Chapter 4

Analysis of $M/G_r^{(a,b)}/1/N$ queue with queue length dependent single and multiple vacation

4.1 Introduction

In vacation queueing models server periodically becomes unavailable from the service center for a random period of time, termed as vacation period of the server. In such systems, server's dormancy period can be used in an optimized manner so that the system become more applicable to a variety of real world stochastic systems. There are several vacation rules are available in the literature (Doshi (1986), Takagi (1988, 1991), Tian and Zhang (2006) and the references therein), viz., single vacation, multiple vacation, Bernoulli vacation, working vacation etc. In most of the research on vacation queues, it has been considered that the server will go for a vacation of random length which is independent of the queue length at the vacation initiation epoch. The vacation queueing models in which the vacation period is modulated depending upon the queue length at vacation initiation epoch is termed as queue length dependent vacation queue (Shin and Pearce (1998), Banik (2013a) and references therein). Finite/infinite buffer $M/G/1$ queue with vacation (single and/or multiple) has been studied by Courtois (1980), Lee (1984), Frey and Takahashi (1997), etc. Bulk service queues with single vacation have been studied by Lee et al. (1992), Gupta and Sikdar (2004a); Sikdar and Gupta (2005a), etc. For recent development on vacation queues one may refer to Banik (2013a), Panda et al. (2016), Sikdar and Samanta (2016) and the references therein. However, in none of the above literature, batch size dependent bulk service queue with queue length dependent single and multiple vacation have been considered.

In this chapter we have studied $M/G_r^{(a,b)}/1/N$ queue with single vacation (SV) and multiple vacation (MV) in an unified way. The service time of the batches vary according to the batch size under service. The vacation rule must be fixed at the beginning of the analysis and is not allowed to change at intermediate stage. We have also considered the vacation time of random length which changes dynamically depending on the number of customers remaining in the queue at vacation initiation epoch. That is, when the server finishes serving a batch and finds less than 'a' customers in the queue, say 'k' $(0 \leq k \leq a-1)$, then the server leaves for a vacation of random length which is considered to be dependent on the number of customers remaining in the queue (i.e., k) at vacation initiation epoch, and is referred as $k^{th}-$typeofvacation taken by the server. On returning from a vacation if the server finds 'a' or more customers waiting in the queue it resumes service with maximum of 'b' customers, otherwise, it will remain dormant or leave for another vacation depending on the vacation rule under consideration, i.e., single vacation or multiple vacation respectively. The model is analyzed using the embedded Markov chain technique, and the joint distribution of queue content and serving batch size, and queue content and vacation type taken by the server at service and vacation completion epoch, respectively, are obtained. The inclusion of batch size dependent service in GBS rule along with queue length dependent vacation makes the transition probability matrix of associated Markov chain more complex and challenging to handle. As a result one can see in Section 4.2.1 that the state space of the embedded Markov chain become large and needs a rigorous analysis in constructing the corresponding transition probability matrix (TPM) for obtaining required joint probabilities at service/vacation completion epoch. Next using the supplementary variable technique we obtain a relation between service/vacation completion epoch and arbitrary epoch joint distributions of queue content and serving batch size, and queue content and vacation type.

A real life example of the queueing model under consideration can be found in business and industrial application. For example consider the management of a bus depot. To be more specific let us consider a bus service center with a single bus, run by a single driver, operates on a particular route. The bus will start from the terminal only when a certain number (lower threshold) of passengers are available, and in a particular trip a maximum number (upper threshold) of passengers are accommodated. Whenever there are fewer number of passengers (less than the required number of passengers to start the bus) are present, the bus driver takes vacation of random duration (with constant mean length). During his vacation it may so happen that enough number of passengers arrive or bus is full and ready to depart. However, it could not depart due to the absence of the bus driver as he has left for vacation. This results in a loss of revenue to the bus owner. Now in order to minimize the loss, the bus owner imposes a penalty on the bus driver for his late arrival from vacation. The imposition of penalty on the bus driver may results in reduced earning. To minimize the loss earning of the driver as well loss in revenue of the owner, instead of taking a vacation of constant mean length by the driver, he may decide to take a vacation of variable mean length depending on the queue length at vacation initiation epoch (i.e., shorter vacation when queue is long and longer vacation when queue is small). This may eventually lead to revenue loss of the bus owner and penalty imposed on the bus driver to be minimal.

The outline of the rest of this chapter is as follows: mathematical description along with the use of embedded Markov chain technique, to obtain the joint distributions at service and vacation completion epoch, is explained in Section 4.2 and Section 4.2.1, respectively. Next in Section 4.2.2, a relation between the joint distributions of service and vacation completion epoch and arbitrary epoch is established with the help of supplementary variable technique. Section 4.3 is assigned for the various performance measures. Numerical results and their discussion for several service/vacation time distributions are presented in Section 4.4. Some conclusions of this chapter are drawn in Section 4.5.

4.2 Model description

We consider a finite buffer single server bulk service vacation queue. The customers arrive to the system according to the Poisson process with rate λ. The bulk service rule is considered to be 'general bulk service rule' (GBS rule) with the minimum threshold limit 'a' and maximum threshold limit 'b'. For more detail on GBS rule readers are referred to the book by Chaudhry and Templeton (1972), which provides detail study on bulk service queue.

We have studied the queueing model under consideration with two type of vacation rules, viz., single vacation (SV) and multiple vacation (MV), in a unified way by defining an indicator variable δ_s as follows.

$$\delta_s = \begin{cases} 1, & \text{for SV rule,} \\ 0, & \text{for MV rule} \end{cases}$$

It should be noted here that,

1. the vacation rule must be decided by the server at the beginning of the system operation. Once the queueing system is started working (operating) with a pre-specified vacation rule, it is not allowed to change the vacation rule.

2. one can obtain the results for $M/G_r^{(a,b)}/1/N$ queue with SV by substituting $\delta_s = 1$ and that of MV by substituting $\delta_s = 0$.

We have also considered the model with queue length dependent vacation. Let us define the model with SV and MV separately as follows:

(a) Let us first suppose that the ***single vacation rule*** is considered.

. In this case, if at the end of the service of a batch server finds that the queue length (k) is greater than a, i.e., $k \geq a$, then it continues service in batches of size r $(a \leq r \leq b)$ customers with service time of random length S_r, otherwise, it leaves for a vacation of random duration $V^{[k]}$, which is dependent on the queue length k $(0 \leq k \leq a-1)$ remaining in the system at vacation initiation epoch. At the end of the vacation, if it finds that the queue length is still less than a, i.e.,

$0 \le k \le a-1$, then server remains dormant till the queue length attains a and then resumes its service.

(b) Let us now suppose that the ***multiple vacation rule*** is considered.

. In this case, server continue service in bathes of size r $(a \le r \le b)$ customers with service time of random length S_r, if queue length k is found to be greater that a, i.e., $k \ge a$, at the end of a service period. Otherwise, it leaves for a vacation of random length $V^{[k]}$ which is dependent on the queue length k $(0 \le k \le a-1)$ at vacation initiation epoch. Now after returning from the vacation, if it finds that the queue length k is still less than a, i.e., $(0 \le k \le a-1)$, then the server leaves for another vacation of random length $V^{[k]}$ and the process of taking consecutive vacations continue till it finds that the queue length is greater than or equal to a at the end of vacation.

Note:- It should be noted here that, whenever server leaves for a vacation of random length $V^{[k]}$, leaving k $(0 \le k \le a-1)$ in the system, it will be termed as **k-th type of vacation** taken by the server or simply **k-th type of vacation**, throughout the chapter.

