**	**0.00000032**
15		**0.00000232**			**0.00000497**			**0.00000575**			**0.00000430**		**0.00000173**

Table 6.5: Service completion epoch joint distributions for MV

n	$r=4$			$r=5$			$r=6$			$r=7$			$\sum_{r=a}^{b}\pi^{+}(n,r)\mathbf{e}$
	$\pi_1^+(n,4)$	$\pi_2^+(n,4)$	$\pi_3^+(n,4)$	$\pi_1^+(n,5)$	$\pi_2^+(n,5)$	$\pi_3^+(n,5)$	$\pi_1^+(n,6)$	$\pi_2^+(n,6)$	$\pi_3^+(n,6)$	$\pi_1^+(n,7)$	$\pi_2^+(n,7)$	$\pi_3^+(n,7)$	
0	**0.00127486**	**0.00336130**	**0.00000000**	**0.00021300**	**0.00065980**	**0.00000000**	**0.00009744**	**0.00032614**	**0.00000000**	**0.00006053**	**0.00020835**	**0.00000000**	**0.00620142**
1	**0.00525969**	**0.00668232**	**0.00000015**	**0.00113346**	**0.00155242**	**0.00000003**	**0.00058989**	**0.00084533**	**0.00000001**	**0.00044571**	**0.00077446**	**0.00000001**	**0.01728348**
2	**0.00744018**	**0.00773907**	**0.00000045**	**0.00184943**	**0.00205052**	**0.00000010**	**0.00105109**	**0.00121484**	**0.00000005**	**0.00117154**	**0.00164362**	**0.00000004**	**0.02416092**
3	**0.00754988**	**0.00714754**	**0.00000079**	**0.00212749**	**0.00213916**	**0.00000019**	**0.00131300**	**0.00137468**	**0.00000011**	**0.00214785**	**0.00270722**	**0.00000012**	**0.02650801**
4	**0.00649900**	**0.00582462**	**0.00000110**	**0.00206377**	**0.00196131**	**0.00000028**	**0.00138051**	**0.00136544**	**0.00000017**	**0.00325828**	**0.00384572**	**0.00000024**	**0.02620042**
5	**0.00507009**	**0.00438407**	**0.00000134**	**0.00180924**	**0.00165748**	**0.00000037**	**0.00131063**	**0.00124928**	**0.00000022**	**0.00438538**	**0.00494599**	**0.00000040**	**0.02481451**
6	**0.00370511**	**0.00312404**	**0.00000152**	**0.00148332**	**0.00132436**	**0.00000044**	**0.00116303**	**0.00108022**	**0.00000028**	**0.00542449**	**0.00591333**	**0.00000062**	**0.02322077**
7	**0.00258511**	**0.00213947**	**0.00000165**	**0.00115970**	**0.00101590**	**0.00000049**	**0.00098378**	**0.00089639**	**0.00000032**	**0.00629383**	**0.00667952**	**0.00000088**	**0.02175704**
8	**0.00174318**	**0.00142235**	**0.00000173**	**0.00087528**	**0.00075564**	**0.00000053**	**0.00080302**	**0.00072099**	**0.00000036**	**0.00716269**	**0.00795486**	**0.00000117**	**0.02144180**
9	**0.00114582**	**0.00092473**	**0.00000177**	**0.00064307**	**0.00054885**	**0.00000056**	**0.00063774**	**0.00056599**	**0.00000039**	**0.00869031**	**0.00940395**	**0.00000151**	**0.02256468**
10	**0.00073912**	**0.00059150**	**0.00000180**	**0.00046270**	**0.00039132**	**0.00000058**	**0.00049568**	**0.00043582**	**0.00000041**	**0.00985674**	**0.01021020**	**0.00000191**	**0.02318778**
11	**0.00047068**	**0.00037436**	**0.00000180**	**0.00032759**	**0.00027505**	**0.00000059**	**0.00037872**	**0.00033045**	**0.00000042**	**0.01027373**	**0.01025989**	**0.00000235**	**0.02269565**
12	**0.00029775**	**0.00023591**	**0.00000180**	**0.00022916**	**0.00019131**	**0.00000060**	**0.00028543**	**0.00024751**	**0.00000044**	**0.00999808**	**0.00970780**	**0.00000278**	**0.02119856**
13	**0.00018848**	**0.00014915**	**0.00000180**	**0.00015899**	**0.00013217**	**0.00000060**	**0.00021284**	**0.00018363**	**0.00000044**	**0.00923162**	**0.00877233**	**0.00000318**	**0.01903523**
14	**0.00012051**	**0.00009556**	**0.00000178**	**0.00010984**	**0.00009107**	**0.00000060**	**0.00015745**	**0.00013531**	**0.00000045**	**0.00818656**	**0.00764955**	**0.00000354**	**0.01655222**
15	**0.00177707**	**0.00153768**	**0.00018257**	**0.00091240**	**0.00080428**	**0.00006297**	**0.00103686**	**0.00092429**	**0.00004869**	**0.04377137**	**0.03967392**	**0.00056046**	**0.09129256**

