Chapter 5

Analysis of $M^X/G_r^{(a,b)}/1/N$ queue with queue length dependent single and multiple vacation

5.1 Introduction

Congestion control mechanisms prevent congestion of the system either before it happens or remove congestion after it has happened. A significant amount of literature on queuing study, focused on congestion control, is found in literature. In modern telecommunication system request for service arrive in batches of varying size and are served in batches of varying size, e.g., e-mail messages. For effective utilization and proper maintenance of the mail server systems a continuous and repetitive virus scan should be performed on a regular basis whenever server found to be idle due to non availability of messages to be served. During virus scan period the server will be unavailable from the system and the period for which server remain unavailable is termed as vacation period. Bulk arrival and bulk service queue with vacation (single vacation and multiple vacation) is an appropriate mathematical model to handle such situation.

Bulk service queues with vacation have been widely studied in past decades by many researchers, see e.g., Krishna et al. (1998), Arumuganathan and Jeyakumar (2005), Samanta et al. (2007a), Sikdar and Gupta (2008), Sikdar et al. (2008), Balasubramanian and Arumuganathan (2011), Jeyakumar and Arumuganathan (2011), Haridass and Arumuganathan (2012a), Haridass and Arumuganathan (2012b), Laxmi et al. (2013), Jeyakumar and Senthilnathan (2014), Sikdar and Samanta (2016), Jeyakumar and Senthilnathan (2016), Singh and Kumar (2017) and the references therein. Only few of the above literature concern both bulk arrival and bulk service for the vacation queues, see e.g., Krishna et al. (1998), Arumuganathan and Jeyakumar (2005), Sikdar and Gupta (2008), Haridass and Arumuganathan (2011), Haridass and Arumuganathan (2012a), Laxmi et al. (2013), Sikdar and Samanta (2016) etc. for continuous time setup, and Samanta et al. (2007a), Gupta et al. (2007) etc. for discrete time setup.

Sikdar and Gupta (2008), Sikdar and Samanta (2016) considered finite buffer continuous time bulk arrival and bulk service queue with single and multiple vacation and obtained queue length distribution at various epochs. Finite buffer discrete time bulk arrival bulk service queue with server vacation (single vacation and multiple vacation) has been studied by Samanta et al. (2007a). They obtained queue length distribution at various epochs. However, from their analysis one can not draw the information regarding the server content and hence their study can not be extended to batch size dependent bulk arrival bulk service queue with vacation. Recently, Banerjee et al. (2011) considered $M^X/G^Y/1/N$ queue and obtained joint distribution of the server content and queue content and then they extend their study to analyze batch size dependent bulk arrival bulk service queue in Banerjee and Gupta (2012).

In this chapter we have studied $M^X/G_r^{(a,b)}/1/N$ queue with single vacation (SV) and multiple vacation (MV) in an unified way. The service time of the batches are considered to be generally distributed and vary according to the batch size under service. The vacation rule must be fixed at the beginning of the analysis and is not allowed to change at intermediate stage. The vacation time is also considered to be generally distributed and it changes dynamically depending on the number of customers remaining in the queue at vacation initiation epoch. That is, at the end of a service if server finds less than 'a' customers in the queue, say 'k' $(0 \le k \le a-1)$, then the server leaves for a vacation of random length which is dependent on the number of customers present in the queue (i.e., k), at vacation initiation epoch and is referred as $k^{th}-$ type of vacation taken by the server. On returning from a vacation if the server finds 'a' or more customers are waiting in the queue it resumes service with maximum capacity of 'b' customers, otherwise, it remain idle or leave for another vacation depending on the vacation rule under consideration, i.e., SV or MV, respectively. The model is analyzed using the embedded Markov chain technique, and the joint distribution of queue content and serving batch size at service completion epoch, and queue content and vacation type taken by the server at vacation completion epoch are obtained. The inclusion of batch size dependent service along with queue length dependent vacation in bulk arrival bulk service queue makes the transition probability matrix of associated Markov chain more complex and challenging to handle. Next using the supplementary variable technique we have obtained a relation between service/vacation completion epoch and arbitrary epoch joint distributions of queue content and serving batch size, and queue content and vacation type. Since the buffer size is considered to be finite and arrivals are in batches of varying size, therefore, whenever buffer becomes full or insufficient buffer space is available in the queue to accept a new batch, the arrival batch will be lost fully or partially. We consider here the partial batch acceptance (or rejection) policy of the arrivals for optimizing the queuing performance.

The outline of the rest of this chapter is as follows: formal description of the model is described in Section 5.2. The joint distributions at service/vacation completion epoch, obtained by using the embedded Markov chain technique, is explained in Section 5.2.1. Next in Section 5.2.2, a relation between the joint distributions of service/vacation completion epoch and arbitrary epoch is established with the help of the supplementary variable technique. Section 5.2.3 is assigned for pre-arrival epoch joint probabilities obtained in terms of arbitrary epoch joint probabilities. Section 5.3 is assigned for the various performance measures. Numerical results and their discussion are presented in Section 5.4. Some conclusions are drawn in Section 5.5.

5.2 Model description

We consider a finite buffer single server bulk arrival and bulk service vacation queue. The customers are arriving to the system in batches of random size according to the compound Poisson process with rate λ. The arriving group size are independently identically distributed

random variables with probability mass function (pmf) $P(G=m)=g_m$, $m=1,2,\ldots$, probability generating function (pgf) $G(z)=\sum_{m=1}^{\infty} g_m z^m$ and mean $\bar{g}$. The bulk service rule is considered to be 'general bulk service' (GBS) rule with the minimum threshold limit 'a' and maximum threshold limit 'b'. For more detail on GBS rule readers are referred to the book by Chaudhry and Templeton (1972). The service time distribution ($S_r(.)$) of a batch of size r ($a \le r \le b$) is considered to be generally distributed with probability distribution function (pdf) $s_r(.)$, Laplace Stilzes transformation (LST) $s_r^*(.)$ and mean service time $\tilde{s_r}$.
After returning from each busy period the server investigate the queue length (k) and if found less than the minimum threshold limit a, then the server leave for a vacation of random length $V^{[k]}$, which depends on the queue length at vacation initiation epoch. We term this as kth type of vacation taken by the server or simply k-th type vacation for future reference. However, if the queue length is found to be greater than or equal to a, then the server will continue its service process in bathes of size r ($a \le r \le b$). Now after vacation completion if the server finds that the queue length (k) is still less than a, then the server will go for another kth type of vacation for the case of **multiple vacation** or stay in dormant till the queue length attains the value a and then resume service for the case of **single vacation.**

In this chapter we have studied SV and MV model in an unified way by defining an indicator δ_s as follows

$$\delta_s = \begin{cases} 1, & \text{for single vacation,} \\ 0, & \text{for multiple vacation.} \end{cases}$$

The vacation time distribution ($V^{[k]}(.)$) is considered to be generally distributed and is dependent on the queue length k ($0 \le k \le a-1$) at vacation initiation epoch, with pdf $v^{[k]}(.)$, LST $v^{[k]*}(.)$ and mean vacation time $\tilde{v}^{[k]}$. The finite buffer size is considered to be $N\,(>b)$.

5.2.1 Probability distribution at service/vacation completion epoch

In this section, we obtain (i) the joint distribution of queue content and server content at service completion epoch, and (ii) the joint distribution of the queue content and the type of the vacation taken by the server at vacation termination epoch. The embedded points are considered to be the service completion epoch points and the vacation completion epoch points and the corresponding steady state joint probabilities are defined as follows

- $p_{n,r}^{+}$ be the joint probability that there are n customers are in the queue at the service completion epoch of a batch of size r , $0 \le n \le N$, $a \le r \le b$,

- $q_n^{[k]+}$ be the joint probability that there are $n+k$ customers are present in the queue at k-th type vacation termination epoch of the server, $0 \le n+k \le N$, $n \ge 0$, $0 \le k \le a-1$,

- $p_n^{+}\left(=\sum_{r=a}^{b} p_{n,r}^{+}\right)$ be the marginal probability that n customers are in the queue at service completion epoch of a batch, $0 \le n \le N$,

- $q_n^{+}\left(=\sum_{k=0}^{min(n,a-1)} q_n^{[k]+}\right)$ be the marginal probability that n customers are in the queue at vacation termination epoch, $0 \le n \le N$.

The unknown quantities $p_{n,r}^{+}$ and $q_n^{[k]+}$ is obtained by solving the system of equations $\Pi\mathscr{P} = \Pi$, where

- $\Pi = (\tilde{\pi}, \tilde{\gamma}) = (\pi_0^{+}, \pi_1^{+}, ..., \pi_N^{+}, \gamma_0^{+}, \gamma_1^{+}, ..., \gamma_N^{+})$,

- $\tilde{\pi}$ and $\tilde{\gamma}$ are the row vectors, each of dimension $(N+1)$, and is defined as

$$\tilde{\pi} = \left(\pi_0^{+}, \pi_1^{+}, ..., \pi_N^{+}\right),\ \tilde{\gamma} = \left(\gamma_0^{+}, \gamma_1^{+}, ..., \gamma_N^{+}\right),$$

- each $\pi_n^{+}(0 \le n \le N)$ are the row vectors of dimension $(b-a+1)$ and is given by

$$\pi_n^{+} = (p_{n,a}^{+}, p_{n,a+1}^{+}, ..., p_{n,b}^{+}),$$

- each γ_n^{+} are the row vectors of dimension $(n+1)$ for $0 \le n \le a-2$, and of dimension a for $a-1 \le n \le N$, and is given by

$$\gamma_n^{+} \equiv \begin{cases} (q_n^{[0]+}, q_{n-1}^{[1]+}, ..., q_0^{[n]+}), & 0 \le n \le a-2, \\ (q_n^{[0]+}, q_{n-1}^{[1]+}, ..., q_{n-a+1}^{[a-1]+}), & a-1 \le n \le N. \end{cases}$$

- $\mathscr{P}$ is the one-step transition probability matrix (TPM) of dimension

$\left((N+1)(b-a+1)+\frac{a(a-1)}{2}+a(N-a+1)\right)$, and is given by

$$\mathscr{P}=\begin{pmatrix}\Phi & \Theta\\ \Lambda & \Psi\end{pmatrix},$$

where, Φ, Θ, Λ and Ψ are block matrices of dimension $(N+1)(b-a+1)\times(N+1)(b-a+1)$, $(N+1)(b-a+1)\times\left(\frac{a(a-1)}{2}+a(N-a+1)\right)$, $\left(\frac{a(a-1)}{2}+a(N-a+1)\right)\times(N+1)(b-a+1)$ and $\left(\frac{a(a-1)}{2}+a(N-a+1)\right)\times\left(\frac{a(a-1)}{2}+a(N-a+1)\right)$, respectively.

The block matrix Φ contains the transition probabilities among the service completion epochs and is given by

$$\Phi=\begin{array}{c} \\ 0\\ 1\\ \vdots\\ a-1\\ a\\ \vdots\\ b\\ b+1\\ \vdots\\ N\end{array}\begin{array}{c}\begin{array}{cccccccc}0 & 1 & \dots & N-b-1 & N-b & \dots & N-1 & N\end{array}\\ \begin{pmatrix}0 & 0 & \dots & 0 & 0 & \dots & 0 & 0\\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0\\ \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots\\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0\\ D_0^{(1)} & D_1^{(1)} & \dots & D_{N-b-1}^{(1)} & D_{N-b}^{(1)} & \dots & D_{N-1}^{(1)} & \bar{D}_N^{(1)}\\ \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots\\ D_0^{(b-a+1)} & D_1^{(b-a+1)} & \dots & D_{N-b-1}^{(b-a+1)} & D_{N-b}^{(b-a+1)} & \dots & D_{N-1}^{(b-a+1)} & \bar{D}_N^{(b-a+1)}\\ 0 & D_0^{(b-a+1)} & \dots & D_{N-b-2}^{(b-a+1)} & D_{N-b-1}^{(b-a+1)} & \dots & D_{N-2}^{(b-a+1)} & \bar{D}_{N-1}^{(b-a+1)}\\ \vdots & \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots\\ 0 & 0 & \dots & 0 & D_0^{(b-a+1)} & \dots & D_{b-1}^{(b-a+1)} & \bar{D}_b^{(b-a+1)}\end{pmatrix}\end{array}.$$

Each 0's and $D_j^{(i)}$ are the square matrices of dimension $(b-a+1)$ and are described as follows

$D_j^{(i)}=e_i^T\otimes\kappa_j^{(i+a-1)},\ 1\le i\le b-a+1,\ 0\le j\le N-1,$

$\bar{D}_N^{(i)}=e_i^T\otimes\kappa_N^{(i+a-1)}+e_{b-a+1}^T\otimes\bar{\kappa}_N^{(i+a-1)},\ 1\le i\le b-a,$

$\bar{D}_j^{(b-a+1)}=e_{b-a+1}^T\otimes\bar{\kappa}_{j-1}^{(b)},\ b\le j\le N.$

In the above expression

- e_i is the column vector of dimension $(b-a+1)$ with 1 at i^{th}-position and 0 elsewhere.

- $\kappa_j^{(r)}$ is the column vector of dimension $(b-a+1)$ consisting of $\xi_j^{(r)}$, where $\xi_j^{(r)}$ is the probability of j arrivals during the service period of a batch of size r and is given by

$$\xi_j^{(r)} = \int_0^\infty \sum_{m=1}^{j} \frac{e^{-\lambda t}(\lambda t)^j}{j!} g_j^{(m)*} dS_r(t), j \geq 0,\, a \leq r \leq b,$$

where $g_j^{(m)*}$ is m-fold convolution of g_j with itself. The corresponding pgf of $\xi_j^{(r)}$ is given by $W^{(r)}(z) = s_r^*(\lambda - \lambda G(z))$, $a \leq r \leq b$,

- $\bar{\kappa}_j^{(r)}$ is the column vector of dimension $(b-a+1)$ consisting of $\left(1-\sum_{i=0}^{j}\xi_i^{(r)}\right)$.