The service time distribution $(S_r(\cdot))$ is generally distributed and dependent on serving batch size r $(a \le r \le b)$ with probability density function (pdf) $s_r(.)$, Laplace-Stieltjes transformation (LST) $s_r^*(\cdot)$ and mean service time $\tilde{s_r}$. The vacation time distribution $(V^{[k]}(\cdot))$ is also generally distributed and depends on queue length k, $(0 \le k \le a-1)$ at vacation initiation epoch with pdf $v^{[k]}(\cdot)$, LST $v^{[k]*}(\cdot)$ and mean vacation time $\tilde{v}^{[k]}$. The finite buffer size is considered to be $N\,(> b)$.

4.2.1 Probability distribution at service/vacation completion epoch

In this section, we obtain (i) the joint distribution of queue content and serving batch size at service completion-epoch, and (ii) the joint distribution of the queue content and the type of the vacation taken by the server at vacation termination epoch by using the embedded Markov chain technique. Towards this end, the embedded points are considered as the service completion epoch and the vacation completion epoch, respectively. The steady state joint probabilities are defined as follows:

- $p_{n,r}^{+}$ be the probability that n customers are in the queue at the service completion epoch of a batch of customers of size r , $0 \leq n \leq N$, $a \leq r \leq b$,
- $q_n^{[k]+}$ be the probability that $n+k$ customers are present in the queue at k^{th}-type vacation termination epoch of the server, $0 \leq k \leq a-1$, $0 \leq n \leq N-k$,
- $p_n^{+}\left(=\sum_{r=a}^{b} p_{n,r}^{+}\right)$ be the probability that n customers are in the queue at service completion epoch of a batch, $0 \leq n \leq N$,
- $q_n^{+}\left(=\sum_{k=0}^{min(n,a-1)} q_{n-k}^{[k]+}\right)$ be the probability that n customers are in the queue at vacation termination epoch of the server, $0 \leq n \leq N$.

The unknown quantities $p_{n,r}^{+}$ and $q_n^{[k]+}$ can be obtained by solving the system of equations $\Pi\mathscr{P} = \Pi$, where

- $\Pi = (\tilde{\pi}, \tilde{\gamma}) = (\pi_0^{+}, \pi_1^{+}, ..., \pi_N^{+}, \gamma_0^{+}, \gamma_1^{+}, ..., \gamma_N^{+})$, i.e., $\tilde{\pi} = \left(\pi_0^{+}, \pi_1^{+}, ..., \pi_N^{+}\right)$ and $\tilde{\gamma} = \left(\gamma_0^{+}, \gamma_1^{+}, ..., \gamma_N^{+}\right)$ are row vectors each of dimension $(N+1)$,
- each $\pi_n^{+}(0 \leq n \leq N)$ is a row vector of dimension $(b-a+1)$ and is given by

$$\pi_n^{+} = (p_{n,a}^{+}, p_{n,a+1}^{+}, ..., p_{n,b}^{+}),$$

- each γ_n^{+} is a row vector of dimension $(n+1)$ for $0 \leq n \leq a-2$, and of dimension a for $a-1 \leq n \leq N$ and is given by

$$\gamma_n^{+} \equiv \begin{cases} (q_n^{[0]+}, q_{n-1}^{[1]+}, ..., q_0^{[n]+}) & 0 \leq n \leq a-2, \\ (q_n^{[0]+}, q_{n-1}^{[1]+}, ..., q_{n-a+1}^{[a-1]+}) & a-1 \leq n \leq N, \end{cases}$$

- $\mathscr{P}$ is the one-step transition probability matrix (TPM) of dimension

$\left((N+1)(b-a+1) + \frac{a(a-1)}{2} + a(N-a+1)\right)$, and is given by

$$\mathscr{P} = \begin{pmatrix} \Phi & \Theta \\ \Lambda & \Psi \end{pmatrix}.$$

The block matrices Φ, Θ, Λ and Ψ, as appeared in $\mathscr{P}$, are of dimension $(N+1)(b-a+1)\times(N+1)(b-a+1)$, $(N+1)(b-a+1)\times\left(\frac{a(a-1)}{2}+a(N-a+1)\right)$, $\left(\frac{a(a-1)}{2}+a(N-a+1)\right)\times(N+1)(b-a+1)$ and $\left(\frac{a(a-1)}{2}+a(N-a+1)\right)\times\left(\frac{a(a-1)}{2}+a(N-a+1)\right)$, respectively. Below we describe these block matrices in detail.

The block matrix Φ contains the transition probabilities among the service completion epochs and is given by

$$\Phi = \begin{array}{c} \\ 0 \\ 1 \\ \vdots \\ a-1 \\ a \\ \vdots \\ b \\ b+1 \\ \vdots \\ N \end{array}\begin{array}{c} \begin{array}{cccccccc} 0 & 1 & \dots & N-b-1 & N-b & \dots & N-1 & N \end{array} \\ \left(\begin{array}{cccccccc} 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ D_0^{(1)} & D_1^{(1)} & \dots & D_{N-b-1}^{(1)} & D_{N-b}^{(1)} & \dots & D_{N-1}^{(1)} & \bar{D}_N^{(1)} \\ \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots \\ D_0^{(b-a+1)} & D_1^{(b-a+1)} & \dots & D_{N-b-1}^{(b-a+1)} & D_{N-b}^{(b-a+1)} & \dots & D_{N-1}^{(b-a+1)} & \bar{D}_N^{(b-a+1)} \\ 0 & D_0^{(b-a+1)} & \dots & D_{N-b-2}^{(b-a+1)} & D_{N-b-1}^{(b-a+1)} & \dots & D_{N-2}^{(b-a+1)} & \bar{D}_{N-1}^{(b-a+1)} \\ \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & D_0^{(b-a+1)} & \dots & D_{b-1}^{(b-a+1)} & \bar{D}_b^{(b-a+1)} \end{array}\right) \end{array},$$

where each 0 and $D_j^{(i)}$ are matrices of dimension $(b-a+1)$ and are given by

$D_j^{(i)} = e_i^T \otimes \kappa_j^{(i+a-1)}$, $1 \le i \le b-a+1, 0 \le j \le N-1,$

$\bar{D}_N^{(i)} = e_i^T \otimes \kappa_N^{(i+a-1)} + e_{b-a+1}^T \otimes \bar{\kappa}_N^{(i+a-1)}$, $1 \le i \le b-a,$

$\bar{D}_j^{(b-a+1)} = e_{b-a+1}^T \otimes \bar{\kappa}_{j-1}^{(b)}$, $b \le j \le N.$

In the above expression

- each e_i is a column vector of dimension $(b-a+1)$ with 1 at i^{th}-position and 0 elsewhere.
- each $\kappa_j^{(r)}$ is a column vector of dimension $(b-a+1)$ consisting of $\xi_j^{(r)}$, where $\xi_j^{(r)}$ represents the probability of j arrivals during service period of a batch of size r and is given by,

$$\xi_j^{(r)} = \int_0^{\infty} \frac{e^{-\lambda t}(\lambda t)^j}{j!} dS_r(t), j \geq 0, a \leq r \leq b,$$

- $\bar{\kappa}_j^{(r)}$ is a column vector of dimension $(b-a+1)$ consisting of $\left(1-\sum_{i=0}^{j}\xi_i^{(r)}\right)$.