Table 6.6: Arbitrary epoch joint distributions for MV when server is busy

n	$r=4$			$r=5$			$r=6$			$r=7$			π_n^{busy}
	$\pi_1(n,4)$	$\pi_2(n,4)$	$\pi_3(n,4)$	$\pi_1(n,5)$	$\pi_2(n,5)$	$\pi_3(n,5)$	$\pi_1(n,6)$	$\pi_2(n,6)$	$\pi_3(n,6)$	$\pi_1(n,7)$	$\pi_2(n,7)$	$\pi_3(n,7)$	
0	**0.01168473**	**0.01645774**	**0.00000001**	**0.00324955**	**0.00571986**	**0.00000001**	**0.00229391**	**0.00442009**	**0.00000001**	**0.00208006**	**0.00407029**	**0.00000001**	**0.04997629**
1	**0.01529998**	**0.01403809**	**0.00000074**	**0.00544866**	**0.00515550**	**0.00000027**	**0.00426453**	**0.00410490**	**0.00000021**	**0.00594697**	**0.00785128**	**0.00000021**	**0.06211134**
2	**0.01249440**	**0.01103294**	**0.00000136**	**0.00475531**	**0.00436809**	**0.00000049**	**0.00386584**	**0.00363643**	**0.00000039**	**0.00965346**	**0.01150479**	**0.00000057**	**0.06131407**
3	**0.00949996**	**0.00815021**	**0.00000183**	**0.00392377**	**0.00351981**	**0.00000068**	**0.00335300**	**0.00309341**	**0.00000055**	**0.01316938**	**0.01485482**	**0.00000109**	**0.05956852**
4	**0.00684558**	**0.00574565**	**0.00000218**	**0.00309643**	**0.00272605**	**0.00000083**	**0.00280301**	**0.00254516**	**0.00000068**	**0.01625781**	**0.01765477**	**0.00000175**	**0.05767990**
5	**0.00473566**	**0.00390777**	**0.00000241**	**0.00235901**	**0.00204601**	**0.00000094**	**0.00227374**	**0.00203783**	**0.00000078**	**0.01869745**	**0.01972474**	**0.00000253**	**0.05578888**
6	**0.00317445**	**0.00258450**	**0.00000256**	**0.00174756**	**0.00149747**	**0.00000102**	**0.00179959**	**0.00159560**	**0.00000087**	**0.02035989**	**0.02098822**	**0.00000340**	**0.05375514**
7	**0.00207624**	**0.00167222**	**0.00000264**	**0.00126581**	**0.00107397**	**0.00000108**	**0.00139582**	**0.00122655**	**0.00000093**	**0.02121856**	**0.02146316**	**0.00000431**	**0.05140129**
8	**0.00133227**	**0.00106367**	**0.00000269**	**0.00090037**	**0.00075768**	**0.00000112**	**0.00106474**	**0.00092858**	**0.00000097**	**0.02897306**	**0.03561240**	**0.00000644**	**0.07064400**
9	**0.00084272**	**0.00066810**	**0.00000271**	**0.00063118**	**0.00052753**	**0.00000114**	**0.00080106**	**0.00069418**	**0.00000101**	**0.03362976**	**0.03185391**	**0.00000795**	**0.06966124**
10	**0.00052794**	**0.00041624**	**0.00000271**	**0.00043745**	**0.00036355**	**0.00000115**	**0.00059590**	**0.00051360**	**0.00000103**	**0.02969199**	**0.02776388**	**0.00000928**	**0.06032472**
11	**0.00032930**	**0.00025861**	**0.00000270**	**0.00030064**	**0.00024870**	**0.00000115**	**0.00043925**	**0.00037685**	**0.00000104**	**0.02556711**	**0.02362962**	**0.00001042**	**0.05116540**
12	**0.00020587**	**0.00016133**	**0.00000268**	**0.00020554**	**0.00016942**	**0.00000115**	**0.00032152**	**0.00027478**	**0.00000104**	**0.02152880**	**0.01969557**	**0.00001136**	**0.04257906**
13	**0.00013016**	**0.00010200**	**0.00000266**	**0.00014030**	**0.00011534**	**0.00000115**	**0.00023419**	**0.00019951**	**0.00000105**	**0.01778094**	**0.01612436**	**0.00001212**	**0.03484377**
14	**0.00008422**	**0.00006617**	**0.00000264**	**0.00009603**	**0.00007883**	**0.00000114**	**0.00017014**	**0.00014459**	**0.00000104**	**0.01444450**	**0.01299994**	**0.00001271**	**0.02810195**
15		**0.003375132**			**0.002068865**			**0.002677865**			**0.127447461**		**0.13556932**