The block matrix Θ contains the transition probabilities from the service completion epoch to the vacation completion epoch and is given by

$$\Theta = \begin{array}{c} \\ 0 \\ 1 \\ \vdots \\ a-1 \\ a \\ \vdots \\ N \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & \dots & a-2 & a-1 & \dots & N-1 & N \end{array} \\ \begin{pmatrix} C_0^{(1)} & C_1^{(1)} & \dots & C_{a-2}^{(1)} & C_{a-1}^{(1)} & \dots & C_{N-1}^{(1)} & \bar{C}_N^{(1)} \\ 0 & C_0^{(2)} & \dots & C_{a-3}^{(2)} & C_{a-2}^{(2)} & \dots & C_{N-2}^{(2)} & \bar{C}_{N-1}^{(2)} \\ \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & C_0^{(a)} & \dots & C_{N-a}^{(a)} & \bar{C}_{N-a+1}^{(a)} \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \end{array}.$$

The $i-j$th element of Θ, i.e., $\Theta_{i,j}$ are also matrices and their dimension is described as follows

$$\Theta_{i,j} \equiv \begin{cases} \text{matrix of dimension } (i+1)\times(b-a+1), & 0 \leq i \leq a-2,\, 0 \leq j \leq N, \\ \text{matrix of dimension } a\times(b-a+1), & a-1 \leq i \leq N,\, 0 \leq j \leq N. \end{cases}$$

The zeros appeared in Θ are matrices of proper dimension and each $C_j^{(k)}$ are described as follows

$C_j^{(k)} = e_k^T \otimes \vartheta_j^{(k-1)}$, $1 \leq k \leq a$, $0 \leq j \leq N-1$,

$\bar{C}_j^{(k)} = e_k^T \otimes \vartheta_j^{(k-1)} + e_a^T \otimes \bar{\vartheta}_j^{(k-1)}$, $1 \leq k \leq a-1$, $N-a+2 \leq j \leq N$, $j+k = N+1$,

$\bar{C}_{N-a+1}^{(a)} = e_a^T \otimes \bar{\vartheta}_{N-a}^{(a-1)}$.

In above expression

- e_i is the column vector of with 1 in the i^{th}-position and 0 elsewhere, and its dimension is chosen in such a way that dimension of each $C_j^{(k)}$ is well defined.

- $\vartheta_j^{(k)}$ is the column vector of dimension $(b-a+1)$ consisting of $\omega_j^{(k)}$, where $\omega_j^{(k)}$ is the probability of j arrivals during the $k^{th}-$type vacation period and is given by

$$\omega_j^{(k)} = \int_0^\infty \sum_{m=1}^{j} \frac{e^{-\lambda t}(\lambda t)^j}{j!} g_j^{(m)*} dV^{[k]}(t), \quad j \geq 0,\ 0 \leq k \leq a-1.$$

  The corresponding pgf of $\omega_j^{(k)}$ is given by $M^{(k)}(z) = v^{[k]*}(\lambda - \lambda G(z)),\ 0 \leq k \leq a-1$.

- $\bar{\vartheta}_j^{(k)}$ is the column vector of dimension $(b-a+1)$ consisting of $\left(1 - \sum_{i=0}^{j} \omega_i^{(k)}\right)$.

The block matrix Λ contains the transition probabilities from the vacation termination epoch to the service completion epoch and is given by

$$\Lambda = \begin{array}{c} \\ 0 \\ 1 \\ \vdots \\ a-1 \\ a \\ \vdots \\ b \\ b+1 \\ \vdots \\ N \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & \dots & N-b-1 & N-b & \dots & N-1 & N \end{array} \\ \begin{pmatrix} \delta_s B_{0,0} & \delta_s B_{0,1} & \dots & \delta_s B_{0,N-b-1} & \delta_s B_{0,N-b} & \dots & \delta_s B_{0,N-1} & \delta_s \bar{B}_{0,N} \\ \delta_s B_{1,0} & \delta_s B_{1,1} & \dots & \delta_s B_{1,N-b-1} & \delta_s B_{1,N-b} & \dots & \delta_s B_{1,N-1} & \delta_s \bar{B}_{1,N} \\ \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \delta_s B_{a-1,0} & \delta_s B_{a-1,1} & \dots & \delta_s B_{a-1,N-b-1} & \delta_s B_{a-1,N-b} & \dots & \delta_s B_{a-1,N-1} & \delta_s \bar{B}_{a-1,N} \\ B_0^{(1)} & B_1^{(1)} & \dots & B_{N-b-1}^{(1)} & B_{N-b}^{(1)} & \dots & B_{N-1}^{(1)} & \bar{B}_N^{(1)} \\ \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots \\ B_0^{(b-a+1)} & B_1^{(b-a+1)} & \dots & B_{N-b-1}^{(b-a+1)} & B_{N-b}^{(b-a+1)} & \dots & B_{N-1}^{(b-a+1)} & \bar{B}_N^{(b-a+1)} \\ 0 & B_0^{(b-a+1)} & \dots & B_{N-b-2}^{(b-a+1)} & B_{N-b-1}^{(b-a+1)} & \dots & B_{N-2}^{(b-a+1)} & \bar{B}_{N-1}^{(b-a+1)} \\ \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & B_0^{(b-a+1)} & \dots & B_{b-1}^{(b-a+1)} & \bar{B}_b^{(b-a+1)} \end{pmatrix} \end{array}.$$

The $i-j$th element of Λ, i.e., $\Lambda_{i,j}$ are also matrices and their dimension is described as follows

$$\Lambda_{i,j} \equiv \begin{cases} \text{matrix of dimension } (i+1)\times(b-a+1), & 0\le i\le a-2,\ 0\le j\le N,\\ \text{matrix of dimension } a\times(b-a+1), & a-1\le i\le N,\ 0\le j\le N. \end{cases}$$

Let us define some notations which will be used in sequel. $y_0=0$, $y_j=\sum_{m=1}^{j} g_j^{(m)*}$, $j\ge 1$, For $0\le n\le a-1$, $\varphi_{n,r}=\sum_{k=0}^{a-n-1} g_{r-n-k}y_k$, $a-n\le r\le N-n-1$, and $\varphi_{n,N-n}=\sum_{k=0}^{a-n-1}\left(\sum_{i=N-k-n}^{\infty} g_i\right)y_k$.
Hence, $B_{n,j}$, as appeared in matrix Λ, are described as follows

$$B_{n,0}=\sum_{i=1}^{b-a+1}\varphi_{n,a-n+i-1}e_i^T\otimes\kappa_0^{(i+a-1)},\ 0\le n\le a-1,$$

$$B_{n,j}=\sum_{i=1}^{b-a+1}\varphi_{n,a-n+i-1}e_i^T\otimes\kappa_j^{(i+a-1)}+\sum_{i=1}^{j}\varphi_{n,b-n+i}e_{b-a+1}^T\otimes\kappa_{j-i}^{(b)},\ 0\le n\le a-1,\ 1\le j\le N-b-1,$$

$$B_{n,j}=\sum_{i=1}^{b-a+1}\varphi_{n,a-n+i-1}e_i^T\otimes\kappa_j^{(i+a-1)}+\sum_{i=1}^{N-b-1}\varphi_{n,b-n+i}e_{b-a+1}^T\otimes\kappa_{j-i}^{(b)}+\varphi_{n,N-n}e_{b-a+1}^T\otimes\kappa_{j-N+b}^{(b)},$$
$$0\le n\le a-1,\ N-b\le j\le N-1,$$

$$\bar{B}_{n,N}=\sum_{i=1}^{b-a}\varphi_{n,a-n+i-1}e_i^T\otimes\kappa_N^{(i+a-1)}+e_{b-a+1}^T\otimes\left(e[n]-\left(\sum_{j=0}^{N-1}B_{n,j}+\sum_{i=1}^{b-a}\varphi_{n,a-n+i-1}e_i^T\otimes\kappa_N^{(i+a-1)}\right)e[b-a]\right),$$
$$0\le n\le a-1,$$

and each $B_j^{(k)}$ are described as follows

$$B_j^{(i)}=e_i^T\otimes\kappa_j^{(i+a-1)},\ 1\le i\le b-a+1,\ 0\le j\le N-1,$$

$$\bar{B}_N^{(i)}=e_i^T\otimes\kappa_N^{(i+a-1)}+e_{b-a+1}^T\otimes\bar{\kappa}_N^{(i+a-1)},\ 1\le i\le b-a,$$

$$\bar{B}_j^{(b-a+1)}=e_{b-a+1}^T\otimes\bar{\kappa}_{j-1}^{(b)},\ b\le j\le N.$$

Where

- each $e[n]$ is the column vector of dimension $n+1$ consisting of 1 at all entries.
- each e_i is the column vector of dimension $(b-a+1)$ with 1 at i^{th}-position and 0 elsewhere.
- each $\kappa_j^{(r)}$ is the column vector, consisting of $\xi_j^{(r)}$, where $\xi_j^{(r)}$ is the probability of j arrivals during the service period of a batch of size r $(a\le r\le b)$ and is defined as previous. The dimension of $\kappa_j^{(r)}$ is taken in such a way that dimension of each $B_j^{(k)}$ and $B_{n,j}$ must be well defined.
- each $\bar{\kappa}_j^{(r)}$ is the column vector of appropriate dimension consisting of $\left(1-\sum_{i=0}^{j}\xi_i^{(r)}\right)$.

Ψ contains the transition probabilities among the vacation termination epochs and is given by

$$\Psi = \begin{array}{c} \\ 0 \\ 1 \\ \vdots \\ a-1 \\ a \\ \vdots \\ N \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & \dots & a-2 & a-1 & \dots & N-1 & N \end{array} \\ \begin{pmatrix} (1-\delta_s)A_0^{(1)} & (1-\delta_s)A_1^{(1)} & \dots & (1-\delta_s)A_{a-2}^{(1)} & (1-\delta_s)A_{a-1}^{(1)} & \dots & (1-\delta_s)A_{N-1}^{(1)} & (1-\delta_s)\bar{A}_N^{(1)} \\ 0 & (1-\delta_s)A_0^{(2)} & \dots & (1-\delta_s)A_{a-3}^{(2)} & (1-\delta_s)A_{a-2}^{(2)} & \dots & (1-\delta_s)A_{N-2}^{(2)} & (1-\delta_s)\bar{A}_{N-1}^{(2)} \\ \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & (1-\delta_s)A_0^{(a)} & \dots & (1-\delta_s)A_{N-a}^{(a)} & (1-\delta_s)\bar{A}_{N-a+1}^{(a)} \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \end{array}.$$

The $i-j$th element of Ψ, i.e., $\Psi_{i,j}$ are also matrices and their dimension is described as follows.

$$\Psi_{i,j} \equiv \begin{cases} \text{matrix of dimension } (i+1)\times(i+1), & 0 \le i,j \le a-2, \\ \text{matrix of dimension } a\times a, & a-1 \le i,j \le N, \end{cases}$$

and each $A_j^{(k)}$ are given in accordance of corresponding indexes where each $A_j^{(k)}$ are given as follows

$A_j^{(k)} = e_k^T \otimes \vartheta_j^{(k-1)}$, $1 \le k \le a$, $0 \le j \le N-1$,

$\bar{A}_j^{(k)} = e_k^T \otimes \vartheta_j^{(k-1)} + e_a^T \otimes \bar{\vartheta}_j^{(k-1)}$, $1 \le k \le a-1$, $N-a+2 \le j \le N$, $j+k = N+1$,

$\bar{A}_{N-a+1}^{(a)} = e_a^T \otimes \bar{\vartheta}_{N-a}^{(a-1)}$.

Where

- each e_i is the column vector of dimension i, $(0 \le i \le a-1)$ with 1 at i^{th}-position and 0 elsewhere and are defined in such a way that the dimension of each $A_j^{(k)}$ must be well defined.

- each $\vartheta_j^{(k)}$ is the column vector of dimension $i\,(1 \le i \le a)$, consisting of $\omega_j^{(k)}$, where $\omega_j^{(k)}$ is the probability of j arrivals during $k^{th}-$type vacation period and is defined as follows

$$\omega_j^{(k)} = \int_0^\infty \sum_{m=1}^{j} \frac{e^{-\lambda t}(\lambda t)^j}{j!} g_j^{(m)*} dV^{[k]}(t), \quad j \ge 0, 0 \le k \le a-1.$$

The dimension of $\vartheta_j^{(k)}$ is defined in such a way that each $A_j^{(k)}$ must be well defined.

- each $\bar{\vartheta}_j^{(k)}$ is the column vector of appropriate dimension consisting of $\left(1-\sum_{i=0}^{j}\omega_i^{(k)}\right)$.

Remark : According to Theorem 3.1 given in Abolnikov and Dukhovny (1991) every Markov chain whose TPM can be represented as a finite positive delta matrix is ergodic. Since the TPM $\mathscr{P}$ of the model considered in this chapter is of finite positive $\triangle_{m,n}$-type matrix, one can conclude that the corresponding Markov chain is ergodic which ensures the existence of steady state distribution.