The block matrix Θ contains the transition probabilities from the service completion epoch to the vacation completion epoch and is given by

$$\Theta = \begin{array}{c} \\ 0 \\ 1 \\ \vdots \\ a-1 \\ a \\ \vdots \\ N \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & \dots & a-2 & a-1 & \dots & N-1 & N \end{array} \\ \begin{pmatrix} C_0^{(1)} & C_1^{(1)} & \dots & C_{a-2}^{(1)} & C_{a-1}^{(1)} & \dots & C_{N-1}^{(1)} & \bar{C}_N^{(1)} \\ 0 & C_0^{(2)} & \dots & C_{a-3}^{(2)} & C_{a-2}^{(2)} & \dots & C_{N-2}^{(2)} & \bar{C}_{N-1}^{(2)} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & C_0^{(a)} & \dots & C_{N-a}^{(a)} & \bar{C}_{N-a+1}^{(a)} \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \end{array}.$$

The $i-j$th element of Θ, i.e., $\Theta_{i,j}$ are matrices of dimensions as described below:

$$\Theta_{i,j} \equiv \begin{cases} \text{matrix of dimension } (i+1)\times(b-a+1), & 0 \leq i \leq a-2, 0 \leq j \leq N, \\ \text{matrix of dimension } a \times (b-a+1), & a-1 \leq i \leq N, 0 \leq j \leq N, \end{cases}$$

and each $C_j^{(k)}$, as appeared in block matrix Θ, is given by
$C_j^{(k)} = e_k^T \otimes \vartheta_j^{(k-1)}$, $1 \leq k \leq a, 0 \leq j \leq N-1$,
$\bar{C}_j^{(k)} = e_k^T \otimes \vartheta_j^{(k-1)} + e_a^T \otimes \bar{\vartheta}_j^{(k-1)}$, $1 \leq k \leq a-1, N-a+2 \leq j \leq N, j+k = N+1$,
$\bar{C}_{N-a+1}^{(a)} = e_a^T \otimes \bar{\vartheta}_{N-a}^{(a-1)}$.
In the above expression

- each e_i is a column vectors with 1 in the i^{th}-position and 0 elsewhere, and its dimension is chosen in such a way that dimension of each $C_j^{(k)}$ is well defined.

- each $\vartheta_j^{(k)}$ is a column vector of dimension $(b-a+1)$ consisting of $\omega_j^{(k)}$, where $\omega_j^{(k)}$

represents the probability of j arrivals during the $k^{th}-$type vacation period and is given by

$$\omega_j^{(k)} = \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^j}{j!} dV^{[k]}(t), \quad j \geq 0, 0 \leq k \leq a-1.$$

- $\bar{\vartheta}_j^{(k)}$ is a column vector of dimension $(b-a+1)$ consisting of $\left(1-\sum_{i=0}^{j}\omega_i^{(k)}\right)$.

The block matrix Λ contains the transition probabilities from the vacation termination epoch to the service completion epoch is given by

$$\Lambda = \begin{array}{c} \\ 0 \\ 1 \\ \vdots \\ a-1 \\ a \\ \vdots \\ b \\ b+1 \\ \vdots \\ N \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & \dots & N-b-1 & N-b & \dots & N-1 & N \end{array} \\ \left(\begin{array}{cccccccc} \delta_s B_0^{(1)} & \delta_s B_1^{(1)} & \dots & \delta_s B_{N-b-1}^{(1)} & \delta_s B_{N-b}^{(1)} & \dots & \delta_s B_{N-1}^{(1)} & \delta_s \bar{B}_N^{(1)} \\ \delta_s B_0^{(1)} & \delta_s B_1^{(1)} & \dots & \delta_s B_{N-b-1}^{(1)} & \delta_s B_{N-b}^{(1)} & \dots & \delta_s B_{N-1}^{(1)} & \delta_s \bar{B}_N^{(1)} \\ \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \delta_s B_0^{(1)} & \delta_s B_1^{(1)} & \dots & \delta_s B_{N-b-1}^{(1)} & \delta_s B_{N-b}^{(1)} & \dots & \delta_s B_{N-1}^{(1)} & \delta_s \bar{B}_N^{(1)} \\ B_0^{(1)} & B_1^{(1)} & \dots & B_{N-b-1}^{(1)} & B_{N-b}^{(1)} & \dots & B_{N-1}^{(1)} & \bar{B}_N^{(1)} \\ \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots \\ B_0^{(b-a+1)} & B_1^{(b-a+1)} & \dots & B_{N-b-1}^{(b-a+1)} & B_{N-b}^{(b-a+1)} & \dots & B_{N-1}^{(b-a+1)} & \bar{B}_N^{(b-a+1)} \\ 0 & B_0^{(b-a+1)} & \dots & B_{N-b-2}^{(b-a+1)} & B_{N-b-1}^{(b-a+1)} & \dots & B_{N-2}^{(b-a+1)} & \bar{B}_{N-1}^{(b-a+1)} \\ \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & B_0^{(b-a+1)} & \dots & B_{b-1}^{(b-a+1)} & \bar{B}_b^{(b-a+1)} \end{array}\right) \end{array}.$$

The $i-j$th element of Λ, i.e., $\Lambda_{i,j}$ are matrices of dimension as described below:

$$\Lambda_{i,j} \equiv \begin{cases} \text{matrix of dimension } (i+1)\times(b-a+1), & 0 \leq i \leq a-2, 0 \leq j \leq N, \\ \text{matrix of dimension } a\times(b-a+1), & a-1 \leq i \leq N, 0 \leq j \leq N, \end{cases}$$

and each $B_j^{(k)}$, as appeared in Λ, is given by

$B_j^{(i)} = e_i^T \otimes \kappa_j^{(i+a-1)}$, $1 \leq i \leq b-a+1, 0 \leq j \leq N-1$,

$\bar{B}_N^{(i)} = e_i^T \otimes \kappa_N^{(i+a-1)} + e_{b-a+1}^T \otimes \bar{\kappa}_N^{(i+a-1)}$, $1 \leq i \leq b-a$,

$\bar{B}_j^{(b-a+1)} = e_{b-a+1}^T \otimes \bar{\kappa}_{j-1}^{(b)}$, $b \leq j \leq N$,

where

- each e_i is a column vector of dimension $(b-a+1)$ with 1 at i^{th}-position and 0 elsewhere,
- each $\kappa_j^{(r)}$ is a column vector consisting of $\xi_j^{(r)}$, where $\xi_j^{(r)}$ represents the probability of j arrivals during the service period of a batch of size r. The dimension of each $\kappa_j^{(r)}$ is chosen in such a way that the dimension of each $B_j^{(k)}$ is well defined,
- $\bar{\kappa}_j^{(r)}$ is a column vector of appropriate dimension consisting of $\left(1-\sum_{i=0}^{j}\xi_i^{(r)}\right)$.

The block matrix Ψ contains the transition probabilities among the vacation termination epochs and is given by

$$\Psi = \begin{array}{c} \\ 0 \\ 1 \\ \vdots \\ a-1 \\ a \\ \vdots \\ N \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & \dots & a-2 & a-1 & \dots & N-1 & N \end{array} \\ \begin{pmatrix} (1-\delta_s)A_0^{(1)} & (1-\delta_s)A_1^{(1)} & \dots & (1-\delta_s)A_{a-2}^{(1)} & (1-\delta_s)A_{a-1}^{(1)} & \dots & (1-\delta_s)A_{N-1}^{(1)} & (1-\delta_s)\bar{A}_N^{(1)} \\ 0 & (1-\delta_s)A_0^{(2)} & \dots & (1-\delta_s)A_{a-3}^{(2)} & (1-\delta_s)A_{a-2}^{(2)} & \dots & (1-\delta_s)A_{N-2}^{(2)} & (1-\delta_s)\bar{A}_{N-1}^{(2)} \\ \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & (1-\delta_s)A_0^{(a)} & \dots & (1-\delta_s)A_{N-a}^{(a)} & (1-\delta_s)\bar{A}_{N-a+1}^{(a)} \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \end{array}.$$

The $i-j$th element of Ψ, i.e., $\Psi_{i,j}$ are matrices of dimension as described below:

$$\Psi_{i,j} \equiv \begin{cases} \text{matrix of dimension } (i+1)\times(i+1), & 0 \le i,j \le a-2, \\ \text{matrix of dimension } a\times a, & a-1 \le i,j \le N. \end{cases}$$

Each $A_j^{(k)}$, as appeared in Ψ, is given by

$A_j^{(k)} = e_k^T \otimes \vartheta_j^{(k-1)}$, $1 \le k \le a$, $0 \le j \le N-1$,

$\bar{A}_j^{(k)} = e_k^T \otimes \vartheta_j^{(k-1)} + e_a^T \otimes \bar{\vartheta}_j^{(k-1)}$, $1 \le k \le a-1$, $N-a+2 \le j \le N$, $j+k = N+1$,

$\bar{A}_{N-a+1}^{(a)} = e_a^T \otimes \bar{\vartheta}_{N-a}^{(a-1)}$.