$L = 13.46785$, $W = 28.51894$, $L_q = 7.445622$, $W_q = 15.76652$, $L_s = 6.376208$, $\zeta = 2.244066$, $L_q^{vac} = 7.0758928$, $\pi_{busy} = 0.9444849$, $\pi_{Block} = 0.5277061$, $\pi_{vac} = 0.0555151$,

Table 6.7: Vacation completion epoch joint distributions for MV

n	$k=0$			$k=1$			$k=2$			$k=3$			$\sum_{k=0}^{min(n,a-1)} \omega^+(n-k,k)\mathbf{e}$
	$\omega_1^+(n,0)$	$\omega_2^+(n,0)$	$\omega_3^+(n,0)$	$\omega_1^+(n-1,1)$	$\omega_2^+(n-1,1)$	$\omega_3^+(n-1,1)$	$\omega_1^+(n-2,2)$	$\omega_2^+(n-2,2)$	$\omega_3^+(n-2,2)$	$\omega_1^+(n-3,3)$	$\omega_2^+(n-3,3)$	$\omega_3^+(n-3,3)$	
0	**0.0027592**	**0.0103974**	**0.0000000**										**0.01315668**
1	**0.0038477**	**0.0014399**	**0.000000458**	**0.0195974**	**0.03923029**	**0.0000000**							**0.0641158**
2	**0.0005324**	**0.0001975**	**0.00000051**	**0.01419281**	**0.005191426**	**0.00000173**	**0.0488195**	**0.082382101**	**0.00000000**				**0.1513179**
3	**0.00007237**	**0.000026388**	**0.00000051**	**0.00187566**	**0.00067976**	**0.00000193**	**0.02852286**	**0.00998610**	**0.000003649**	**0.0971913568**	**.150779922**	**0.0000000**	**0.28914061**
4	**0.00000986**	**0.00000353**	**0.00000051**	**0.00024345**	**0.00008669**	**0.00000193**	**0.00345350**	**0.00119769**	**0.00000402**	**0.04624238**	**0.01435155**	**0.00000670**	**0.06560182**
5	**0.00000167**	**0.00000060**	**0.00000050**	**0.00003184**	**0.00001114**	**0.00000190**	**0.00041086**	**0.00013993**	**0.00000400**	**0.00440112**	**0.00135270**	**0.00000720**	**0.00636346**
6	**0.00000063**	**0.00000023**	**0.00000049**	**0.00000539**	**0.00000188**	**0.00000186**	**0.00004986**	**0.00001669**	**0.00000394**	**0.00041287**	**0.00012448**	**0.00000711**	**0.00062542**
7	**0.00000049**	**0.00000018**	**0.00000048**	**0.00000218**	**0.00000078**	**0.00000183**	**0.00000858**	**0.00000289**	**0.00000387**	**0.00004172**	**0.00001238**	**0.00000698**	**0.00008237**
8	**0.00000047**	**0.00000018**	**0.00000047**	**0.00000178**	**0.00000065**	**0.00000180**	**0.00000400**	**0.00000138**	**0.00000380**	**0.00000848**	**0.00000256**	**0.00000684**	**0.00003240**
9	**0.00000046**	**0.00000017**	**0.00000046**	**0.00000171**	**0.00000062**	**0.00000177**	**0.00000346**	**0.00000121**	**0.00000373**	**0.00000554**	**0.00000171**	**0.00000671**	**0.00002754**
10	**0.00000045**	**0.00000017**	**0.00000046**	**0.00000168**	**0.00000061**	**0.00000174**	**0.00000335**	**0.00000117**	**0.00000366**	**0.00000520**	**0.00000161**	**0.00000657**	**0.00002666**
11	**0.00000045**	**0.00000017**	**0.00000045**	**0.00000165**	**0.00000060**	**0.00000171**	**0.00000328**	**0.00000115**	**0.00000359**	**0.00000508**	**0.00000157**	**0.00000644**	**0.00002613**
12	**0.00000044**	**0.00000016**	**0.00000044**	**0.00000162**	**0.00000059**	**0.00000168**	**0.00000322**	**0.00000113**	**0.00000353**	**0.00000498**	**0.00000154**	**0.00000632**	**0.00002564**
13	**0.00000043**	**0.00000016**	**0.00000043**	**0.00000159**	**0.00000058**	**0.00000165**	**0.00000317**	**0.00000110**	**0.00000346**	**0.00000488**	**0.00000151**	**0.00000619**	**0.00002515**
14	**0.00000042**	**0.00000016**	**0.00000042**	**0.00000156**	**0.00000057**	**0.00000162**	**0.00000311**	**0.00000108**	**0.00000340**	**0.00000478**	**0.00000148**	**0.00000607**	**0.00002468**
15	**0.00002401**	**0.00000893**	**0.00002413**	**0.00008738**	**0.00003177**	**0.00009047**	**0.00016861**	**0.00005868**	**0.00018455**	**0.00023825**	**0.00007358**	**0.00030221**	**0.00129255**