5.2.2 Probability distribution at arbitrary epoch

In this section, we obtain the joint distribution of queue content and vacation type taken by the server when server is in vacation state, and the joint distribution of queue content and the serving batch size when server is in busy state, at arbitrary epoch. Towards this end, we define the corresponding stochastic processes as follows

- $N_q(t) \equiv$ the number of customers present in the queue, at time t,
- $N_s(t) \equiv$ the number of customers in service when server is busy, at time t,
- $\chi(t) \equiv$ the state of the server, at time t, and is defined as follows

$$\chi(t) = \begin{cases} 0, & \text{if server is in dormant state,} \\ k, & \text{if server is in k}^{\text{th}} - \text{type of vacation } (0 \le k \le a-1), \\ r, & \text{if server is busy in serving batch of size } r \ (a \le r \le b), \end{cases}$$

- $\widetilde{U}(t) \equiv$ the remaining service time of a batch of customers under service, at time t, if any.
- $\widetilde{V}(t) \equiv$ the remaining vacation time of the server, at time t, if any.

Let us define the following state probabilities, at time t, as follows.

- $P_{n,0}(t) \equiv prob.\{N_q(t)=n,\ \chi(t)=0\}, \quad 0 \le n \le a-1,$

- $P_{n,r}(x,t)dx \equiv prob.\{N_q(t)=n, N_s(t)=r, x \le \widetilde{U}(t) \le x+dx, \chi(t)=r\}, \quad 0 \le n \le N,$
  $a \le r \le b, x \ge 0,$

- $Q_n^{[k]}(x,t)dx \equiv prob.\{N_q(t)=n+k, , x \le \widetilde{V}(t) \le x+dx, \chi(t)=k\}, \quad 0 \le n+k \le N,$
  $n \ge 0, 0 \le k \le a-1, x \ge 0.$

Now relating the state of the system at time t and $t+dt$ we obtain the Kolmogrov equations of the model under consideration is obtained as follows

$$\frac{d}{dt}P_{0,0}(t) = -\delta_s\lambda P_{0,0}(t)+\delta_s Q_0^{[0]}(0,t), \tag{5.1}$$

$$\frac{d}{dt}P_{n,0}(t) = -\delta_s\lambda P_{n,0}(t)+\delta_s\lambda\sum_{i=1}^{n}g_iP_{n-i,0}(t)+\delta_s\sum_{k=0}^{n}Q_{n-k}^{[k]}(0,t), 1 \le n \le a-1, \tag{5.2}$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)P_{0,r}(x,t) = -\lambda P_{0,r}(x,t)+\delta_s\lambda\sum_{i=0}^{a-1}g_{r-i}P_{i,0}(t)s_r(x)+\sum_{k=0}^{a-1}Q_{r-k}^{[k]}(0,t)s_r(x)$$
$$+\sum_{k=a}^{b}P_{r,k}(0,t)s_r(x), \qquad a \le r \le b, \tag{5.3}$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)P_{n,r}(x,t) = -\lambda P_{n,r}(x,t)+\lambda\sum_{i=1}^{n}g_iP_{n-i,r}(x,t), a \le r \le b-1, 1 \le n \le N-1, \tag{5.4}$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)P_{n,b}(x,t) = -\lambda P_{n,b}(x,t)+\lambda\sum_{i=1}^{n}g_iP_{n-i,b}(x,t)+\delta_s\lambda\sum_{i=0}^{a-1}g_{n+b-i}P_{i,0}(t)s_b(x)$$
$$+\sum_{k=0}^{a-1}Q_{n+b-k}^{[k]}(0,t)s_b(x)+\sum_{r=a}^{b}P_{n+b,r}(0,t)s_b(x), 1 \le n \le N-b-1, \tag{5.5}$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)P_{N-b,b}(x,t) = -\lambda P_{N-b,b}(x,t)+\lambda\sum_{i=1}^{N-b}g_iP_{N-b-i,b}(x,t)+\delta_s\lambda\sum_{i=0}^{a-1}\left(\sum_{j=N-i}^{\infty}g_j\right)P_{i,0}(t)s_b(x)$$
$$+\sum_{k=0}^{a-1}Q_{N-k}^{[k]}(0,t)s_b(x)+\sum_{r=a}^{b}P_{N,r}(0,t)s_b(x), \tag{5.6}$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)P_{n,b}(x,t) = -\lambda P_{n,b}(x,t)+\lambda\sum_{i=1}^{n}g_iP_{n-i,b}(x,t), N-b+1 \le n \le N-1, \tag{5.7}$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)P_{N,r}(x,t) = \lambda\sum_{i=1}^{N}\left(\sum_{j=i}^{\infty}g_j\right)P_{N-i,r}(x,t), a \le r \le b, \tag{5.8}$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)Q_0^{[k]}(x,t) = -\lambda Q_0^{[k]}(x,t)+\Bigg(\sum_{r=a}^{b}P_{k,r}(0,t)+$$
$$(1-\delta_s)\sum_{j=0}^{k}Q_{k-j}^{[j]}(0,t)\Bigg)v^{[k]}(x), 0 \le k \le a-1, \tag{5.9}$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)Q_n^{[k]}(x,t) = -\lambda Q_n^{[k]}(x,t)+\lambda\sum_{i=1}^{n}g_iQ_{n-i}^{[k]}(x,t), 1 \le n \le N-1,$$
$$0 \le k \le min(a-1, N-n-1), \tag{5.10}$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)Q_{N-k}^{[k]}(x,t) = \lambda\sum_{i=1}^{N-k}\left(\sum_{j=i}^{\infty}g_j\right)Q_{N-k-i}^{[k]}(x,t), 0 \le k \le a-1. \tag{5.11}$$

In steady-state, as $t \to \infty$, we define

$$\begin{aligned}
\lim_{t\to\infty} P_{n,0}(t) &= P_{n,0},\ 0 \le n \le a-1,\\
\lim_{t\to\infty} P_{n,r}(x,t) &= P_{n,r}(x),\ 0 \le n \le N,\ a \le r \le b,\\
\lim_{t\to\infty} Q_n^{[k]}(x,t) &= Q_n^{[k]}(x),\ 0 \le n+k \le N,\ 0 \le k \le a-1,\ n \ge 0.
\end{aligned}$$

The corresponding steady state equations are obtained from equations (5.1)-(5.11) as follows

$$0 = -\delta_s \lambda P_{0,0} + \delta_s Q_0^{[0]}(0), \tag{5.12}$$

$$0 = -\delta_s \lambda P_{n,0} + \delta_s \lambda \sum_{i=1}^{n} g_i P_{n-i,0} + \delta_s \sum_{k=0}^{n} Q_{n-k}^{[k]}(0),\ 1 \le n \le a-1, \tag{5.13}$$

$$\begin{aligned}
-\frac{\partial}{\partial x} P_{0,r}(x) &= -\lambda P_{0,r}(x) + \delta_s \lambda \sum_{i=0}^{a-1} g_{r-i} P_{i,0} s_r(x) + \sum_{k=0}^{a-1} Q_{r-k}^{[k]}(0) s_r(x)\\
&\quad + \sum_{k=a}^{b} P_{r,k}(0) s_r(x), \qquad a \le r \le b,
\end{aligned} \tag{5.14}$$

$$-\frac{\partial}{\partial x} P_{n,r}(x) = -\lambda P_{n,r}(x) + \lambda \sum_{i=1}^{n} g_i P_{n-i,r}(x),\ a \le r \le b-1,\ 1 \le n \le N-1, \tag{5.15}$$

$$\begin{aligned}
-\frac{\partial}{\partial x} P_{n,b}(x) &= -\lambda P_{n,b}(x) + \lambda \sum_{i=1}^{n} g_i P_{n-i,b}(x) + \delta_s \lambda \sum_{i=0}^{a-1} g_{n+b-i} P_{i,0} s_b(x)\\
&\quad + \sum_{k=0}^{a-1} Q_{n+b-k}^{[k]}(0) s_b(x) + \sum_{r=a}^{b} P_{n+b,r}(0) s_b(x),\ 1 \le n \le N-b-1,
\end{aligned} \tag{5.16}$$

$$\begin{aligned}
-\frac{\partial}{\partial x} P_{N-b,b}(x) &= -\lambda P_{N-b,b}(x) + \lambda \sum_{i=1}^{N-b} g_i P_{N-b-i,b}(x) + \delta_s \lambda \sum_{i=0}^{a-1} \left(\sum_{j=N-i}^{\infty} g_j \right) P_{i,0} s_b(x)\\
&\quad + \sum_{k=0}^{a-1} Q_{N-k}^{[k]}(0) s_b(x) + \sum_{r=a}^{b} P_{N,r}(0) s_b(x),
\end{aligned} \tag{5.17}$$

$$-\frac{\partial}{\partial x} P_{n,b}(x) = -\lambda P_{n,b}(x) + \lambda \sum_{i=1}^{n} g_i P_{n-i,b}(x),\ N-b+1 \le n \le N-1, \tag{5.18}$$

$$-\frac{\partial}{\partial x} P_{N,r}(x) = \lambda \sum_{i=1}^{N} \left(\sum_{j=i}^{\infty} g_j \right) P_{N-i,r}(x),\ a \le r \le b, \tag{5.19}$$

$$-\frac{\partial}{\partial x} Q_0^{[k]}(x) = -\lambda Q_0^{[k]}(x) + \left(\sum_{r=a}^{b} P_{k,r}(0) + (1-\delta_s) \sum_{j=0}^{k} Q_{k-j}^{[j]}(0) \right) v^{[k]}(x),\ 0 \le k \le a-1, \tag{5.20}$$

$$\begin{aligned}
-\frac{\partial}{\partial x} Q_n^{[k]}(x) &= -\lambda Q_n^{[k]}(x) + \lambda \sum_{i=1}^{n} g_i Q_{n-i}^{[k]}(x),\ 1 \le n \le N-1,\\
&\quad 0 \le k \le min(a-1, N-n-1),
\end{aligned} \tag{5.21}$$

$$-\frac{\partial}{\partial x} Q_{N-k}^{[k]}(x) = \lambda \sum_{i=1}^{N-k} \left(\sum_{j=i}^{\infty} g_j \right) Q_{N-k-i}^{[k]}(x),\ 0 \le k \le a-1. \tag{5.22}$$

Further, let us define LST of few terms, for $Re\,\theta \geq 0$, as follows

$$\begin{aligned}
\int_0^\infty e^{-\theta x} P_{n,r}(x)dx &= P_{n,r}^*(\theta),\ 0 \leq n \leq N,\ a \leq r \leq b,\\
\int_0^\infty e^{-\theta x} Q_n^{[k]}(x)dx &= Q_n^{[k]*}(\theta),\ 0 \leq n+k \leq N,\ 0 \leq k \leq a-1,\\
\int_0^\infty e^{-\theta x} s_r(x)dx &= s_r^*(\theta),\ a \leq r \leq b,\\
\int_0^\infty e^{-\theta x} v^{[k]}(x)dx &= v^{[k]*}(\theta),\ 0 \leq k \leq a-1,
\end{aligned}$$

The following two results followed immediately from the above definitions.

$$\begin{aligned}
P_{n,r} \equiv P_{n,r}^*(0) &= \int_0^\infty P_{n,r}(x)dx,\\
Q_n^{[k]} \equiv Q_n^{[k]*}(0) &= \int_0^\infty Q_n^{[k]}(x)dx.
\end{aligned}$$

Multiplying (5.14)-(5.22) by $e^{-\theta x}$ and integrating with respect to x over 0 to ∞, we get

$$(\lambda-\theta)P_{0,r}^*(\theta) = \delta_s\lambda\sum_{i=0}^{a-1} g_{r-i}P_{i,0}s_r^*(\theta) + \sum_{k=0}^{a-1} Q_{r-k}^{[k]}(0)s_r^*(\theta) + \sum_{k=a}^{b} P_{r,k}(0)s_r^*(\theta) - P_{0,r}(0),\quad a+1 \leq r \leq b, \tag{5.23}$$

$$(\lambda-\theta)P_{n,r}^*(\theta) = \lambda\sum_{i=1}^{n} g_iP_{n-i,r}^*(\theta) - P_{n,r}(0),\ a \leq r \leq b-1,\ 1 \leq n \leq N-1, \tag{5.24}$$

$$(\lambda-\theta)P_{n,b}^*(\theta) = \lambda\sum_{i=1}^{n} g_iP_{n-i,b}^*(\theta) + \delta_s\lambda\sum_{i=0}^{a-1} g_{n+b-i}P_{i,0}s_b^*(\theta) + \sum_{k=0}^{a-1} Q_{n+b-k}^{[k]}(0)s_b^*(\theta) + \sum_{r=a}^{b} P_{n+b,r}(0)s_b^*(\theta) - P_{n,b}(0),\ 1 \leq n \leq N-b-1, \tag{5.25}$$

$$(\lambda-\theta)P_{N-b,b}^*(\theta) = \lambda\sum_{i=1}^{N-b} g_iP_{N-b-i,b}^*(\theta) + \delta_s\lambda\sum_{i=0}^{a-1}\left(\sum_{j=N-i}^{\infty} g_j\right)P_{i,0}s_b^*(\theta) + \sum_{k=0}^{a-1} Q_{N-k}^{[k]}(0)s_b^*(\theta) + \sum_{r=a}^{b} P_{N,r}(0)s_b^*(\theta) - P_{N-b,b}(0),\ 1 \leq n \leq N-b-1, \tag{5.26}$$

$$(\lambda-\theta)P_{n,b}^*(\theta) = \lambda\sum_{i=1}^{n} g_iP_{n-i,b}^*(\theta) - P_{n,b}(0),\ N-b+1 \leq n \leq N-1, \tag{5.27}$$

$$-\theta P_{N,r}^*(\theta) = \lambda\sum_{i=1}^{N}\left(\sum_{j=i}^{\infty} g_j\right)P_{N-i,r}^*(\theta) - P_{N,r}(0),\ a \leq r \leq b, \tag{5.28}$$

$$(\lambda-\theta)Q_0^{[k]*}(\theta) = \left(\sum_{r=a}^{b} P_{k,r}(0) + (1-\delta_s)\sum_{j=0}^{k} Q_{k-j}^{[j]}(0)\right)v^{[k]*}(\theta) - Q_0^{[k]}(0),\quad 0 \leq k \leq a-1, \tag{5.29}$$

$$(\lambda-\theta)Q_n^{[k]*}(\theta) = \lambda\sum_{i=1}^{n} g_iQ_{n-i}^{[k]*}(\theta) - Q_n^{[k]}(0),\quad 1 \leq n \leq N-1,\ 0 \leq k \leq min(a-1, N-n-1), \tag{5.30}$$

$$-\theta Q_{N-k}^{[k]*}(\theta) = \lambda \sum_{i=1}^{N-k} \left(\sum_{j=i}^{\infty} g_j \right) Q_{N-k-i}^{[k]*}(\theta) - Q_{N-k}^{[k]}(0) \,, 0 \le k \le a-1. \tag{5.31}$$

Now using (5.12)-(5.13) and (5.23)-(5.31), we derive the following three important results in Lemma 5.1 to Lemma 5.3 which will be used in sequel.