In above expression

- each e_i is a column vector with 1 in the i^{th}-position and 0 elsewhere, and its dimension is chosen in such a way that dimension of each $A_j^{(k)}$ is well defined.

- each $\vartheta_j^{(k)}$ is a column vector consisting of $\omega_j^{(k)}$, where $\omega_j^{(k)}$ is the probability of j arrivals during the $k^{th}-$type vacation period and is given by

$$\omega_j^{(k)} = \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^j}{j!} dV^{[k]}(t), \quad j \geq 0,\ 0 \leq k \leq a-1.$$

The dimension of each $\vartheta_j^{(k)}$ is chosen in such a way that the dimension of each $A_j^{(k)}$ is well defined.

- $\bar{\vartheta}_j^{(k)}$ is a column vector of appropriate dimension consisting of $\left(1-\sum_{i=0}^{j}\omega_i^{(k)}\right)$.

Remark : According to Theorem 3.1 given in Abolnikov and Dukhovny (1991) every Markov chain whose TPM can be represented as a finite positive delta matrix is ergodic. Since the TPM $\mathscr{P}$ of the model considered in this chapter is of finite positive $\triangle_{m,n}$-type matrix, one can conclude that the corresponding Markov chain is ergodic which ensures the existence of steady state distribution.

4.2.2 Probability distribution at arbitrary epoch

In this section, we obtain the joint distribution of queue content and vacation type when server is in vacation, and the joint distribution of queue content and the serving batch size when server is busy in serving customers, at arbitrary epoch. Towards this end, we define the following notations, at time t.

- $N_q(t) \equiv$ the number of customers present in the queue,
- $N_s(t) \equiv$ the number of customers in service when server is busy,
- $\chi(t) \equiv$ the state of the server, defined as,

$$\chi(t) = \begin{cases} 0, & \text{if server is in dormant state,} \\ k, & \text{if server is in } k^{\text{th}} - \text{type of vacation, } 0 \leq k \leq a-1, \\ r, & \text{if server is busy in serving batch of size } r,\ a \leq r \leq b, \end{cases}$$

- $U(t) \equiv$ the remaining service time of a batch of customers under service, if any,

- $\widetilde{V}(t) \equiv$ the remaining vacation time of the server, if any.

Let us define the following state probabilities, at time t

- $P_{n,0}(t) \equiv prob.\{N_q(t) = n,\ \chi(t) = 0\}, \quad 0 \le n \le a-1,$
- $P_{n,r}(x,t)dx \equiv prob.\{N_q(t) = n, N_s(t) = r, x \le U(t) \le x+dx, \chi(t) = r\}, \quad 0 \le n \le N,$ $a \le r \le b, x \ge 0,$
- $Q_n^{[k]}(x,t)dx \equiv prob.\{N_q(t) = n+k, , x \le \widetilde{V}(t) \le x+dx, \chi(t) = k\}, \quad 0 \le k \le a-1, 0 \le$ $n \le N-k, x \ge 0.$

Now relating the state of the system at time t and $t+dt$ we obtain the Kolmogrov equations of the model under consideration as follows:

$$\frac{d}{dt}P_{0,0}(t) = -\delta_s\lambda P_{0,0}(t) + \delta_s Q_0^{[0]}(0,t), \tag{4.1}$$

$$\frac{d}{dt}P_{n,0}(t) = -\delta_s\lambda P_{n,0}(t) + \delta_s\lambda P_{n-1,0}(t) + \delta_s\sum_{k=0}^{n} Q_{n-k}^{[k]}(0,t),\ 1 \le n \le a-1, \tag{4.2}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) P_{0,a}(x,t) = -\lambda P_{0,a}(x,t) + \delta_s\lambda P_{a-1,0}(t)s_a(x) + \sum_{k=0}^{a-1} Q_{a-k}^{[k]}(0,t)s_a(x) + \sum_{r=a}^{b} P_{a,r}(0,t)s_a(x), \tag{4.3}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) P_{0,r}(x,t) = -\lambda P_{0,r}(x,t) + \sum_{k=0}^{a-1} Q_{r-k}^{[k]}(0,t)s_r(x) + \sum_{k=a}^{b} P_{r,k}(0,t)s_r(x),\ a+1 \le r \le b, \tag{4.4}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) P_{n,r}(x,t) = -\lambda P_{n,r}(x,t) + \lambda P_{n-1,r}(x,t),\ a \le r \le b-1, 1 \le n \le N-1, \tag{4.5}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) P_{n,b}(x,t) = -\lambda P_{n,b}(x,t) + \lambda P_{n-1,b}(x,t) + \sum_{k=0}^{a-1} Q_{n+b-k}^{[k]}(0,t)s_b(x) + \sum_{r=a}^{b} P_{n+b,r}(0,t)s_b(x),\ 1 \le n \le N-b, \tag{4.6}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) P_{n,b}(x,t) = -\lambda P_{n,b}(x,t) + \lambda P_{n-1,b}(x,t),\ N-b+1 \le n \le N-1, \tag{4.7}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) P_{N,r}(x,t) = \lambda P_{N-1,r}(x,t),\ a \le r \le b, \tag{4.8}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) Q_0^{[k]}(x,t) = -\lambda Q_0^{[k]}(x,t) + \left(\sum_{r=a}^{b} P_{k,r}(0,t) + (1-\delta_s)\sum_{j=0}^{k} Q_{k-j}^{[j]}(0,t)\right) v^{[k]}(x),$$

$$0 \le k \le a-1, \tag{4.9}$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)Q_n^{[k]}(x,t) = -\lambda Q_n^{[k]}(x,t)+\lambda Q_{n-1}^{[k]}(x,t),$$
$$1 \le n \le N-1,\ 0 \le k \le min(a-1,N-n-1), \tag{4.10}$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)Q_{N-k}^{[k]}(x,t) = \lambda Q_{N-k-1}^{[k]}(x,t),\ 0 \le k \le a-1. \tag{4.11}$$

In steady-state, as $t \to \infty$, we define

$$\lim_{t\to\infty} P_{n,0}(t) = P_{n,0},\ 0 \le n \le a-1,$$
$$\lim_{t\to\infty} P_{n,r}(x,t) = P_{n,r}(x),\ 0 \le n \le N,\ a \le r \le b,$$
$$\lim_{t\to\infty} Q_n^{[k]}(x,t) = Q_n^{[k]}(x),\ 0 \le k \le a-1,\ 0 \le n \le N-k.$$