Table 6.8: Arbitrary epoch joint distributions for MV when server is in vacation

n	$k=0$			$k=1$			$k=2$			$k=3$			$\omega_n^{[vac]}$
	$\omega_1(n,0)$	$\omega_2(n,0)$	$\omega_3(n,0)$	$\omega_1(n-1,1)$	$\omega_2(n-1,1)$	$\omega_3(n-1,1)$	$\omega_1(n-2,2)$	$\omega_2(n-2,2)$	$\omega_3(n-2,2)$	$\omega_1(n-3,3)$	$\omega_2(n-3,3)$	$\omega_3(n-3,3)$	
0	**0.00031080**	**0.00117109**	**0.00000000**										**0.00148189**
1	**0.00043276**	**0.00016085**	**0.00000005**	**0.00212947**	**0.00426233**	**0.00000000**							**0.00698546**
2	**0.00005882**	**0.00002153**	**0.00000006**	**0.00153952**	**0.00055917**	**0.00000019**	**0.00495471**	**0.00836084**	**0.00000000**				**0.01549484**
3	**0.00000778**	**0.00000279**	**0.00000006**	**0.00019980**	**0.00007143**	**0.00000021**	**0.00289095**	**0.00100516**	**0.00000037**	**0.00823521**	**0.01277889**	**0.00000001**	**0.02519266**
4	**0.00000103**	**0.00000036**	**0.00000006**	**0.00002521**	**0.00000884**	**0.00000021**	**0.00034378**	**0.00011761**	**0.00000041**	**0.00391867**	**0.00120850**	**0.00000057**	**0.00562526**
5	**0.00000017**	**0.00000006**	**0.00000006**	**0.00000321**	**0.00000111**	**0.00000021**	**0.00003976**	**0.00001334**	**0.00000041**	**0.00036659**	**0.00011114**	**0.00000061**	**0.00053667**
6	**0.00000007**	**0.00000002**	**0.00000005**	**0.00000054**	**0.00000019**	**0.00000020**	**0.00000471**	**0.00000155**	**0.00000040**	**0.00003344**	**0.00000993**	**0.00000060**	**0.00005171**
7	**0.00000005**	**0.00000002**	**0.00000005**	**0.00000023**	**0.00000008**	**0.00000020**	**0.00000082**	**0.00000027**	**0.00000039**	**0.00000331**	**0.00000097**	**0.00000059**	**0.00000699**
8	**0.00000005**	**0.00000002**	**0.00000005**	**0.00000019**	**0.00000007**	**0.00000020**	**0.00000040**	**0.00000014**	**0.00000038**	**0.00000069**	**0.00000021**	**0.00000058**	**0.00000298**
9	**0.00000005**	**0.00000002**	**0.00000005**	**0.00000018**	**0.00000007**	**0.00000019**	**0.00000035**	**0.00000012**	**0.00000038**	**0.00000046**	**0.00000014**	**0.00000057**	**0.00000259**
10	**0.00000005**	**0.00000002**	**0.00000005**	**0.00000018**	**0.00000007**	**0.00000019**	**0.00000034**	**0.00000012**	**0.00000037**	**0.00000044**	**0.00000013**	**0.00000056**	**0.00000251**
11	**0.00000005**	**0.00000002**	**0.00000005**	**0.00000018**	**0.00000006**	**0.00000019**	**0.00000033**	**0.00000011**	**0.00000036**	**0.00000043**	**0.00000013**	**0.00000055**	**0.00000246**
12	**0.00000005**	**0.00000002**	**0.00000005**	**0.00000017**	**0.00000006**	**0.00000018**	**0.00000032**	**0.00000011**	**0.00000036**	**0.00000042**	**0.00000013**	**0.00000054**	**0.00000241**
13	**0.00000005**	**0.00000002**	**0.00000005**	**0.00000017**	**0.00000006**	**0.00000018**	**0.00000032**	**0.00000011**	**0.00000035**	**0.00000041**	**0.00000013**	**0.00000052**	**0.00000237**
14	**0.00000005**	**0.00000002**	**0.00000005**	**0.00000017**	**0.00000006**	**0.00000018**	**0.00000031**	**0.00000011**	**0.00000034**	**0.00000040**	**0.00000012**	**0.00000051**	**0.00000233**
15		**0.00000636**			**0.00002253**			**0.00004140**			**0.00005171**		**0.000122**