Lemma 5.1. The probabilities $\left(p_{n,r}^{+}, P_{n,r}(0)\right)$ and $\left(q_n^{[k]+}, Q_n^{[k]}(0)\right)$ are connected by the following relation

$$p_{n,r}^{+} = \sigma P_{n,r}(0),\, 0 \le n \le N,\, a \le r \le b, \tag{5.32}$$

$$q_n^{[k]+} = \sigma Q_n^{[k]}(0) \,, 0 \le n+k \le N,\, n \ge 0,\, 0 \le k \le a-1, \tag{5.33}$$

where, $\sigma^{-1} = \sum_{n=0}^{N} \sum_{r=a}^{b} P_{n,r}(0) + \sum_{n=0}^{a-1} \sum_{k=0}^{n} Q_{n-k}^{[k]}(0) + \sum_{n=a}^{N} \sum_{k=0}^{a-1} Q_{n-k}^{[k]}(0)$.

Proof. Using Bayes' theorem, for $0 \le n \le N$ and $a \le r \le b$, we have

$p_{n,r}^{+} =$ prob.$\{n$ customers are in the queue at the service completion epoch of a batch of size $r\}$

$=$ prob.$\{n$ customers are in the queue just prior to the service completion epoch of a batch of size $r \mid \le N$ customers are in the queue just prior to the service completion epoch of a batch of size $a \le r \le b$ or vacation completion epoch of k-th type vacation with $0 \le k \le a-1.\}$

$$= \frac{P_{n,r}(0)}{\sum_{n=0}^{N} \sum_{r=a}^{b} P_{n,r}(0) + \sum_{n=0}^{a-1} \sum_{k=0}^{n} Q_{n-k}^{[k]}(0) + \sum_{n=a}^{N} \sum_{k=0}^{a-1} Q_{n-k}^{[k]}(0)}.$$

With the similar argument we obtain

$$q_n^{[k]+} = \frac{Q_n^{[k]}(0)}{\sum_{n=0}^{N} \sum_{r=a}^{b} P_{n,r}(0) + \sum_{n=0}^{a-1} \sum_{k=0}^{n} Q_{n-k}^{[k]}(0) + \sum_{n=a}^{N} \sum_{k=0}^{a-1} Q_{n-k}^{[k]}(0)}, \quad 0 \le k \le a-1, 0 \le n \le N-k.$$

Lemma 5.2. The steady state dormancy period probabilities $P_{n,0}\,(0 \le n \le a-1)$ for the case of single vacation are given by

$$\lambda P_{n,0} = \sum_{m=0}^{n} \sum_{k=0}^{m} l_{n,m} Q_{m-k}^{[k]}(0) \,;\; 0 \le n \le a-1, \tag{5.34}$$

where

$l_{n,n} = 1\ (0 \le n \le a-1),\ l_{n,n-1} = g_1\ (1 \le n \le a-1),$

$l_{n,i} = \sum_{j=i+1}^{n-1} l_{n,j} g_{j-i} + g_{n-i} \quad (2 \le n \le a-1, 0 \le i \le n-2).$

Proof. Using (5.12) in (5.13) we obtain the desired result (5.34).

Lemma 5.3. The value of σ^{-1}, as appearing in Lemma 5.1, is given by

$$\sigma^{-1} = \sum_{n=0}^{N}\sum_{r=a}^{b} P_{n,r}(0) + \sum_{k=0}^{a-1}\sum_{n=0}^{N-k} Q_n^{[k]}(0) = \frac{1-\delta_s \sum_{n=0}^{a-1} P_{n,0}}{w}, \tag{5.35}$$

where, $w = \sum_{n=0}^{a-1}\left[p_n^+ \tilde{v}^{[n]} + \sum_{k=0}^{n} q_{n-k}^{[k]+}\left(\delta_s\left(\sum_{m=a}^{b} C_{m,n}\tilde{s}_m + \sum_{j=n}^{a-1} l_{j,n}\left(\sum_{k=b+1-j}^{\infty} g_k\right)\tilde{s}_b\right) + (1-\delta_s)\tilde{v}^{[n]}\right)\right]$
$+\sum_{n=a}^{b}\left(\sum_{k=0}^{a-1} q_{n-k}^{[k]+} + p_n^+\right)\tilde{s}_n + \sum_{n=b+1}^{N}\left(\sum_{k=0}^{a-1} q_{n-k}^{[k]+} + p_n^+\right)\tilde{s}_b,$

and $C_{m,n} = \begin{cases} g_{m-n}, & n = a-1, \\ \sum_{j=n+1}^{a-1} C_{m,j} g_{j-n} + g_{m-n}, & 0 \le n \le a-2, \end{cases} \quad a \le m \le b.$

Proof. Using (5.34) in (5.23), (5.25) and (5.26) and summing over the range of n and r we obtain

$$\begin{aligned}\left(\sum_{n=0}^{N}\sum_{r=a}^{b} P_{n,r}^*(\theta) + \sum_{k=0}^{a-1}\sum_{n=0}^{N-k} Q_n^{[k]*}(\theta)\right) &= \sum_{n=0}^{a-1}\left(\frac{1-v^{[n]*}(\theta)}{\theta}\sum_{r=a}^{b} P_{n,r}(0) + \frac{1-m(\theta)}{\theta}\sum_{k=0}^{n} Q_{n-k}^{[k]}(0)\right) \\ &+ \sum_{n=a}^{b}\frac{1-s_n^*(\theta)}{\theta}\left(\sum_{r=a}^{b} P_{n,r}(0) + \sum_{k=0}^{a-1} Q_{n-k}^{[k]}(0)\right) \\ &+ \frac{1-s_b^*(\theta)}{\theta}\sum_{n=b+1}^{N}\left(\sum_{r=a}^{b} P_{n,r}(0) + \sum_{k=0}^{a-1} Q_{n-k}^{[k]}(0)\right),\end{aligned} \tag{5.36}$$

where

$m(\theta) = \left[\delta_s\left(\sum_{m=a}^{b} C_{m,n} s_m^*(\theta) + \sum_{j=n}^{a-1} l_{j,n}\left(\sum_{k=b+1-j}^{\infty} g_k\right) s_b^*(\theta)\right) + (1-\delta_s) v^{[n]*}(\theta)\right].$

Taking limit as $\theta \to 0$ in above expression and using L'Hôpital's rule and the normalizing condition

$$\delta_s \sum_{n=0}^{a-1} P_{n,0} + \sum_{n=0}^{N}\sum_{r=a}^{b} P_{n,r} + \sum_{k=0}^{a-1}\sum_{n=0}^{N-k} Q_n^{[k]} = 1 \tag{5.37}$$

we obtain

$$1-\delta_s\sum_{n=0}^{a-1} P_{n,0} = \sum_{n=0}^{a-1}\left(\tilde{v}^{[n]}\sum_{r=a}^{b} P_{n,r}(0) + \left[\delta_s\left(\sum_{m=a}^{b} C_{m,n}\tilde{s}_m + \sum_{j=n}^{a-1} l_{j,n}\left(\sum_{k=b+1-j}^{\infty} g_k\right)\tilde{s}_b\right)\right.\right.$$

$$+(1-\delta_s)\tilde{v}^{[n]}\Bigg]\sum_{k=0}^{n}Q_{n-k}^{[k]}(0)\Bigg)+\sum_{n=a}^{b}\left(\sum_{r=a}^{b}P_{n,r}(0)+\sum_{k=0}^{a-1}Q_{n-k}^{[k]}(0)\right)\tilde{s_n}$$
$$+\sum_{n=b+1}^{N}\left(\sum_{r=a}^{b}P_{n,r}(0)+\sum_{k=0}^{a-1}Q_{n-k}^{[k]}(0)\right)\tilde{s_b}. \tag{5.38}$$

Now using Lemma 5.1 in (5.38), after little algebraic manipulation, we obtain the desired result (5.35).

Theorem 5.1. The steady state probabilities $\left\{P_{n,0},\, P_{n,r},\, Q_n^{[k]}\right\}$ and $\left\{p_n^+,\, p_{n,r}^+,\, q_n^{[k]+}\right\}$ are related by the following relations

$$P_{n,0} = E^{-1}\sum_{m=0}^{n} l_{n,m}\sum_{k=0}^{m} q_{m-k}^{[k]+},\ 0\le n\le a-1, \tag{5.39}$$

$$P_{0,r} = E^{-1}\left[\delta_s\sum_{i=0}^{a-1} g_{r-i}\sum_{m=0}^{i} l_{i,m}\sum_{k=0}^{m} q_{m-k}^{[k]+}+\sum_{k=0}^{a-1} q_{r-k}^{[k]+}+p_r^+-p_{0,r}^+\right],\ a\le r\le b, \tag{5.40}$$

$$P_{n,r} = \sum_{i=1}^{n} g_i P_{n-i,r}-E^{-1}p_{n,r}^+,\ 1\le n\le N-1,\ a\le r\le b-1, \tag{5.41}$$

$$P_{n,b} = \sum_{i=1}^{n} g_i P_{n-i,b}+\delta_s\sum_{i=0}^{a-1} g_{n+b-i}P_{i,0}+E^{-1}\left[\sum_{k=0}^{a-1} q_{n+b-k}^{[k]+}+p_{n+b}^+-p_{n,b}^+\right],$$
$$1\le n\le N-b-1, \tag{5.42}$$

$$P_{N-b,b} = \sum_{i=1}^{n} g_i P_{n-i,b}+\delta_s\sum_{i=0}^{a-1}\left(\sum_{j=N-i}^{\infty} g_j\right)P_{i,0}+E^{-1}\left[\sum_{k=0}^{a-1} q_{N-k}^{[k]+}+p_N^+-p_{N-b,b}^+\right], \tag{5.43}$$

$$P_{n,b} = \sum_{i=1}^{n} g_i P_{n-i,b}-E^{-1}p_{n,b}^+,\ N-b+1\le n\le N-1, \tag{5.44}$$

$$Q_0^{[k]} = E^{-1}\left[p_k^+ +(1-\delta_s)\sum_{j=0}^{k} q_{k-j}^{[j]+}-q_0^{[k]+}\right],\ 0\le k\le a-1, \tag{5.45}$$

$$Q_n^{[k]} = \sum_{i=1}^{n} g_i Q_{n-i}^{[k]}-E^{-1}q_n^{[k]+},\ 0\le k\le a-1,\ 1\le n\le N-a, \tag{5.46}$$

$$Q_{N-j}^{[k]} = \sum_{i=1}^{N-j} g_i Q_{N-j-i}^{[k]}-E^{-1}q_{N-j}^{[k]+},\ 1\le j\le a-1,\ 0\le k\le j-1, \tag{5.47}$$

where $E=\lambda w+\delta_s\Sigma_{n=0}^{a-1}\sum_{m=0}^{n} l_{n,m}\sum_{k=0}^{m} q_{m-k}^{[k]+}$.

Proof : Multiplying (5.12) by σ, using Lemma 5.1 and Lemma 5.2, after algebraic manipulation, we obtain

$$\lambda w P_{0,0} = \left(1-\sum_{n=0}^{a-1} P_{n,0}\right) q_0^{[0]+}. \tag{5.48}$$

With similar argument, from (5.34), we obtain

$$\lambda w P_{n,0} = \left(1 - \sum_{n=0}^{a-1} P_{n,0}\right) \sum_{m=0}^{n} l_{n,m} \sum_{k=0}^{m} q_{m-k}^{[k]+}, \; 0 \le n \le a-1. \tag{5.49}$$

Using (5.48) in (5.49) we obtain

$$P_{n,0} = \left(\frac{P_{0,0}}{q_0^{[0]+}}\right) \sum_{m=0}^{n} l_{n,m} \sum_{k=0}^{m} q_{m-k}^{[k]+}, \; 0 \le n \le a-1. \tag{5.50}$$

Using (5.50) in (5.48) yields

$$P_{0,0} = \frac{q_0^{[0]+}}{\lambda w + \sum_{n=0}^{a-1} \sum_{m=0}^{n} l_{n,m} \sum_{k=0}^{m} q_{m-k}^{[k]+}}. \tag{5.51}$$

Now using (5.51) in (5.49) yields the desired result (5.39).