The corresponding steady state equations are obtained from (4.1)-(4.11) as follows

$$0 = -\delta_s\lambda P_{0,0}+\delta_s Q_0^{[0]}(0), \tag{4.12}$$

$$0 = -\delta_s\lambda P_{n,0}+\delta_s\lambda P_{n-1,0}+\delta_s\sum_{k=0}^{n} Q_{n-k}^{[k]}(0),\ 1 \le n \le a-1, \tag{4.13}$$

$$-\frac{\partial}{\partial x}P_{0,a}(x) = -\lambda P_{0,a}(x)+\delta_s\lambda P_{a-1,0}s_a(x)+\sum_{k=0}^{a-1} Q_{a-k}^{[k]}(0)s_a(x)+\sum_{r=a}^{b} P_{a,r}(0)s_a(x), \tag{4.14}$$

$$-\frac{\partial}{\partial x}P_{0,r}(x) = -\lambda P_{0,r}(x)+\sum_{k=0}^{a-1} Q_{r-k}^{[k]}(0)s_r(x)+\sum_{k=a}^{b} P_{r,k}(0)s_r(x),\ a+1 \le r \le b, \tag{4.15}$$

$$-\frac{\partial}{\partial x}P_{n,r}(x) = -\lambda P_{n,r}(x)+\lambda P_{n-1,r}(x),\ a \le r \le b-1,\ 1 \le n \le N-1, \tag{4.16}$$

$$-\frac{\partial}{\partial x}P_{n,b}(x) = -\lambda P_{n,b}(x)+\lambda P_{n-1,b}(x)+\sum_{k=0}^{a-1} Q_{n+b-k}^{[k]}(0)s_b(x)+\sum_{r=a}^{b} P_{n+b,r}(0)s_b(x),$$
$$1 \le n \le N-b, \tag{4.17}$$

$$-\frac{\partial}{\partial x}P_{n,b}(x) = -\lambda P_{n,b}(x)+\lambda P_{n-1,b}(x),\ N-b+1 \le n \le N-1, \tag{4.18}$$

$$-\frac{\partial}{\partial x}P_{N,r}(x) = \lambda P_{N-1,r}(x),\ a \le r \le b, \tag{4.19}$$

$$-\frac{\partial}{\partial x}Q_0^{[k]}(x) = -\lambda Q_0^{[k]}(x)+\left(\sum_{r=a}^{b} P_{k,r}(0)+(1-\delta_s)\sum_{j=0}^{k} Q_{k-j}^{[j]}(0)\right)\nu^{[k]}(x),$$
$$0 \le k \le a-1, \tag{4.20}$$

$$-\frac{\partial}{\partial x}Q_n^{[k]}(x) = -\lambda Q_n^{[k]}(x)+\lambda Q_{n-1}^{[k]}(x),$$
$$1 \le n \le N-1,\ 0 \le k \le min(a-1,N-n-1), \tag{4.21}$$

$$-\frac{\partial}{\partial x}Q_{N-k}^{[k]}(x) = \lambda Q_{N-k-1}^{[k]}(x),\ 0 \le k \le a-1. \tag{4.22}$$

Further, let us define for $Re\,\theta \geq 0$

$$\begin{aligned}
\int_0^\infty e^{-\theta x}P_{n,r}(x)dx &= P^*_{n,r}(\theta);\ 0\leq n\leq N,\ a\leq r\leq b,\\
\int_0^\infty e^{-\theta x}Q_n^{[k]}(x)dx &= Q_n^{[k]*}(\theta);\ 0\leq k\leq a-1,\ 0\leq n\leq N-k,\\
\int_0^\infty e^{-\theta x}s_r(x)dx &= s^*_r(\theta);\ a\leq r\leq b,\\
\int_0^\infty e^{-\theta x}v^{[k]}(x)dx &= v^{[k]*}(\theta);\ 0\leq k\leq a-1.
\end{aligned}$$

The following two results followed immediately from the above definitions and will be used in sequel.

$$\begin{aligned}
P_{n,r}\equiv P^*_{n,r}(0) &= \int_0^\infty P_{n,r}(x)dx,\\
Q_n^{[k]}\equiv Q_n^{[k]*}(0) &= \int_0^\infty Q_n^{[k]}(x)dx,
\end{aligned}$$

Multiplying (4.14)-(4.22) by $e^{-\theta x}$ and integrating with respect to x from 0 to ∞ we find

$$(\lambda-\theta)P^*_{0,a}(\theta) = \delta_s\lambda P_{a-1,0}s^*_a(\theta)+\sum_{k=0}^{a-1}Q_{a-k}^{[k]}(0)s^*_a(\theta)+\sum_{r=a}^{b}P_{a,r}(0)s^*_a(\theta)-P_{0,a}(0), \tag{4.23}$$

$$(\lambda-\theta)P^*_{0,r}(\theta) = \sum_{k=0}^{a-1}Q_{r-k}^{[k]}(0)s^*_r(\theta)+\sum_{k=a}^{b}P_{r,k}(0)s^*_r(\theta)-P_{0,r}(0),\ a+1\leq r\leq b, \tag{4.24}$$

$$(\lambda-\theta)P^*_{n,r}(\theta) = \lambda P^*_{n-1,r}(\theta)-P_{n,r}(0);\ a\leq r\leq b-1,\ 1\leq n\leq N-1, \tag{4.25}$$

$$(\lambda-\theta)P^*_{n,b}(\theta) = \lambda P^*_{n-1,b}(\theta)+\sum_{k=0}^{a-1}Q_{n+b-k}^{[k]}(0)s^*_b(\theta)+\sum_{r=a}^{b}P_{n+b,r}(0)s^*_b(\theta)-P_{n,b}(0),\ 1\leq n\leq N-b, \tag{4.26}$$

$$(\lambda-\theta)P^*_{n,b}(\theta) = \lambda P^*_{n-1,b}(\theta)-P_{n,b}(0),\ N-b+1\leq n\leq N-1, \tag{4.27}$$

$$-\theta P^*_{N,r}(\theta) = \lambda P^*_{N-1,r}(\theta)-P_{N,r}(0),\ a\leq r\leq b, \tag{4.28}$$

$$(\lambda-\theta)Q_0^{[k]*}(\theta) = \left(\sum_{r=a}^{b}P_{k,r}(0)+(1-\delta_s)\sum_{j=0}^{k}Q_{k-j}^{[j]}(0)\right)v^{[k]*}(\theta)-Q_0^{[k]}(0), 0\leq k\leq a-1, \tag{4.29}$$

$$(\lambda-\theta)Q_n^{[k]*}(\theta) = \lambda Q_{n-1}^{[k]*}(\theta)-Q_n^{[k]}(0),\ 1\leq n\leq N-1,\ 0\leq k\leq min(a-1,N-n-1), \tag{4.30}$$

$$-\theta Q_{N-k}^{[k]*}(\theta) = \lambda Q_{N-k-1}^{[k]*}(\theta)-Q_{N-k}^{[k]}(0),\ 0\leq k\leq a-1. \tag{4.31}$$

Now using (4.12)-(4.13) and (4.23)-(4.31), we first derive following results for use in sequel.

Lemma 4.1. The probabilities $\left(p_{n,r}^{+}, P_{n,r}(0)\right)$ and $\left(q_n^{[k]+}, Q_n^{[k]}(0)\right)$ are connected by following relations

$$p_{n,r}^{+} = \sigma P_{n,r}(0),\, 0 \le n \le N,\, a \le r \le b, \tag{4.32}$$

$$q_n^{[k]+} = \sigma Q_n^{[k]}(0)\,, 0 \le k \le a-1,\, 0 \le n \le N-k, \tag{4.33}$$

where, $\sigma^{-1} = \sum_{n=0}^{N}\sum_{r=a}^{b} P_{n,r}(0) + \sum_{n=0}^{a-1}\sum_{k=0}^{n} Q_{n-k}^{[k]}(0) + \sum_{n=a}^{N}\sum_{k=0}^{a-1} Q_{n-k}^{[k]}(0)$.
Equation (4.32) gives us the relation between queue content and server content joint distribution at service completion epoch with the joint distribution of queue and server content when the service is about to complete. Similarly, (4.33) gives us the relation between queue content and vacation type joint distribution at vacation completion epoch with the joint distribution of queue content and vacation type when the vacation is about to complete. σ^{-1} gives us the mean number of service completion or vacation termination per unit time, i.e., mean departure rate from busy state or vacation state.