Table 6.9: Mean service time and mean vacation time for Figs. 6.1-6.6

Service		Vacation		
batch size (r)	$\tilde{s_r}$	queue length (k)	$\tilde{v}^{[k]}$(Case 1)	$\tilde{v}^{[k]}$(Case 2)
6	5.3538	0	2.9176	2.9176
7	6.2462	1	2.3341	2.9176
8	7.1384	2	1.9451	2.9176
9	8.0307	3	1.6672	2.9176
		4	1.4588	2.9176
		5	1.2967	2.9176

The assumptions for vacation rates, in Case 1 and Case 2, are made in such a way that for Case 2 the server always takes a vacation with constant vacation rate $\frac{1}{\alpha_0 u_0^{-1}\mathbf{e}}$ irrespective of the queue length at vacation initiation epoch, and for Case 1 the server will start a vacation with vacation rate $\frac{1}{\alpha_0 u_0^{-1}\mathbf{e}}$, when it finds an empty queue and start a vacation with higher vacation rate (i.e., $\frac{1}{\alpha_k u_k^{-1}\mathbf{e}} > \frac{1}{\alpha_{k-1} u_{k-1}^{-1}\mathbf{e}}$, $k = 1, 2, ...a-1$.) depending on queue length. These assumptions ensure us that due to queue length dependent vacation (Case 1) the server is modulating the length of the vacation periods in such a way that the server is taking a longer vacation for empty queue and shorter vacation when queue is non empty.

Figure 6.1- Figure 6.6 present the various performance indices with the variation in the queue capacity N for above considered cases under both vacation models SV and MV. It is clearly observed from Figure 6.1- Figure 6.4 and Figure 6.6 that the measures L, W, L_q^{vac}, ω_{vac} and π_{Block} corresponding to Case 1 is dominated by the corresponding measures of Case 2, while in Figure 6.5 the probability π_{busy} for Case 2 is dominated by Case 1. The above observation is similar for both the vacation models SV and MV. Thus we can conclude that the queue length dependent vacation policy helps in reducing the congestion in a vacation queuing system, and increases the availability of the server to our system, in terms of busy probability π_{busy}.

It is also observed from Figure 6.1- Figure 6.4 and Figure 6.6 that the measures (L, W, L_q^{vac}, ω_{vac} and π_{Block}) corresponding to SV model are lesser than the the corresponding measures for MV model. Which is quite obvious, because in SV model the availability of the server to the system is much more in compare to MV model. It can be depicted through Figure 6.5, i.e., $(\pi_{busy})_{SV} > (\pi_{busy})_{MV}$.

Another observation can be made from Figure 6.1- Figure 6.2 that with the increase in value of N, the measures L and W are increasing. Again Figure 6.3- Figure 6.4 show that with the increasing value of N, initially the values of L_q^{vac} and ω_{vac} decreases rapidly, however these decrements is very slow for large value of N. Figure 6.5 and Figure 6.6 reveal that for large value of N, the measures π_{busy} and π_{Block} are almost constant which implies that the effect of N is insignificant for larger N. Also from Figure 6.6 we can say that when $N \to \infty, \pi_{Block}$ reduces to zero. Hence under the above observation extracted from Figure 6.1- Figure 6.6 we can conclude that for large queue capacity the considered finite buffer vacation queueing system will behave like an infinite buffer vacation queuing system.