Now setting $\theta = 0$ in (5.23)-(5.27) and (5.29)-(5.30) and then multiplying the equations by σ, using Lemma 5.1 we obtain

$$\lambda\sigma P_{0,r} = \delta_s \lambda\sigma \sum_{i=0}^{a-1} g_{r-i} \sum_{m=0}^{i} l_{i,m} \sum_{k=0}^{m} q_{m-k}^{[k]+} + \sum_{k=0}^{a-1} q_{r-k}^{[k]+} + p_r^+ - p_{0,r}^+, \; a \le r \le b, \tag{5.52}$$

$$\lambda\sigma P_{n,r} = \lambda\sigma \sum_{i=1}^{n} g_i P_{n-i,r} - p_{n,r}^+, \; 1 \le n \le N-1, a \le r \le b-1, \tag{5.53}$$

$$\begin{aligned}\lambda\sigma P_{n,b} &= \lambda\sigma \sum_{i=1}^{n} g_i P_{n-i,b} + \lambda\sigma\delta_s \sum_{i=0}^{a-1} g_{n+b-i} P_{i,0} + \sum_{k=0}^{a-1} q_{n+b-k}^{[k]+} \\ &\quad + p_{n+b}^+ - p_{n,b}^+, \qquad 1 \le n \le N-b-1,\end{aligned} \tag{5.54}$$

$$\lambda\sigma P_{N-b,b} = \lambda\sigma \sum_{i=1}^{n} g_i P_{n-i,b} + \lambda\sigma\delta_s \sum_{i=0}^{a-1} \left(\sum_{j=N-i}^{\infty} g_j\right) P_{i,0} + \sum_{k=0}^{a-1} q_{N-k}^{[k]+} + p_N^+ - p_{N-b,b}^+, \tag{5.55}$$

$$\lambda\sigma Q_0^{[k]} = \sum_{r=a}^{b} p_{k,r}^+ + (1-\delta_s) \sum_{j=0}^{k} q_{k-j}^{[j]+} - q_0^{[k]+}, \; 0 \le k \le a-1, \tag{5.56}$$

$$\lambda\sigma Q_n^{[k]} = \lambda\sigma \sum_{i=1}^{n} g_i Q_{n-i}^{[k]} - q_n^{[k]+}, \; 0 \le k \le a-1, 1 \le n \le N-a. \tag{5.57}$$

Then using Lemma 5.2 in (5.52)-(5.57) and solving recursively, after algebraic manipulations, we obtain the desired results (5.40)-(5.47).

Remark. It may be noted here that the probabilities $P_{N,r}(a \le r \le b)$ and $Q_{N-k}^{[k]}(0 \le k \le a-1)$ could not be obtained from Theorem 5.1 by using the normalizing condition given in (5.37). However, these probabilities is obtained using a slightly different approach as explained in next section.

5.2.2.1 Evaluation of $P_{N,r}\,(a \le r \le b)$

To obtain $P_{N,r}\,(a \le r \le b)$ we use equation (5.28). Differentiating (5.28) with respect to θ and then setting $\theta = 0$ we get

$$P_{N,r} = -\lambda \sum_{i=1}^{N} \left(\sum_{k=i}^{\infty} g_k \right) P^{*(1)}_{N-i,r}(0)\,,\; a \le r \le b, \tag{5.58}$$

where $P^{*(1)}_{N-i,r}(0)$ is the derivative of $P^{*}_{N-i,r}(\theta)$ with respect to θ at $\theta = 0$. Now to get $P^{*(1)}_{N-i,r}(0)$, differentiate (5.24)-(5.27) with respect to θ and set $\theta = 0$. Hence, we obtain

$$\lambda P^{*(1)}_{n,r}(0) = P_{n,r} + \lambda \sum_{i=1}^{n} g_i P^{*(1)}_{n-i,r}(0)\,,\; a \le r \le b-1\,, 1 \le n \le N-1, \tag{5.59}$$

$$\begin{aligned} \lambda P^{*(1)}_{n,b}(0) &= P_{n,b} + \lambda \sum_{i=1}^{n} g_i P^{*(1)}_{n-i,b}(0) - \lambda \delta_s \sum_{i=0}^{a-1} g_{n+b-i} P_{i,0} \tilde{s_b} \\ &\quad - \sum_{k=0}^{a-1} Q^{[k]}_{n+b-k}(0) \tilde{s_b} - \sum_{r=a}^{b} P_{n+b,r}(0) \tilde{s_b}\,, 1 \le n \le N-b-1, \end{aligned} \tag{5.60}$$

$$\begin{aligned} \lambda P^{*(1)}_{N-b,b}(0) &= P_{N-b,b} + \lambda \sum_{i=1}^{N-b} g_i P^{*(1)}_{N-b-i,b}(0) - \lambda \delta_s \sum_{i=0}^{a-1} \left(\sum_{j=N-i}^{\infty} g_j \right) P_{i,0} \tilde{s_b} \\ &\quad - \sum_{k=0}^{a-1} Q^{[k]}_{N-k}(0) \tilde{s_b} - \sum_{r=a}^{b} P_{N,r}(0) \tilde{s_b}, \end{aligned} \tag{5.61}$$

$$\lambda P^{*(1)}_{n,b}(0) = P_{n,b} + \lambda \sum_{i=1}^{n} g_i P^{*(1)}_{n-i,b}(0)\,,\; N-b+1 \le n \le N-1. \tag{5.62}$$

From (5.59)-(5.62) one can obtain the values of $P^{*(1)}_{N-i,r}(0)\;(a \le r \le b)$ recursively in terms of $P^{*(1)}_{0,r}(0)\;(a \le r \le b)$. Hence, $P_{N,r}\;(a \le r \le b)$ we obtained in terms of $P^{*(1)}_{0,r}(0)\;(a \le r \le b)$ from (5.58). Now to obtain $P^{*(1)}_{0,r}(0)\;(a \le r \le b)$ we differentiate (5.23) with respect to θ and then setting $\theta = 0$ we get

$$\lambda P^{*(1)}_{0,r}(0) = P_{0,r} - \lambda \delta_s \sum_{i=0}^{a-1} g_{r-i} P_{i,0} \tilde{s_r} - \sum_{k=0}^{a-1} Q^{[k]}_{r-k}(0) \tilde{s_r} - \sum_{k=a}^{b} P_{r,k}(0) \tilde{s_r}\,,\; a \le r \le b. \tag{5.63}$$

As $P^{*(1)}_{0,r}(0)\;(a \le r \le b)$ is known from (5.63), $P_{N,r}\;(a \le r \le b)$ can be obtained from (5.58) in known terms.

5.2.2.2 Evaluation of $Q_{N-k}^{[k]}\,(0\leq k\leq a-1)$

Differentiating (5.31) with respect to θ and then setting $\theta=0$ we get

$$Q_{N-k}^{[k]}=-\lambda\sum_{i=1}^{N-k}\left(\sum_{n=i}^{\infty}g_n\right)Q_{N-k-i}^{[k]*(1)}(0)\,,\;0\leq k\leq a-1, \tag{5.64}$$

where $Q_{N-k-i}^{[k]*(1)}(0)$ is the derivative of $Q_{N-k-i}^{[k]*}(\theta)$ with respect to θ at $\theta=0$. Now to get $Q_{N-k-i}^{[k]*(1)}(0)$ we differentiate (5.30) with respect to θ and set $\theta=0$ which yields the following expression

$$\lambda Q_n^{[k]*(1)}(0)\;=\;Q_n^{[k]}+\lambda\sum_{i=1}^{n}g_iQ_{n-i}^{[k]*(1)}(0)\,,\;1\leq n\leq N-1,\;0\leq k\leq min(a-1,N-n-1). \tag{5.65}$$

From (5.65) we obtain the values of $Q_{N-k-i}^{[k]*(1)}(0)\,(0\leq k\leq a-1)$ recursively in terms of $Q_0^{[k]*(1)}(0)\,(0\leq k\leq a-1)$ and hence, $Q_{N-k}^{[k]}\,(0\leq k\leq a-1)$ is known in terms of $Q_0^{[k]*(1)}(0)\,(0\leq k\leq a-1)$ from (5.64). Now to obtain $Q_0^{[k]*(1)}(0)\,(0\leq k\leq a-1)$ differentiate (5.29) with respect to θ and then setting $\theta=0$ we get

$$\lambda Q_0^{[k]*(1)}(0)\;=\;Q_0^{[k]}-\left(\sum_{r=a}^{b}P_{k,r}(0)+(1-\delta_s)\sum_{j=0}^{k}Q_{k-j}^{[j]}(0)\right)\tilde{v}^{[k]}\,,\;0\leq k\leq a-1. \tag{5.66}$$

Now $Q_{N-k}^{[k]}\,(0\leq k\leq a-1)$ is obtained from (5.64) in known terms.

Henceforth, we have completely obtained the joint distributions of the queue content and server content, joint distribution of queue content and type of the vacation taken by the server. Next we obtain some significant marginal probability distributions as follows

- the distribution of queue content $p_n^{queue}\,(0\leq n\leq N)$ is given by

$$p_n^{queue}=\begin{cases}\delta_sP_{n,0}+\sum\limits_{r=a}^{b}P_{n,r}+\sum\limits_{k=0}^{n}Q_{n-k}^{[k]}, & 0\leq n\leq a-1,\\ \sum\limits_{r=a}^{b}P_{n,r}+\sum\limits_{k=0}^{a-1}Q_{n-k}^{[k]}, & a\leq n\leq N.\end{cases}$$

- the distribution of the system content (including number of customers with the server)

p_n^{sys} $(0 \le n \le N+b)$ is given by

$$p_n^{sys} = \begin{cases} \delta_s P_{n,0} + \sum_{k=0}^{n} Q_{n-k}^{[k]}, & 0 \le n \le a-1, \\ \sum_{r=a}^{min(b,n)} P_{n-r,r} + \sum_{k=0}^{a-1} Q_{n-k}^{[k]}, & a \le n \le N, \\ \sum_{r=a}^{b} P_{n-r,r}, & N+1 \le n \le N+a, \\ \sum_{r=n-N}^{b} P_{n-r,r}, & N+a+1 \le n \le N+b. \end{cases}$$

- the probability that the server is in dormant state is given by $P_{dor} = \sum_{n=0}^{a-1} P_{n,0}$, probability that the server is busy is given by $P_{busy} = \sum_{r=a}^{b} \sum_{n=0}^{N} P_{n,r}$, probability that the server is in vacation state is given by $Q_{vac} = \sum_{k=0}^{a-1} \sum_{n=0}^{N-k} Q_n^{[k]}$.
- the conditional probability distribution that the server is in $k^{th}-$type vacation given that the server is in vacation state is given by

$$\zeta_k = \sum_{n=0}^{N-k} Q_n^{[k]} / Q_{vac}, \quad 0 \le k \le a-1.$$

- the conditional probability distribution of the server content given that the server is busy is given by

$$p_r^{ser} = \sum_{n=0}^{N} P_{n,r} / P_{busy}, \; a \le r \le b.$$

- the probability of the number of customers in the queue when server is busy is given by $p_n^{busy} = \sum_{r=a}^{b} P_{n,r}$, $0 \le n \le N$.
- the probability of the number of customers in the queue when server is in vacation is given by

$$q_n^{[vac]} = \begin{cases} \sum_{k=0}^{n} Q_{n-k}^{[k]}, & 0 \le n \le a-1, \\ \sum_{k=0}^{a-1} Q_{n-k}^{[k]}, & a \le n \le N. \end{cases}$$

5.2.3 Probability distribution at pre-arrival epoch of a random customer

In this section, we obtain the joint distribution of queue content and server content, joint distribution of queue content and type of the vacation taken by the server at pre-arrival epoch of a random customer in terms of arbitrary epoch joint distributions.

Without loss of generality, we assume that the customers within each arriving batch are randomly ordered and that they enter the system according to their order. Let us define

$P^-_{n,0} \equiv$ the probability that a random customer of an arriving batch finds n customers in the queue and server is idle $(0 \le n \le N)$ (for SV)

$P^-_{n,r} \equiv$ the probability that a random customer of an arriving batch finds n customers in the queue and server is busy in serving a batch of size r, $0 \le n \le N$, $a \le r \le b$.

$Q_n^{[k]-} \equiv$ the probability that a random customer of an arriving batch finds $n+k$ customers in the queue and server is in k-th type of vacation , $0 \le n \le N$, $0 \le k \le a-1$.

Thus the joint probabilities $\{P^-_{n,0}$, $P^-_{n,r}$, $Q_n^{[k]-}\}$ and $\{P_{n,0}$, $P_{n,r}$, $Q_n^{[k]}\}$ are related by

$$
\begin{aligned}
P^-_{n,0} &= \delta_s \sum_{j=0}^{n} g^-_{n-j} P_{j,0},\ 0 \le n \le a-1, \\
P^-_{n,0} &= \delta_s \sum_{j=0}^{a-1} g^-_{n-j} P_{j,0},\ a \le n \le N-1, \\
P^-_{N,0} &= \delta_s \sum_{j=0}^{a-1} \sum_{k=N-j}^{\infty} g^-_k P_{j,0}, \\
P^-_{n,r} &= \sum_{j=0}^{n} g^-_{n-j} P_{j,r},\ 0 \le n \le N-1,\ a \le r \le b, \\
P^-_{N,r} &= \sum_{j=0}^{N} \sum_{k=N-j}^{\infty} g^-_k P_{j,r},\ a \le r \le b, \\
Q_n^{[k]-} &= \sum_{j=0}^{n} g^-_{n-j} Q_j^{[k]},\ 0 \le k \le a-1,\ 0 \le n \le N-k-1, \\
Q_n^{[k]-} &= \sum_{j=0}^{n} \sum_{m=n-j}^{\infty} g^-_m Q_j^{[k]},\ 0 \le k \le a-1,\ n = N-k,
\end{aligned}
$$

where, $g^-_j = \dfrac{1}{\bar{g}} \sum_{i=j+1}^{\infty} g_i$, $j \ge 0$, is the probability of j customers are ahead of an arbitrary customer in his batch.