Proof. Using Bayes' theorem, for $0 \le n \le N$, $a \le r \le b$ we have

$p_{n,r}^{+}$ = prob.$\{n$ customers are in the queue at the service completion epoch of a batch of size $r\}$

= prob.$\{n$ customers are in the queue just prior to the service completion epoch of a batch of size $r \,|\le N$ customers are in the queue just prior to the service completion epoch of a batch of size $a \le r \le b$ or vacation completion epoch of k-th type vacation with $0 \le k \le a-1.\}$

$$= \frac{P_{n,r}(0)}{\sum_{n=0}^{N}\sum_{r=a}^{b} P_{n,r}(0) + \sum_{n=0}^{a-1}\sum_{k=0}^{n} Q_{n-k}^{[k]}(0) + \sum_{n=a}^{N}\sum_{k=0}^{a-1} Q_{n-k}^{[k]}(0)},$$

With the similar argument one can write, for $0 \le k \le a-1$, $0 \le n \le N-k$

$$q_n^{[k]+} = \frac{Q_n^{[k]}(0)}{\sum_{n=0}^{N}\sum_{r=a}^{b} P_{n,r}(0) + \sum_{n=0}^{a-1}\sum_{k=0}^{n} Q_{n-k}^{[k]}(0) + \sum_{n=a}^{N}\sum_{k=0}^{a-1} Q_{n-k}^{[k]}(0)}.$$

Lemma 4.2. In case of single vacation, the dormant state probabilities $P_{n,0}$, are given by

$$\lambda P_{n,0} = \sum_{i=0}^{n}\sum_{k=0}^{i} Q_{i-k}^{[k]}(0)\,,\ 0 \le n \le a-1, \tag{4.34}$$

The left hand side of (4.34) represents the input rate when server is in dormant state and right hand side represents vacation termination rate.

Proof. Using (4.12) in (4.13), we get our desired result (4.34).

Lemma 4.3. The value of σ^{-1} as appeared in Lemma 4.1 is given by,

$$\sigma^{-1} = \sum_{n=0}^{N}\sum_{r=a}^{b} P_{n,r}(0) + \sum_{n=0}^{a-1}\sum_{k=0}^{n} Q_{n-k}^{[k]}(0) + \sum_{n=a}^{N}\sum_{k=0}^{a-1} Q_{n-k}^{[k]}(0) = \frac{1-\delta_s\sum_{n=0}^{a-1}P_{n,0}}{g}, \tag{4.35}$$

where $g = \sum_{n=0}^{a-1}\left(p_n^+\tilde{v}^{[n]} + \sum_{k=0}^{n} q_{n-k}^{[k]+}\left(\delta_s\tilde{s_a} + (1-\delta_s)\tilde{v}^{[n]}\right)\right) + \sum_{n=a}^{b}\left(\sum_{k=0}^{a-1} q_{n-k}^{[k]+} + p_n^+\right)\tilde{s_n} + \sum_{n=b+1}^{N}\left(\sum_{k=0}^{a-1} q_{n-k}^{[k]+} + p_n^+\right)\tilde{s_b}$.

Proof. Using (4.34) in (4.23) we get

$$(\lambda-\theta)P_{0,a}^*(\theta) = \delta_s\sum_{i=0}^{a-1}\sum_{k=0}^{i} Q_{i-k}^{[k]}(0)s_a^*(\theta) + \sum_{k=0}^{a-1} Q_{a-k}^{[k]}(0)s_a^*(\theta) + \sum_{r=a}^{b} P_{a,r}(0)s_a^*(\theta) - P_{0,a}(0), \tag{4.36}$$

Summing (4.24)-(4.31) and (4.36) we get

$$\begin{aligned}\left(\sum_{n=0}^{N}\sum_{r=a}^{b} P_{n,r}^*(\theta) + \sum_{n=0}^{a-1}\sum_{k=0}^{n} Q_{n-k}^{[k]*}(\theta) + \sum_{n=a}^{N}\sum_{k=0}^{a-1} Q_{n-k}^{[k]*}(\theta)\right) &= \sum_{n=0}^{a-1}\left(\frac{1-v^{[n]*}(\theta)}{\theta}\sum_{r=a}^{b} P_{n,r}(0) + \right.\\ &\left.\frac{1-\delta_s s_a^*(\theta)-(1-\delta_s)v^{[n]*}(\theta)}{\theta}\sum_{k=0}^{n} Q_{n-k}^{[k]}(0)\right) \\ &+ \sum_{n=a}^{b}\frac{1-s_n^*(\theta)}{\theta}\left(\sum_{r=a}^{b} P_{n,r}(0) + \sum_{k=0}^{a-1} Q_{n-k}^{[k]}(0)\right) \\ &+ \frac{1-s_b^*(\theta)}{\theta}\sum_{n=b+1}^{N}\left(\sum_{r=a}^{b} P_{n,r}(0) + \sum_{k=0}^{a-1} Q_{n-k}^{[k]}(0)\right),\end{aligned} \tag{4.37}$$

Taking limit as $\theta \to 0$ in above expression and using L'Hôpital's rule, the normalizing condition

$$\delta_s\sum_{n=0}^{a-1} P_{n,0} + \sum_{n=0}^{N}\sum_{r=a}^{b} P_{n,r} + \sum_{n=0}^{a-1}\sum_{k=0}^{n} Q_{n-k}^{[k]} + \sum_{n=a}^{N}\sum_{k=0}^{a-1} Q_{n-k}^{[k]} = 1 \tag{4.38}$$

we get

$$1-\delta_s\sum_{n=0}^{a-1} P_{n,0} = \sum_{n=0}^{a-1}\left(\tilde{v}^{[n]}\sum_{r=a}^{b} P_{n,r}(0) + \left(\delta_s\tilde{s_a} + (1-\delta_s)\tilde{v}^{[n]}\right)\sum_{k=0}^{n} Q_{n-k}^{[k]}(0)\right)$$

$$+\sum_{n=a}^{b}\left(\sum_{r=a}^{b}P_{n,r}(0)+\sum_{k=0}^{a-1}Q_{n-k}^{[k]}(0)\right)\tilde{s_n} \qquad (4.39)$$
$$+\sum_{n=b+1}^{N}\left(\sum_{r=a}^{b}P_{n,r}(0)+\sum_{k=0}^{a-1}Q_{n-k}^{[k]}(0)\right)\tilde{s_b}.$$

Now using Lemma 4.1 in (4.39) and after little algebraic manipulation, we obtain the desired result (4.35).

Theorem 4.1. The steady state probabilities $\left\{P_{n,0},\, P_{n,r},\, Q_n^{[k]}\right\}$ and $\left\{p_n^+,\, p_{n,r}^+,\, q_n^{[k]+}\right\}$ are related by the following relation

$$P_{n,0} = E^{-1}\sum_{i=0}^{n}\sum_{k=0}^{i}q_{i-k}^{[k]+},\ 0\le n\le a-1, \qquad (4.40)$$
$$P_{n,a} = E^{-1}\left[\delta_s\sum_{i=0}^{a-1}\sum_{k=0}^{i}q_{i-k}^{[k]+}+\sum_{k=0}^{a-1}q_{a-k}^{[k]+}+p_a^+-\sum_{i=0}^{n}p_{i,a}^+\right],\ 0\le n\le N-1, \qquad (4.41)$$
$$P_{n,r} = E^{-1}\left[\sum_{k=0}^{a-1}q_{r-k}^{[k]+}+p_r^+-\sum_{i=0}^{n}p_{i,r}^+\right],\ 0\le n\le N-1,\ a+1\le r\le b-1, \qquad (4.42)$$
$$P_{n,b} = E^{-1}\left[\sum_{i=b}^{min(b+n,N)}\left(\sum_{k=0}^{a-1}q_{i-k}^{[k]+}+p_i^+\right)-\sum_{i=0}^{n}p_{i,b}^+\right],\ 0\le n\le N-1, \qquad (4.43)$$
$$Q_n^{[k]} = E^{-1}\left[p_k^+ +(1-\delta_s)\sum_{j=0}^{k}q_{k-j}^{[j]+}-\sum_{i=0}^{n}q_i^{[k]+}\right],$$
$$0\le n\le N-1,\ 0\le k\le min(N-n-1,a-1), \qquad (4.44)$$

where, $E=\lambda g+\delta_s\sum_{n=0}^{a-1}\sum_{i=0}^{n}\sum_{k=0}^{i}q_{i-k}^{[k]+}$.