Another numerical example is taken to observe the effect of vacation time distribution (VTD) for our current model, which deals with three qualitatively different PH type distributions for vacation time, viz., EXV (exponential), ERV (E_2) and HEV (HE_2), for a fixed PH type distribution of service time. Towards this end, the input parameters are taken as $a = 5$, $b = 7$, the matrix parameters of MAP are $C = \begin{pmatrix} -4.657 & 1.761 \\ 1.128 & -3.941 \end{pmatrix}$, $D = \begin{pmatrix} 1.657 & 1.239 \\ 0.872 & 1.941 \end{pmatrix}$ so that $\varpi = \begin{bmatrix} 0.40 & 0.60 \end{bmatrix}$ and $\lambda^* = 2.8462$. The service time distribution is considered as PHD with irreducible representation (β_r, h_r), where $\beta_r = \begin{pmatrix} 0.7 & 0.3 \end{pmatrix}$, $h_r = \begin{pmatrix} -0.5\mu_r & 0.4\mu_r \\ 1.5\mu_r & -2.5\mu_r \end{pmatrix}$ with $\mu_r = 1.5 + \frac{1}{r-1}$ $(a \leq r \leq b)$ and the different PH type distributions for vacation time with irreducible representation (α_k, u_k) are as follows

For $\nu_k = \frac{1}{a-k}$ $(0 \leq k \leq a-1)$,

(i). ERV : $\alpha_k = \begin{pmatrix} 1.0 & 0 \end{pmatrix}$, $u_k = \begin{pmatrix} -2.0\nu_k & 2.0\nu_k \\ 0 & -2.0\nu_k \end{pmatrix}$.

(ii). EXV : $\alpha_k = \begin{pmatrix} 1.0 \end{pmatrix}$, $u_k = \begin{pmatrix} -\nu_k \end{pmatrix}$.

(iii). HEV : $\alpha_k = \begin{pmatrix} 0.9 & 0.1 \end{pmatrix}$, $u_k = \begin{pmatrix} -1.9\nu_k & 0 \\ 0 & -0.19\nu_k \end{pmatrix}$.

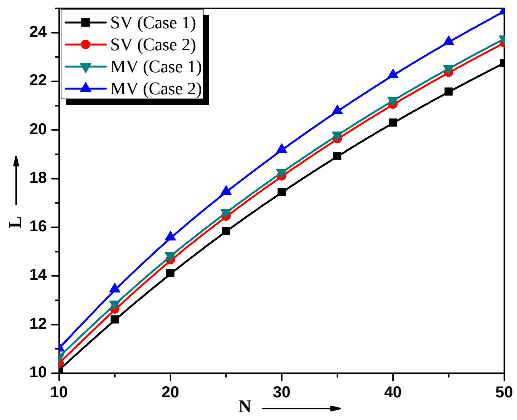

Figure 6.1: Effect of N on L.

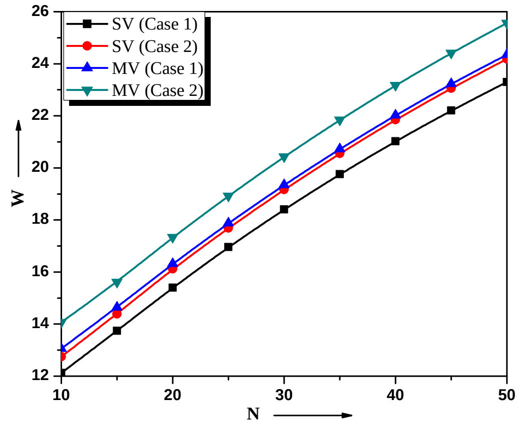

Figure 6.2: Effect of N on W.

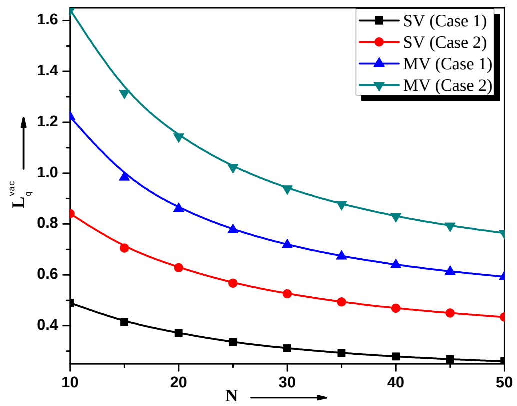

Figure 6.3: Effect of N on L_q^{vac}.

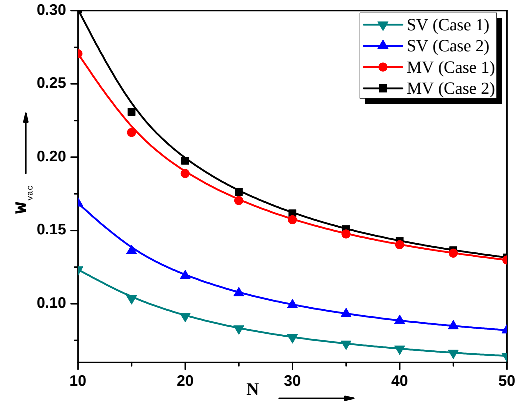

Figure 6.4: Effect of N on ω_{vac}.