Once we get the joint distribution of the number of customers in the queue and number with the server at arrival-epoch of a random customer, the other distributions such as distribution of the number of customers in the queue when server is busy at arrival-epoch of a random customer and distribution of the number of customers with the server at arrival epoch of a random customer, can be easily obtained.

5.3 Performance measure

As all the state probabilities are known, the performance measures of the present model are evaluated and presented as follows.

(i) Average queue length is given by $(L_q) = \sum_{n=0}^{N} np_n^{queue}$.

(ii) Average system length is given by $(L) = \sum_{n=0}^{N+b} np_n^{sys}$.

(iii) Average queue length when server is in vacation is given by $(L_q^{vac}) = \sum_{n=0}^{N} n.q_n^{[vac]}/Q_{vac}$.

(iv) Average number of customers with the server is given by $(L_s) = \sum_{r=a}^{b} rp_r^{ser}$.

(v) Average vacation type (average number of customer in the queue at vacation initiation epoch) is given by $(\zeta) = \sum_{k=0}^{a-1} k\zeta_k$.

Next we obtain the blocking probabilities of the first customer, an arbitrary customer and the last customer of an arriving batch as follows.

Blocking probability of the first customer in a batch

Let P_{BF} be the probability that the first customer in a batch (and therefore the whole batch) is being lost upon arrival. The first customer is being lost if there is no waiting space, i.e., there have been N customers in the queue upon arrival. Hence, $P_{BF} = p_N^{queue}$.

Blocking probability of an arbitrary customer in a batch

Let P_{BA} be the probability that an arbitrary customer in a batch is being lost upon arrival. An arbitrary customer in a batch is being lost if he finds n $(0 \le n \le N)$ customers in the queue upon arrival and k $(\ge N-n)$ customers ahead of him in his batch. Hence, we have $P_{BA} = \sum_{n=0}^{N} \sum_{k=N-n}^{\infty} g_k^- p_n^{queue}$.

Blocking probability of the last customer in a batch

Let P_{BL} be the probability that the last customer in a batch is being lost upon arrival. The last customer in a batch is being lost if he finds n $(0 \leq n \leq N)$ customers in the queue upon arrival and his batch size is k $(\geq N-n+1)$. Hence, we have $P_{BL} = \sum_{n=0}^{N} \sum_{k=N-n+1}^{\infty} g_k p_n^{queue}$.

Finally, using Little's law the average waiting time of an arbitrary customer in the queue (W_q) as well as the system (W) is given by $(W_q) = L_q/\bar{\lambda}$ and $(W) = L/\bar{\lambda}$, respectively, where $\bar{\lambda}$ is the effective arrival rate of the system and is given by $\bar{\lambda} = \lambda \bar{g}(1-P_{BA})$.

5.4 Numerical results

This section consists of four fold numerical investigations which extracts several important features of the considered model. In the first part, the joint probabilities at various epoch, viz., service/vacation completion epoch, pre-arrival epoch and arbitrary epoch, are presented in tabular form along with useful performance measures. In the second part, we perform the sensitive analysis of the performance measures by varying the queue capacity N. In the third part, we have presented the effect of 'a' (lower threshold limit of the serving capacity of the server) on the selected performance measures by fixing the range of the serving capacity of the server, i.e., $b-a$. A cost function is defined in the fourth part, named as total system cost (TSC), in terms of the decision variables a, b and N to discuss several optimal control policy of the considered model.

Here below we are defining few abbreviation which are used in sequel.

SV : single vacation

MV : multiple vacation

STD : service time distribution

VTD : vacation time distribution

IP : inversely proportional

DP : directly proportional

Let us now focus ourselves to the first part of this section. Here, we have considered $M/G_r^{(4,7)}/1/18$ queue with SV (Table 5.2 to Table 5.4) and MV (Table 5.5 to Table 5.7) with

E_4 STD and deterministic VTD. The arrival rate is taken as $\lambda = 1.0$. The service rates are taken as $\mu_r = \mu/r$ $(a \leq r \leq b)$ with $\mu = 7.5$ and the vacation rates are taken as $\nu_k = (k+1)\nu$ $(0 \leq k \leq a-1)$ with $\nu = 0.75$. The input parameter for the service rates and the vacation rates for Table 5.2- Table 5.7 are taken as presented in Table 5.1.

Table 5.1: Service and vacation rates for Table 5.2-Table 5.7

Service rate		Vacation rate	
batch size (r)	IP	queue length (k)	DP
4	1.875	0	0.75
5	1.50	1	1.5
6	1.25	2	2.25
7	1.0714	3	3.0

Table 5.2-Table 5.4 present the service/vacation completion epoch joint probabilities, arbitrary epoch joint probabilities and pre-arrival epoch joint probabilities, respectively, for SV and Table 5.5-Table 5.7 present the same distributions for MV. These results are presented here to show the numerical compatibility of our analytical results. The important performance measures of the queueing model under consideration are also presented at the bottom of the Table 5.3 and Table 5.6.

Let us now turn our attention to some other numerical examples of this section presented in the form of graphs. Towards this end, we consider the input parameters as given in Set I to Set III.

Set I : $a = 3,\ b = 7,\ \lambda = 0.9,\ g_2 = 0.15, g_3 = 0.20, g_4 = 0.25, g_5 = 0.10, g_6 = 0.05, g_7 = 0.05, g_8 = 0.10, g_9 = 0.05, g_{10} = 0.05,\ \mu_r = \mu + \frac{1}{r-1}$ with $\mu = 0.5$, $\nu_k = \nu - \frac{1}{k+0.5}$ with $\nu = 3.0$, deterministic STD and exponential VTD , varying N from 10 to 200.

Set II : $b = a+4$, $N = 40$ and all other input parameters are same as given in Set I.

Set III : $a = 5,\ b = 8,\ \lambda = 0.7,\ g_3 = 0.10, g_4 = 0.20, g_5 = 0.25, g_6 = 0.25, g_7 = 0.10, g_8 = 0.05, g_9 = 0.05,\ \mu_r = \mu/r$ with $\mu = 5.5$, $\nu_k = (\frac{k}{4}+1)\nu$ with $\nu = 0.75$, E_4 STD and deterministic VTD, varying N from 20 to 150.

The input parameters presented in Set I are used for Figure 5.1- Figure 5.4, Set II are used

Table 5.2: Joint distributions at service/vacation completion epoch for $M^X/G_r^{(4,7)}/1/18$ queue with SV, E_4 STD, deterministic VTD, $\lambda = 1.0$ and $g_i = 0.1\,(1 \le i \le 10)$.

n	$p_{n,4}^{+}$	$p_{n,5}^{+}$	$p_{n,6}^{+}$	$p_{n,7}^{+}$	p_{n}^{+}	$q_{n}^{[0]+}$	$q_{n-1}^{[1]+}$	$q_{n-2}^{[2]+}$	$q_{n-3}^{[3]+}$	q_{n}^{+}
0	0.044872	0.030205	0.026939	0.023850	0.125865	0.033178				0.033178
1	0.002112	0.001726	0.001796	0.026056	0.031689	0.004424	0.016270			0.020694
2	0.002174	0.001788	0.001871	0.028627	0.034459	0.004719	0.001085	0.022095		0.027898
3	0.002237	0.001851	0.001948	0.031356	0.037393	0.005027	0.001121	0.000982	0.026793	0.033923
4	0.002302	0.001916	0.002028	0.045999	0.052246	0.005348	0.001158	0.001004	0.000893	0.008403
5	0.002369	0.001983	0.002110	0.027301	0.033764	0.005684	0.001196	0.001026	0.000908	0.008813
6	0.002437	0.002052	0.002196	0.026555	0.033240	0.006034	0.001234	0.001048	0.000923	0.009239
7	0.002507	0.002123	0.002284	0.025207	0.032121	0.006399	0.001274	0.001071	0.000938	0.009682
8	0.002578	0.002196	0.002375	0.025441	0.032590	0.006779	0.001314	0.001094	0.000954	0.010141
9	0.002651	0.002271	0.002469	0.026228	0.033619	0.007175	0.001355	0.001118	0.000969	0.010617
10	0.002726	0.002348	0.002566	0.027034	0.034673	0.007588	0.001397	0.001142	0.000985	0.011112
11	0.000691	0.000701	0.000870	0.068416	0.070678	0.003594	0.001440	0.001166	0.001001	0.007201
12	0.000644	0.000659	0.000823	0.021313	0.023440	0.003451	0.000399	0.001190	0.001017	0.006058
13	0.000596	0.000614	0.000773	0.020202	0.022185	0.003287	0.000372	0.000233	0.001033	0.004925
14	0.000544	0.000566	0.000718	0.018884	0.020713	0.003101	0.000343	0.000215	0.000156	0.003815
15	0.000490	0.000515	0.000658	0.016454	0.018118	0.002890	0.000312	0.000196	0.000143	0.003542
16	0.000433	0.000460	0.000594	0.015414	0.016902	0.002655	0.000280	0.000176	0.000130	0.003242
17	0.000373	0.000402	0.000525	0.014433	0.015733	0.002394	0.000247	0.000156	0.000116	0.002913
18	0.000311	0.000340	0.000450	0.100067	0.101167	0.002105	0.000212	0.000135	0.011562	0.014013

Table 5.3: Joint distributions at arbitrary epoch for $M^X/G_r^{(4,7)}/1/18$ queue with SV, E_4 STD, deterministic VTD, $\lambda = 1.0$ and $g_i = 0.1\,(1 \le i \le 10)$
.

n	$P_{n,0}$	$P_{n,4}$	$P_{n,5}$	$P_{n,6}$	$P_{n,7}$	p_n^{busy}	$Q_n^{[0]}$	$Q_{n-1}^{[1]}$	$Q_{n-2}^{[2]}$	$Q_{n-3}^{[3]}$	$q_n^{[vac]}$	p_n^{queue}
0	0.032598	0.028648	0.025303	0.028416	0.030786	0.113155	0.091069				0.091069	0.236822
1	0.023592	0.000790	0.000835	0.001077	0.032609	0.035311	0.004760	0.015150			0.019911	0.078814
2	0.033030	0.000808	0.000857	0.001111	0.034823	0.037600	0.004947	0.000449	0.012149		0.017545	0.088175
3	0.042252	0.000826	0.000881	0.001146	0.037146	0.040000	0.005139	0.000459	0.000250	0.010415	0.016262	0.098514
4		0.000845	0.000905	0.001183	0.054746	0.057679	0.005337	0.000468	0.000254	0.000164	0.006223	0.063902
5		0.000864	0.000929	0.001220	0.028698	0.031711	0.005541	0.000478	0.000257	0.000166	0.006442	0.038153
6		0.000884	0.000955	0.001258	0.026652	0.029749	0.005751	0.000488	0.000261	0.000168	0.006667	0.036416
7		0.000904	0.000980	0.001297	0.023879	0.027060	0.005967	0.000498	0.000265	0.000169	0.006899	0.033959
8		0.000924	0.001007	0.001338	0.023219	0.026487	0.006190	0.000508	0.000268	0.000171	0.007138	0.033625
9		0.000945	0.001034	0.001379	0.023277	0.026635	0.006420	0.000518	0.000272	0.000173	0.007383	0.034018
10		0.000966	0.001062	0.001422	0.023341	0.026790	0.006657	0.000529	0.000276	0.000175	0.007636	0.034426
11		0.000197	0.000256	0.000389	0.076786	0.077628	0.002140	0.000540	0.000280	0.000177	0.003136	0.080764
12		0.000183	0.000239	0.000365	0.014316	0.015103	0.002018	0.000101	0.000284	0.000179	0.002581	0.017684
13		0.000168	0.000221	0.000340	0.013356	0.014086	0.001886	0.000093	0.000037	0.000180	0.002197	0.016283
14		0.000153	0.000202	0.000314	0.012273	0.012942	0.001744	0.000085	0.000034	0.000018	0.001882	0.014824
15		0.000137	0.000183	0.000285	0.010413	0.011018	0.001591	0.000077	0.000031	0.000017	0.001716	0.012734
16		0.000120	0.000162	0.000255	0.009606	0.010143	0.001427	0.000068	0.000028	0.000015	0.001538	0.011681
17		0.000103	0.000140	0.000223	0.008866	0.009331	0.001252	0.000059	0.000024	0.000013	0.001349	0.01068
18		0.000328	0.000503	0.000890	0.051438	0.053160	0.005054	0.000188	0.000079	0.000047	0.005368	0.058528
Total	0.131472 (P_{dor})	0.038793	0.036654	0.043908	0.53623	0.655588 (P_{busy})	0.16489	0.020756	0.015049	0.012247	0.212942 (Q_{vac})	1.000000

$L = 9.861022$, $W = 2.005536$, $L_q = 5.505508$, $W_q = 1.119711$, $P_{BF} = 0.0585277$, $P_{BA} = 0.106018$, $P_{BL} = 0.1450246$,
$L_s = 6.643675$, $\zeta = 0.411344$, $L_q^{vac} = 3.472227$.