Proof. From equation (4.12), using Lemma 4.1, Lemma 4.2 and equation (4.34), after little algebraic manipulation we obtain the desired result (4.40).

Next setting $\theta=0$ in (4.24)-(4.27), (4.29)-(4.30) and (4.36) and using Lemma 4.1 and 4.2, after some algebraic manipulations we obtain the desired results (4.41)-(4.44).

Remark. It should be noted here that the probabilities $P_{N,r}(a\le r\le b)$ and $Q_{N-k}^{[k]}(0\le k\le a-1)$ cannot be obtained using the normalizing condition given in (4.38). However, one can obtain them using a slightly different approach which is explained in the following subsections.

4.2.2.1 Evaluation of $P_{N,r}\,(a \le r \le b)$

In this section we will evaluate $P_{N,r}\,(a \le r \le b)$ with the help of equation (4.28). We denote $P_{n,r}^{*(1)}(0)$ to be the derivative of $P_{n,r}^{*}(\theta)$ with respect to θ at $\theta = 0$.

Now differentiating (4.23)-(4.27) with respect to θ and setting $\theta = 0$ we get

$$\lambda P_{0,a}^{*(1)}(0) = P_{0,a} - \delta_s \lambda P_{a-1,0}\tilde{s_a} - \sum_{k=0}^{a-1} Q_{a-k}^{[k]}(0)\tilde{s_a} - \sum_{r=a}^{b} P_{a,r}(0)\tilde{s_a}, \tag{4.45}$$

$$\lambda P_{0,r}^{*(1)}(0) = P_{0,r} - \sum_{k=0}^{a-1} Q_{r-k}^{[k]}(0)\tilde{s_r} - \sum_{k=a}^{b} P_{r,k}(0)\tilde{s_r}\,,\ a+1 \le r \le b, \tag{4.46}$$

$$\lambda P_{n,r}^{*(1)}(0) = P_{n,r} + \lambda P_{n-1,r}^{*(1)}(0)\,;\ a \le r \le b-1\,,\ 1 \le n \le N-1, \tag{4.47}$$

$$\lambda P_{n,b}^{*(1)}(0) = P_{n,b} + \lambda P_{n-1,b}^{*(1)}(0) - \sum_{k=0}^{a-1} Q_{n+b-k}^{[k]}(0)\tilde{s_b} - \sum_{r=a}^{b} P_{n+b,r}(0)\tilde{s_b}\,;\quad 1 \le n \le N-b, \tag{4.48}$$

$$\lambda P_{n,b}^{*(1)}(0) = P_{n,b} + \lambda P_{n-1,b}^{*(1)}(0)\,;\ N-b+1 \le n \le N-1. \tag{4.49}$$

With the help of the fact that

$$P_{n,r}(0) = \sigma^{-1} p_{n,r}^{+},\ 0 \le n \le N,\ a \le r \le b, \tag{4.50}$$

$$Q_n^{[k]}(0) = \sigma^{-1} q_n^{[k]+},\ 0 \le k \le a-1,\ 0 \le n \le N-k, \tag{4.51}$$

which is easily seen from Lemma 4.1, equations (4.45)-(4.49) gives us a recursive method to calculate $P_{n,r}^{*(1)}(0)$, $0 \le n \le N-1$, $a \le r \le b$ in known terms.

Now differentiating (4.28) with respect to θ and then setting $\theta = 0$ we get $P_{N,r}\,(a \le r \le b)$ in completely known terms as follows

$$P_{N,r} = -\lambda P_{N-1,r}^{*(1)}(0)\,;\ a \le r \le b, \tag{4.52}$$

4.2.2.2 Evaluation of $Q_{N-k}^{[k]}\,(0 \le k \le a-1)$

In order to obtain $Q_{N-k}^{[k]}\,(0 \le k \le a-1)$, let us denote $Q_n^{[k]*(1)}(0)$ to be the derivative of $Q_n^{[k]*}(\theta)$ with respect to θ at $\theta = 0$. Now differentiating (4.29)-(4.30) with respect to θ and set $\theta = 0$

we get

$$\lambda Q_0^{[k]*(1)}(0) = Q_0^{[k]} - \left(\sum_{r=a}^{b} P_{k,r}(0) + (1-\delta_s)\sum_{j=0}^{k} Q_{k-j}^{[j]}(0)\right)\tilde{v}^{[k]},\ 0 \le k \le a-1, \quad (4.53)$$

$$\lambda Q_n^{[k]*(1)}(0) = Q_n^{[k]} + \lambda Q_{n-1}^{[k]*(1)}(0),$$
$$1 \le n \le N-1,\ 0 \le k \le min(a-1, N-n-1), \quad (4.54)$$

using (4.50)-(4.51) and Lemma 4.1, equations (4.53)-(4.54) gives us a recursive method to calculate $Q_n^{[k]*(1)}(0)$ $(0 \le n \le N-1, 0 \le k \le min(a-1, N-n-1))$ in known terms. Now differentiating (4.31) with respect to θ and then setting $\theta = 0$ we obtain $Q_{N-k}^{[k]}$ $(0 \le k \le a-1)$ in completely known terms as follows

$$Q_{N-k}^{[k]} = -\lambda Q_{N-k-1}^{[k]*(1)}(0),\ 0 \le k \le a-1, \quad (4.55)$$

Henceforth, we have completely obtained the joint distribution of queue and server content, joint distribution of queue content as well as type of the vacation taken by the server. Now we may proceed to obtain other significant distribution, which are useful in computing various performance measures, as follows:

- the distribution of queue content, p_n^{queue} $(0 \le n \le N)$, is given by

$$p_n^{queue} = \begin{cases} \delta_s P_{n,0} + \sum_{r=a}^{b} P_{n,r} + \sum_{k=0}^{n} Q_{n-k}^{[k]}, & 0 \le n \le a-1, \\ \sum_{r=a}^{b} P_{n,r} + \sum_{k=0}^{a-1} Q_{n-k}^{[k]} & a \le n \le N. \end{cases}$$

- the distribution of the system content (including number of customers with the server), p_n^{sys} $(0 \le n \le N+b)$, is given by

$$p_n^{sys} = \begin{cases} \delta_s P_{n,0} + \sum_{k=0}^{n} Q_{n-k}^{[k]}, & 0 \le n \le a-1, \\ \sum_{r=a}^{min(b,n)} P_{n-r,r} + \sum_{k=0}^{a-1} Q_{n-k}^{[k]}, & a \le n \le N, \\ \sum_{r=a}^{b} P_{n-r,r}, & N+1 \le n \le N+a, \\ \sum_{r=n-N}^{b} P_{n-r,r}, & N+a+1 \le n \le N+b. \end{cases}$$

- the probability that the server is in dormant state (P_{dor}), busy state $\left(P_{busy}\right)$ and in vacation state (Q_{vac}) are given by $P_{dor}=\sum_{n=0}^{a-1}P_{n,0}$, $P_{busy}=\sum_{r=a}^{b}\sum_{n=0}^{N}P_{n,r}$ and $Q_{vac}=\sum_{k=0}^{a-1}\sum_{n=0}^{N-k}Q_n^{[k]}$, respectively.