The above PHD of VTD are taken in such a way that each PHD having the equal mean time for a particular k-th type of vacation.

Figure 6.7- Figure 6.11 illustrate the effect of N on various performance measures under the above considered three kinds of vacation time distributions (VTD). It can be seen from Figure 6.7- Figure 6.9 that the similar behavior of π_{Block}, π_{busy} and ω_{vac} is observed as in Figure 6.4- Figure 6.6, i.e., enlarging the queue capacity (N) reduces the probabilities π_{Block} and ω_{vac} while π_{busy} increases. Also the relations $(\pi_{Block})_{SV} < (\pi_{Block})_{MV}$, $(\omega_{vac})_{SV} < (\omega_{vac})_{MV}$ and $(\pi_{busy})_{SV} > (\pi_{busy})_{MV}$ hold good as previous one for each corresponding VTD. The effect of different kinds of VTD (having equal mean) can be seen from Figure 6.7- Figure 6.9 for both the models SV and MV. From Figure 6.7 and Figure 6.9 we observe that for a fixed N the measures π_{Block} and ω_{vac} are largest for HEV, and smallest for ERV. Figure 6.8 reveals the reverse nature, i.e., the measure π_{busy} is largest for ERV while it is smallest for HEV. The

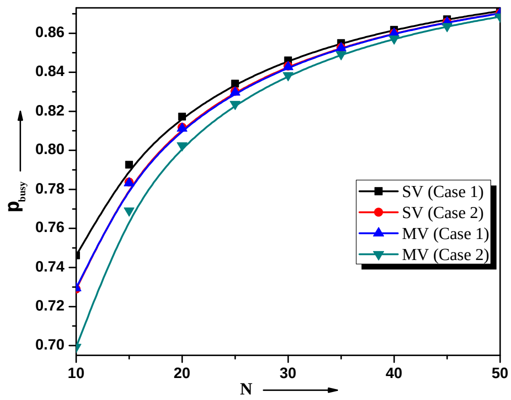

Figure 6.5: Effect of N on π_{busy}.

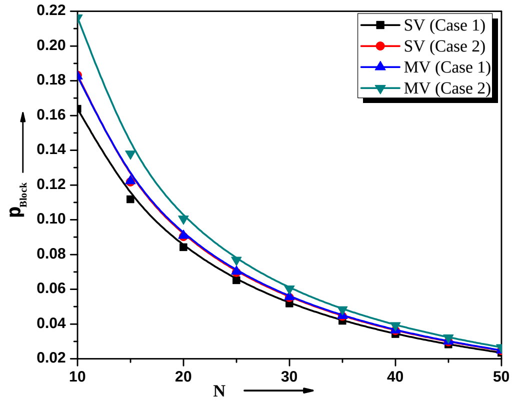

Figure 6.6: Effect of N on π_{Block}.

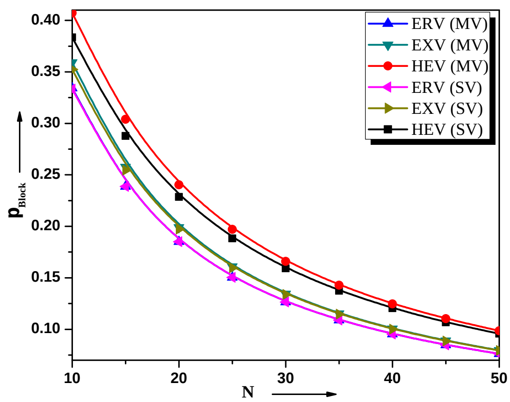

Figure 6.7: Effect of N on π_{Block}.

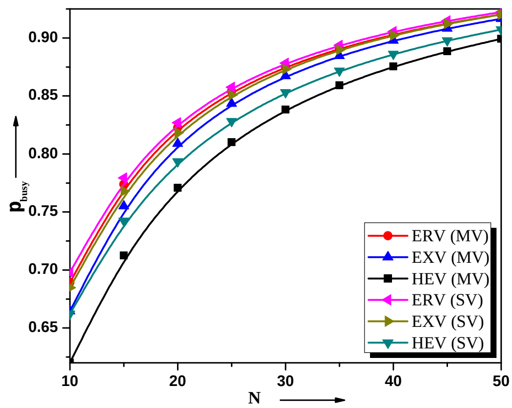

Figure 6.8: Effect of N on π_{busy}.