Table 5.4: Joint distributions at pre-arrival epoch for $M^X/G_r^{(4,7)}/1/18$ queue with SV, E_4 STD, deterministic VTD, $\lambda = 1.0$ and $g_i = 0.1\,(1 \le i \le 10)$
.

n	$P^-_{n,0}$	$P^-_{n,4}$	$P^-_{n,5}$	$P^-_{n,6}$	$P^-_{n,7}$	P^-_n	$Q^{[0]-}_n$	$Q^{[1]-}_{n-1}$	$Q^{[2]-}_{n-2}$	$Q^{[3]-}_{n-3}$	Q^-_n
0	0.005927	0.005209	0.004601	0.005167	0.005598	0.026501	0.016558				0.016558
1	0.009624	0.004832	0.004292	0.004846	0.010967	0.034560	0.015768	0.002755			0.018522
2	0.014607	0.004443	0.003973	0.004512	0.016146	0.043681	0.014925	0.002561	0.002209		0.019694
3	0.020667	0.004044	0.003642	0.004164	0.021114	0.053630	0.014027	0.002361	0.002033	0.001894	0.020314
4	0.018277	0.003632	0.003300	0.003801	0.028606	0.057617	0.013071	0.002154	0.001854	0.001734	0.018813
5	0.015887	0.003209	0.002946	0.003424	0.030367	0.055833	0.012056	0.001940	0.001671	0.001572	0.017239
6	0.013496	0.002774	0.002579	0.003032	0.031235	0.053116	0.010978	0.001720	0.001484	0.001407	0.015588
7	0.011106	0.002326	0.002200	0.002624	0.031114	0.049369	0.009835	0.001492	0.001292	0.001239	0.013859
8	0.008715	0.001865	0.001807	0.002200	0.030438	0.045026	0.008624	0.001257	0.001097	0.001069	0.012047
9	0.006325	0.001392	0.001402	0.001759	0.029351	0.040229	0.007342	0.001015	0.000897	0.000896	0.010150
10	0.003935	0.000905	0.000982	0.001300	0.027853	0.034975	0.005987	0.000766	0.000693	0.000720	0.008165
11	0.002137	0.000781	0.000857	0.001145	0.036207	0.041127	0.005345	0.000509	0.000485	0.000541	0.006879
12	0.000768	0.000666	0.000739	0.000998	0.032399	0.035571	0.004728	0.000437	0.000272	0.000359	0.005797
13	0.000000	0.000560	0.000630	0.000860	0.028790	0.030840	0.004141	0.000371	0.000230	0.000175	0.004917
14	0.000000	0.000463	0.000528	0.000732	0.025417	0.027139	0.003587	0.000310	0.000192	0.000147	0.004235
15	0.000000	0.000375	0.000436	0.000614	0.022477	0.023903	0.003071	0.000254	0.000157	0.000121	0.003603
16	0.000000	0.000298	0.000354	0.000508	0.019724	0.020883	0.002596	0.000204	0.000126	0.000098	0.003024
17	0.000000	0.000231	0.000282	0.000414	0.017145	0.018073	0.002169	0.000160	0.000098	0.000077	0.002503
18	0.000000	0.000789	0.001102	0.001808	0.091286	0.094986	0.010083	0.000494	0.000258	0.000197	0.011032

Table 5.5: Joint distributions at service/vacation completion epoch for $M^X/G_r^{(4,7)}/1/18$ queue with MV, E_4 STD, deterministic VTD, $\lambda = 1.0$ and $g_i = 0.1\,(1 \le i \le 10)$.

n	$p^+_{n,4}$	$p^+_{n,5}$	$p^+_{n,6}$	$p^+_{n,7}$	p^+_n	$q^{[0]+}_n$	$q^{[1]+}_{n-1}$	$q^{[2]+}_{n-2}$	$q^{[3]+}_{n-3}$	q^+_n
0	0.033004	0.021176	0.019182	0.017247	0.090609	0.032434				0.032434
1	0.001553	0.001210	0.001279	0.018937	0.022979	0.004324	0.028810			0.033134
2	0.001599	0.001253	0.001332	0.020931	0.025115	0.004613	0.001921	0.056553		0.063086
3	0.001646	0.001298	0.001387	0.023044	0.027374	0.004914	0.001985	0.002513	0.092986	0.102398
4	0.001693	0.001343	0.001444	0.037022	0.041503	0.005228	0.002050	0.002569	0.003100	0.012947
5	0.001742	0.001390	0.001503	0.021145	0.025781	0.005556	0.002117	0.002626	0.003151	0.013450
6	0.001793	0.001439	0.001563	0.021010	0.025804	0.005898	0.002185	0.002684	0.003203	0.013971
7	0.001844	0.001489	0.001626	0.020439	0.025397	0.006255	0.002255	0.002742	0.003256	0.014509
8	0.001896	0.001540	0.001691	0.020606	0.025733	0.006627	0.002327	0.002801	0.003310	0.015064
9	0.001950	0.001592	0.001758	0.021241	0.026540	0.007014	0.002400	0.002861	0.003364	0.015639
10	0.002005	0.001646	0.001827	0.021872	0.027350	0.007418	0.002474	0.002922	0.003418	0.016232
11	0.000508	0.000492	0.000619	0.058030	0.059649	0.003513	0.002550	0.002984	0.003473	0.012521
12	0.000474	0.000462	0.000586	0.017164	0.018687	0.003374	0.000707	0.003047	0.003529	0.010657
13	0.000438	0.000431	0.000550	0.016424	0.017844	0.003213	0.000658	0.000597	0.003586	0.008054
14	0.000400	0.000397	0.000511	0.015527	0.016835	0.003031	0.000607	0.000550	0.000543	0.004731
15	0.000361	0.000361	0.000469	0.013571	0.014761	0.002826	0.000553	0.000501	0.000497	0.004377
16	0.000319	0.000322	0.000423	0.012797	0.013861	0.002596	0.000497	0.000451	0.000450	0.003994
17	0.000275	0.000282	0.000374	0.012035	0.012965	0.002340	0.000437	0.000399	0.000403	0.003579
18	0.000228	0.000238	0.000321	0.083295	0.084082	0.002058	0.000375	0.000345	0.013575	0.016353

Table 5.6: Joint distributions at arbitrary epoch for $M^X/G_r^{(4,7)}/1/18$ queue with MV, E_4 STD, deterministic VTD, $\lambda = 1.0$ and $g_i = 0.1\,(1 \le i \le 10)$
.

n	$P_{n,4}$	$P_{n,5}$	$P_{n,6}$	$P_{n,7}$	p_n^{busy}	$Q_n^{[0]}$	$Q_{n-1}^{[1]}$	$Q_{n-2}^{[2]}$	$Q_{n-3}^{[3]}$	$q_n^{[vac]}$	p_n^{queue}
0	0.026507	0.022316	0.025454	0.028006	0.102284	0.111993				0.111993	0.214277
1	0.000731	0.000736	0.000965	0.029819	0.032251	0.005854	0.033748			0.039602	0.071853
2	0.000748	0.000756	0.000995	0.032046	0.034545	0.006083	0.001001	0.039118		0.046202	0.080747
3	0.000765	0.000777	0.001027	0.034373	0.036941	0.006319	0.001022	0.000805	0.045468	0.053615	0.090556
4	0.000782	0.000798	0.001059	0.055867	0.058506	0.006563	0.001043	0.000817	0.000716	0.009138	0.067645
5	0.000800	0.000820	0.001093	0.028144	0.030856	0.006814	0.001065	0.000828	0.000724	0.009430	0.040286
6	0.000818	0.000842	0.001127	0.026868	0.029654	0.007072	0.001087	0.000840	0.000731	0.009730	0.039384
7	0.000836	0.000865	0.001162	0.024906	0.027769	0.007338	0.001109	0.000852	0.000739	0.010038	0.037807
8	0.000855	0.000888	0.001198	0.024189	0.027130	0.007613	0.001132	0.000864	0.000747	0.010355	0.037485
9	0.000874	0.000912	0.001235	0.024237	0.027258	0.007895	0.001155	0.000876	0.000755	0.010681	0.037939
10	0.000893	0.000936	0.001274	0.024260	0.027363	0.008186	0.001178	0.000888	0.000763	0.011015	0.038378
11	0.000182	0.000225	0.000348	0.082884	0.083640	0.002631	0.001202	0.000900	0.000771	0.005505	0.089145
12	0.000169	0.000211	0.000327	0.014562	0.015269	0.002482	0.000225	0.000913	0.000779	0.004399	0.019668
13	0.000156	0.000195	0.000305	0.013728	0.014384	0.002320	0.000208	0.000120	0.000788	0.003435	0.017819
14	0.000142	0.000179	0.000281	0.012773	0.013374	0.002145	0.000190	0.000110	0.000080	0.002525	0.015899
15	0.000127	0.000161	0.000255	0.010881	0.011425	0.001957	0.000171	0.000100	0.000073	0.002301	0.013726
16	0.000111	0.000143	0.000228	0.010112	0.010594	0.001755	0.000152	0.000089	0.000066	0.002062	0.012657
17	0.000095	0.000123	0.000199	0.009378	0.009795	0.001540	0.000132	0.000078	0.000059	0.001808	0.011604
18	0.000303	0.000444	0.000797	0.054486	0.056030	0.006215	0.000419	0.000254	0.000207	0.007095	0.063125
Total	0.035894	0.032327	0.039330	0.541519	0.649069 (P_{busy})	0.202775	0.046238	0.048452	0.053466	0.350931 (Q_{vac})	1.000000

$L = 10.247680$, $W = 2.105104$, $L_q = 5.915862$, $W_q = 1.215251$, $P_{BF} = 0.063125$, $P_{BA} = 0.114906$, $P_{BL} = 0.157832$, $L_s = 6.673894$, $\zeta = 0.864959$, $L_q^{vac} = 3.458383$.

Table 5.7: Joint distributions at pre-arrival epoch for $M^X/G_r^{(4,7)}/1/18$ queue with MV, E_4 STD, deterministic VTD, $\lambda = 1.0$ and $g_i = 0.1\,(1 \le i \le 10)$
.

n	$P^-_{n,4}$	$P^-_{n,5}$	$P^-_{n,6}$	$P^-_{n,7}$	P^-_n	$Q^{[0]-}_n$	$Q^{[1]-}_{n-1}$	$Q^{[2]-}_{n-2}$	$Q^{[3]-}_{n-3}$	Q^-_n
0	0.004820	0.004058	0.004628	0.005092	0.018597	0.020362				0.020362
1	0.004470	0.003786	0.004341	0.010005	0.022601	0.019391	0.006136			0.025526
2	0.004111	0.003504	0.004041	0.014780	0.026436	0.018354	0.005704	0.007112		0.031171
3	0.003741	0.003212	0.003729	0.019395	0.030078	0.017250	0.005258	0.006547	0.008267	0.037322
4	0.003361	0.002910	0.003405	0.027294	0.036970	0.016075	0.004798	0.005970	0.007570	0.034413
5	0.002969	0.002598	0.003067	0.029136	0.037771	0.014826	0.004322	0.005380	0.006862	0.031390
6	0.002566	0.002275	0.002716	0.030235	0.037792	0.013500	0.003831	0.004777	0.006142	0.028250
7	0.002152	0.001940	0.002350	0.030488	0.036931	0.012095	0.003324	0.004161	0.005411	0.024990
8	0.001726	0.001594	0.001970	0.030158	0.035449	0.010605	0.002801	0.003531	0.004667	0.021605
9	0.001288	0.001236	0.001575	0.029398	0.033497	0.009029	0.002262	0.002888	0.003911	0.018090
10	0.000837	0.000866	0.001165	0.028200	0.031069	0.007362	0.001706	0.002231	0.003143	0.014442
11	0.000723	0.000756	0.001026	0.037730	0.040234	0.006573	0.001133	0.001561	0.002362	0.011628
12	0.000617	0.000652	0.000894	0.033872	0.036035	0.005814	0.000974	0.000876	0.001569	0.009234
13	0.000518	0.000555	0.000770	0.030181	0.032025	0.005092	0.000826	0.000742	0.000763	0.007424
14	0.000428	0.000466	0.000655	0.026692	0.028242	0.004411	0.000689	0.000619	0.000641	0.006360
15	0.000347	0.000385	0.000550	0.023642	0.024924	0.003776	0.000565	0.000506	0.000530	0.005377
16	0.000275	0.000312	0.000455	0.020767	0.021809	0.003193	0.000454	0.000405	0.000428	0.004479
17	0.000214	0.000249	0.000371	0.018062	0.018896	0.002667	0.000355	0.000315	0.000338	0.003675
18	0.000730	0.000972	0.001620	0.096391	0.099713	0.012400	0.001100	0.000831	0.000862	0.015193

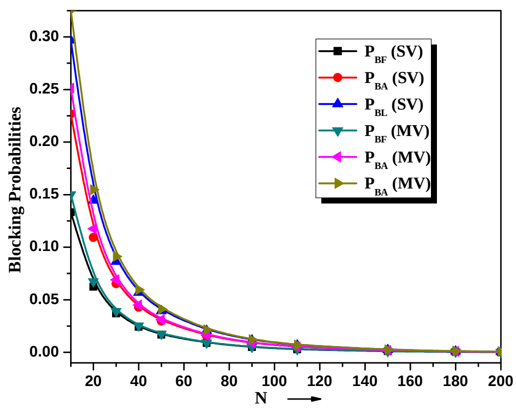

Figure 5.1: Effect of N on P_{Block}.