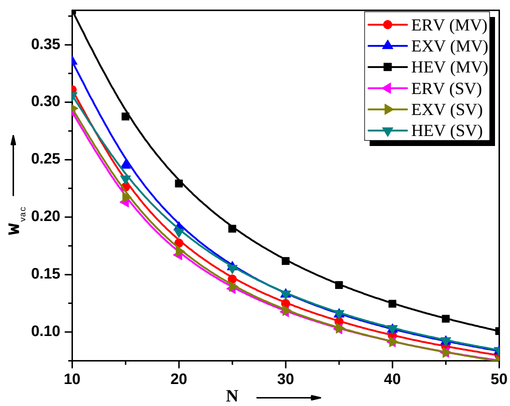

Figure 6.9: Effect of N on ω_{vac}.

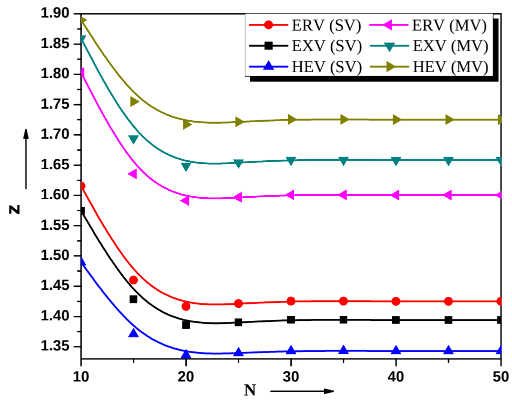

Figure 6.10: Effect of N on ζ.

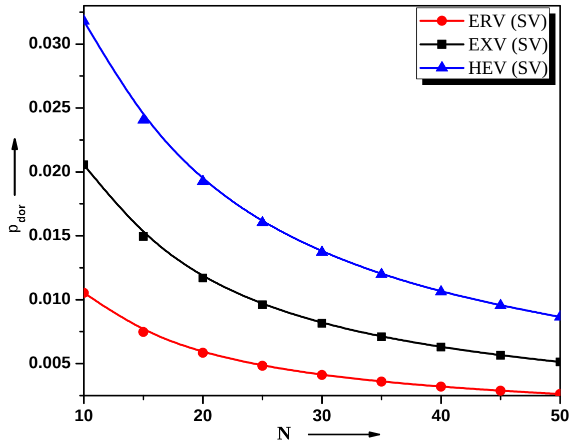

Figure 6.11: Effect of N on $\mathbf{p}_{dor}$.

reason behind that the coefficient of variation for ERV is less than 1 while for HEV it is greater than 1.

An interesting performance measure ζ is plotted in Figure 6.10. It depicts the effect of N on expected number of customer present in the queue when a vacation starts (or expected vacation type ζ) for different kinds of VTD. It shows a relation $(\zeta)_{SV} < (\zeta)_{MV}$ which is quite obvious, as server is taking more than one vacation with a same or higher vacation type, depending on the queue length, in case of MV. Also it is observed that the larger value of N (> 20) shows the insignificant change in the value of ζ. Further a relation can be extracted for various VTD as follows (separately for SV and MV) :

(i) for SV $(\zeta)_{HEV} < (\zeta)_{EXV} < (\zeta)_{ERV}$ and

(ii) For MV $(\zeta)_{HEV} > (\zeta)_{EXV} > (\zeta)_{ERV}$.

Figure 6.11 accomplish to support the raising interest of reverse nature of relation (i) and (ii), which depicts the effect of various kind of VTD on $\mathbf{p}_{dor}$ for the case of SV policy. It shows that $(\mathbf{p}_{dor})_{HEV} > (\mathbf{p}_{dor})_{EXV} > (\mathbf{p}_{dor})_{ERV}$, which implies that in case of ERV server waits (for customers to start the service) much lesser time than in case of HEV. In case of MV these dormancy period are utilized in terms of recurring vacations. Thus the fraction of time that the server is in $k-th$ type vacation is largest for HEV, and hence it justifies the relation (ii).

6.6 Concluding remarks

In this chapter, we have considered $MAP/G_r^{(a,b)}/1/N$ queue with queue length dependent vacation (single vacation and multiple vacation). The service time depends on the size of the batches under service and the vacation time depends on the queue length at vacation initiation epoch. The service time distribution and vacation time distribution both are considered to be generally distributed. Using the supplementary variable technique and the embedded Markov chain technique we analytically obtained all the required joint distributions at various epoch. However, for computation purpose we have considered PH type service and PH type vacation time distribution and presented an efficient procedure. Several illustrative numerical examples to show the impact of the various parameters on selected system performance measures is also presented. It is established that the implementation of queue length dependent vacation in batch size dependent bulk service queue with *MAP* further reduces the congestion. The analysis carried out in this chapter may be extended to analyze queuing models involving *BMAP*.