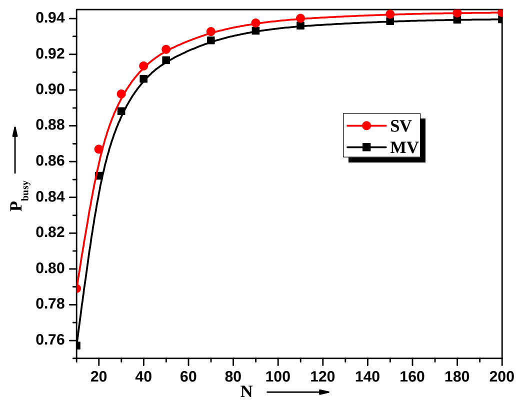

Figure 5.2: Effect of N on P_{busy}.

in Figure 5.5- Figure 5.7 and Set III are used in Figure 5.8. Figure 5.1- Figure 5.4 illustrate the effect of buffer size N on blocking probabilities (P_{BF}, P_{BA}, P_{BL}), P_{busy}, Q_{vac}, L_q and W. It is noticed that the values of blocking probabilities, Q_{vac}, L_q and W for SV is lower for SV than the corresponding values for MV. However, the values of P_{busy} is higher for SV than the corresponding values for MV, irrespective of the values of N. These behavior are according to our intuitive expectations, because of the increase in chance of server's availability to the system for SV model in comparison to MV model. Further, it is observed from Figure 5.1- Figure 5.4 that enlarging the buffer size reduces the blocking probabilities and Q_{vac}, however, increases P_{busy}, L_q and W. It is also observed that for large N (i.e., $N > 110$) these performance measures become insensitive, which implies the fact that for larger values of N our finite buffer model behaves like an infinite buffer queueing system. Figure 5.1- Figure 5.4 reveal an interesting observation that for $N > 60$ the rate of change of various measures is much lower.

Figure 5.5- Figure 5.6 illustrate the effect of minimum threshold limit a on L_q, W and blocking probabilities (P_{BF}, P_{BA}, P_{BL}) by considering fixed service capacity (i.e., $b - a = 4$). Similar observation as seen in Figure 5.1- Figure 5.4 is observed in Figure 5.5- Figure 5.6. The values of L_q, W and blocking probabilities (P_{BF}, P_{BA}, P_{BL}) corresponding to SV are lesser in compare to values correspond to MV. It is observed from Figure 5.5 that for $a < 10$ the values of L_q and W decreases rapidly, however, for $a > 10$ these values increases rapidly for both the considered model, i.e., SV and MV. The analogous behavior is noted for blocking probabilities (P_{BF}, P_{BA}, P_{BL}); when $a < 14$ in the virtue of MV and $a < 22$ in the virtue of SV.

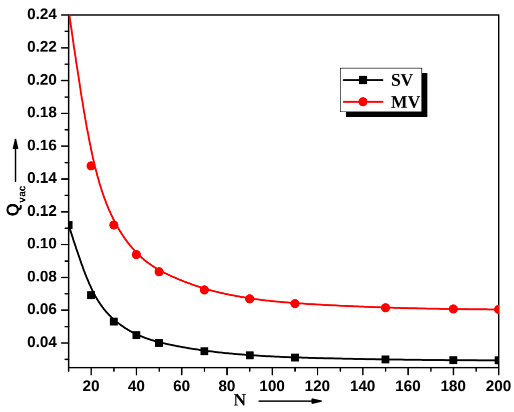

Figure 5.3: Effect of N on Q_{vac}.

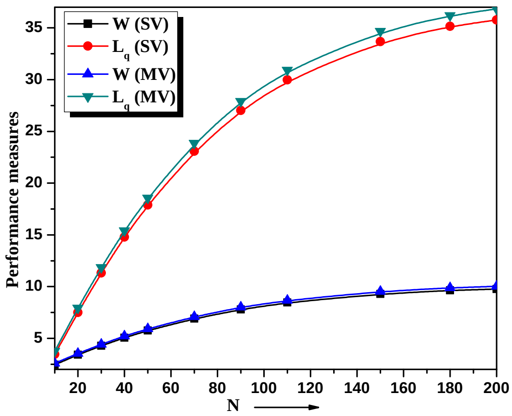

Figure 5.4: Effect of N on Performance measures.

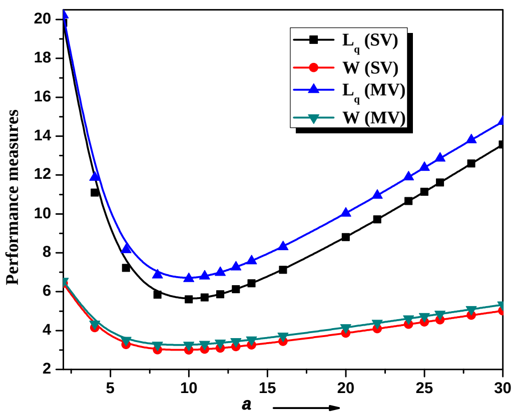

Figure 5.5: Effect of a on Performance measures.

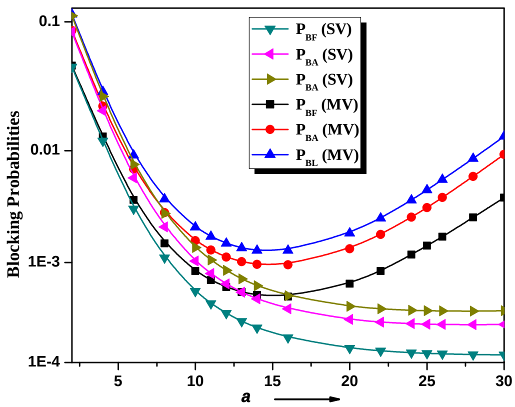

Figure 5.6: Effect of a on P_{Block}.

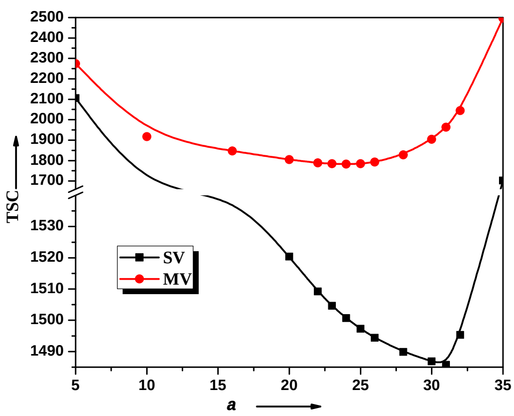

Figure 5.7: Effect of a on TSC.

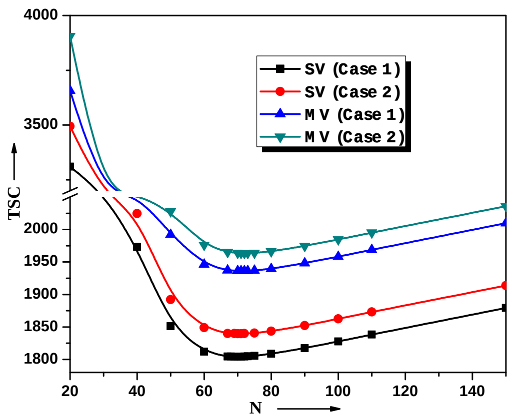

Figure 5.8: Effect of N on TSC for Case 1 and Case 2.

In the following section we carry out a numerical investigation to optimize the values of a, b and N which may eventually minimize the total system cost. A comparative study of the TSC for queue length dependent vacation with queue length independent vacation is also presented. The purpose of such comparison is to see whether queue length dependent vacation is leaving any effect on the total system cost or not.

5.4.1 Optimal control problem

In this section, we develop the expression for total system cost (TSC) per unit time, which is a function of the decision variables a, b and N. Our goal is to find optimal values of the control parameters a, b and N which may minimize TSC function. Towards this end, we define the following state dependent costs incurred at various stages as follows.

- $C_h(n)$ be the unit time cost for holding n customers in the queue. Thus in long run the holding cost is given by $\sum_{n=1}^{N} C_h(n) p_n^{queue}$.
- C_r be the unit time service cost of a batch of size r. Thus in long run the total service cost is given by $\sum_{r=a}^{b} C_r \sum_{n=0}^{N} P_{n,r}$.
- C_k be the unit time cost when server is in $k-$th type of vacation (cost occurs in terms of penalty). Thus in long run the total penalty incurred is given by $\sum_{k=0}^{a-1} C_k \sum_{n=0}^{N-k} Q_n^{[k]}$.
- In case of SV, C_d be the fixed operating cost per unit time while system is dormant. Thus the associated long run operating cost when server is dormant is $C_d P_{dor}$.
- C_l be the fixed cost for each lost order while there is no space to take more order, i.e., queue capacity is full. Thus in long run the lost cost is $C_l \lambda^* P_{BA}$, where λ^* is the mean arrival rate of customers given by $\lambda^* = \lambda \bar{g}$.
- $C_s(N)$ is per unit time cost for holding the buffer size N (this cost penalizes when the system capacity is expanded in excess).

Therefore, in long run, the total system cost defined as

$$TSC(a,b,N) = \sum_{n=1}^{N} C_h(n) p_n^{queue} + \sum_{r=a}^{b} C_r \sum_{n=0}^{N} P_{n,r} + \sum_{k=0}^{a-1} C_k \sum_{n=0}^{N-k} Q_n^{[k]} + \delta_s C_d P_{dor} + C_l \lambda^* P_{BA} + C_s(N). \tag{5.67}$$

Hence, for each selection of any two fixed parameter among a, b and N, the optimal value of remaining parameter can be determined by satisfying certain inequality, e.g., for optimal value of N (denote by N^*) we must have a relation of the following form

$$TSC(a,b,N^*-1) \geq TSC(a,b,N^*) \leq TSC(a,b,N^*+1).$$

Here we consider the following two examples :

Example 1. Let us considered the following different cost parameters

1.	*Unit time cost for holding queue capacity of N, $C_s(N)$*	$12N^{3/5}$ *unit*
2.	*Holding cost per unit time for n customers, $C_h(n)$*	$5n+12n^{3/5}$ *unit*
3.	*Serving cost per unit time of a batch of size r, C_r*	$1500+10r$ *unit*
4.	*Penalty cost incurred per unit time when server is in $k-th$ type of vacation, C_k*	$2000-50k$ *unit*
5.	*Operating cost per unit time while system is dormant, C_d*	*1000 unit*
6.	*Cost for each lost customer, C_l*	*8000 unit*

Let us suppose that the range of server capacity is fixed at 4. Now we want to find out the optimal values of the threshold limits, a and b for which TSC is minimized. For this purpose we have considered the above cost structure and the input parameter as presented in Set *II*. In Figure 5.7, we have presented the effect of a on TSC, which reveals that the optimal value of a is 24 for MV while it is 31 for SV. Hence, the optimal batch size for MV is (24, 28) and that for SV (31, 35) and the corresponding min TSC are 1782.439 unit and 1485.736 unit for MV and SV respectively.

Next we have considered another example to find out the optimal value of N for fixed values a and b, with the above cost parameters as presented in *Example 1.*

Example 2. We have considered the costs parameter as presented in *Example 1.* The input parameters are taken as given in Set III. For the purpose of comparative study we consider the following two cases.

Case 1. The queue length dependent vacation rates are taken as $\nu_k = (\frac{k}{4}+1)\nu$ $(0 \le k \le a-1)$.

Case 2. The queue length independent vacation rate are taken as $\nu_k = \nu_0$ $(0 \le k \le a-1)$.

Remark : For Case 2, i.e., queue length independent vacation, the cost for server's vacation (cost occurs in terms of penalty) is considered to be C_0 per unit time, thus in long run the total penalty incurred due to vacation is given by $C_0 \sum_{k=0}^{a-1} \sum_{n=0}^{N-k} Q_n^{[k]}$, and required modification in TSC is also done.

The assumptions for the vacation rates for Case 1 and Case 2 are made in such a way that for Case 2 the server always takes a vacation with constant rate ν_0, irrespective of the queue length at vacation initiation epoch, and for Case 1 the server will start a vacation with rate ν_0, when it finds an empty queue and start a vacation with higher rate (i.e., $\nu_k > \nu_{k-1}$, $k = 1, 2, \ldots a-1$) depending on queue length. These assumptions ensure us that due to queue length dependent vacation (Case 1) the server is modulating the length of the vacation periods in such a way that the server is taking a longer vacation for empty queue and shorter vacation as queue length increases. In Figure 5.8 we have presented the effect of N on TSC for Case 1 and Case 2. Figure 5.8 reveals the fact (in terms of TSC) that queue length dependent vacation is much effective over queue length independent vacation as it minimizes TSC for Case 1. It is also observed from Figure 5.8 that for SV the optimal value of N is 70 and 71 under Case 1 and Case 2, respectively and the corresponding TSC are 1804.161 units and 1839.74 units respectively. For MV model it is 71 for Case 1 and 72 for Case 2 and the corresponding TSC are 1936.125 units and 1963.572 units respectively. Further it is noticed that $(TSC)_{SV} < (TSC)_{MV}$, which is as of our intuitive expectation.

5.5 Concluding remarks

In this chapter, we perform analysis of a finite buffer bulk arrival bulk service vacation queueing system with queue length dependent vacation (single vacation and multiple vacation). The service time, which depends on the size of the batches under service, is considered to be generally distributed and the vacation time, which depends on the queue length at vacation initiation epoch, is also considered to be generally distributed. We present a procedure to obtain steady state joint distribution of number of customers in the queue and number of customers with the server, joint distribution of number of customers in the queue and vacation type taken by the server, at various epochs, e.g., pre-arrival, arbitrary and service/vacation completion epochs. Several illustrative numerical results are presented to show the impact of the various parameters on system performance measures. We have considered an example of an optimal control problem which yields the fact that total system cost is minimal when queue length dependent vacation is considered along with batch size dependent service in bulk arrival bulk service queue with vacation. The analysis presented in this chapter can be extended to analyze more complex queuing models involving correlated arrival processes, i.e, batch Markovian arrival process $(BMAP)$.