- the conditional probability distribution that the server is in $k^{th}-$type vacation given that the server is in vacation is given by $\zeta_k=\sum_{n=0}^{N-k}Q_n^{[k]}/Q_{vac}$ $(0\le k\le a-1)$.

- the conditional probability distribution of the server content given that the server is busy, is given by $p_r^{ser}=\sum_{n=0}^{N}P_{n,r}/P_{busy}\,(a\le r\le b)$.

4.3 Performance measure

The performance measures of the present model are evaluated and presented as follows:

1. Average queue length $(L_q)=\sum_{n=0}^{N}np_n^{queue}$.

2. Average system length $(L)=\sum_{n=0}^{N+b}np_n^{sys}$.

3. Average number of customers with the server $(L_s)=\sum_{r=a}^{b}rp_r^{ser}$.

4. Average vacation type (average number of customer in the queue at vacation initiation epoch) $(\zeta)=\sum_{k=0}^{a-1}k\zeta_k$.

5. Probability of blocking is given by $P_{Block}=\sum_{r=a}^{b}P_{N,r}+\sum_{k=0}^{a-1}Q_{N-k}^{[k]}$.

6. Using Little's law, the average waiting time of a customer in the queue $(W_q)=L_q/\bar{\lambda}$ as well as the system $(W)=L/\bar{\lambda}$, where $\bar{\lambda}$ is the effective arrival rate of the system and is given by $\bar{\lambda}=\lambda\,(1-P_{Block})$.

7. Average queue length when server is in dormancy (L_q^{dor}), busy (L_q^{busy}) and in vacation (L_q^{vac}) are obtained as $L_q^{dor}=\sum_{n=0}^{a-1}nP_{n,0}/P_{dor}$, $L_q^{busy}=\sum_{n=0}^{N}n.p_n^{busy}/P_{busy}$ and $L_q^{vac}=\sum_{n=0}^{N}\sum_{k=0}^{min(n,a-1)}nQ_{n-k}^{[k]}/Q_{vac}$.

8. Average queue length when server is busy with $r\,(a \le r \le b)$ customers $(L_{q,r}^{busy}) = \sum_{n=0}^{N} nP_{n,r}/P_{busy}$.

9. Average queue length when server is in $k^{th}-$type vacation $(L_{q,k}^{vac}) = \sum_{n=k}^{N} n.Q_{n-k}^{[k]}/Q_{vac}$.

4.4 Numerical results

In this section, we have presented few numerical results in form of self explanatory tables and graphs to adjudicate the analytical results obtained in previous section. For this purpose, we have considered $M/G_r^{(a,b)}/1/N$ queue with single vacation (SV) and multiple vacation (MV). Specifically, in Table 4.2 to 4.5 we have considered $M/G_r^{(4,7)}/1/15$ queue with SV (Table 4.2 and 4.3) and MV (Table 4.4 and 4.5), E_2 service time distribution (STD) and deterministic vacation time distribution (VTD) with $\lambda = 2.1$. The service rates of a batch of size r is considered as $\mu_r = \mu/r$ $(a \le r \le b)$, which is represented by IP (inversely proportional) and the queue length dependent vacation rate is considered as $\nu_k = (k+1)\nu$ $(0 \le k \le a-1)$, which is represented by DP (directly proportional). The inputs of the service rates and vacation rates for Table 4.2-4.5 are taken as given in Table 4.1.

Table 4.1: Service and vacation rates for Table 4.2-4.5

Service rate		Vacation rate	
batch size (r)	IP	queue length (k)	DP
4	1.073864	0	0.729167
5	0.859091	1	1.458333
6	0.715909	2	2.1875
7	0.613636	3	2.916667

Table 4.2 and Table 4.3 present the service/vacation completion-epoch joint probabilities and arbitrary epoch joint probabilities, respectively, for SV, where as, Table 4.4 and Table 4.5 present the same results for MV. These results are presented here to show the numerical compatibility of our analytical results. The important performance measures of the queueing

model under consideration are also presented at the bottom of the Table 4.3 and Table 4.5. Column 2 to 5 of Table 4.2 (Table 4.4) present the joint distribution of queue length and serving batch size at service completion epoch and column 7 to 10 present the joint distribution of queue length and vacation type at vacation completion epoch. The 6th column and last column (i.e., the 11th column) of Table 4.2 (Table 4.4) display the marginal distribution of the number of customers in the queue at service compilation epoch and vacation termination epoch, respectively. Column 3 to 6 of Table 4.3 and column 2 to 5 of Table 4.5 present the joint distribution of queue length and serving batch size at arbitrary epoch, and column 8 to 11 of Table 4.3 and column 7 to 10 of Table 4.5 present the joint distribution of queue length and type of the vacation at arbitrary epoch. The last but one row of column 2 to 5 of Table 4.3 (Table 4.5) display the distribution of the number of customer with the server, column 7 to 10 display the distribution of the queue length at vacation initiation epoch. The last column, i.e., the 13th column of Table 4.3 (12th column of Table 4.5) display the queue length distribution at arbitrary epoch irrespective of the state of the server. The 7th column and 12th column of Table 4.3 (6th column and 11th column of Table 4.5) display the queue length distribution at arbitrary epoch during server's busy and vacation period, respectively.

After tabular representation we present few numerical results in the form of graphs. In Figures 4.1-4.4, the behavior of the performance measures of the queuing model under consideration for SV as well as MV is examined. For this purpose the input parameters are considered as $a = 4$, $b = 7$ and $\lambda = 3.0$. The service rates (μ_r, $a \leq r \leq b$) and vacation rates (ν_k, $0 \leq k \leq a-1$) are chosen here in such a way that the mean service time of a batch of size r, and mean vacation time of k-th type of vacation will remain same, irrespective of the different service time distribution (STD), e.g., exponential (EXS), deterministic (DTS), Erlang (ERS)), and different vacation time distribution (VTD), e.g., exponential (EXV), deterministic (DTV), Erlang (ERV). The batch size dependent service rates are considered as $\mu_r = \mu/r$ ($a \leq r \leq b$), i.e., inversely proportional (IP), and the queue length dependent vacation rates are considered as $\nu_k = (k+1)\nu$ ($0 \leq k \leq a-1$), i.e., directly proportional (DP). Hence the other input parameters, as considered for Figures 4.1-4.4, are given in Table 4.6.

Table 4.2: Joint distributions at departure epoch for $M/G_r^{(4,7)}/1/15$ queue with SV, E_2 STD, deterministic VTD and $\lambda = 2.1$.

n	$p_{n,4}^{+}$	$p_{n,5}^{+}$	$p_{n,6}^{+}$	$p_{n,7}^{+}$	p_n^{+}	$q_n^{[0]+}$	$q_{n-1}^{[1]+}$	$q_{n-2}^{[2]+}$	$q_{n-3}^{[3]+}$	q_n^{+}
0	0.111261	0.012533	0.004830	0.002029	0.130652	0.007334				0.007334
1	0.110011	0.013786	0.005744	0.003688	0.133228	0.021122	0.031565			0.052688
2	0.081581	0.011373	0.005123	0.004511	0.102588	0.030416	0.045454	0.039280		0.115150
3	0.053776	0.008340	0.004061	0.004630	0.070808	0.029199	0.032727	0.037709	0.034466	0.134101
4	0.033232	0.005734	0.003018	0.004301	0.046286	0.021024	0.015709	0.018100	0.024815	0.079648
5	0.019715	0.003784	0.002154	0.003746	0.029399	0.012110	0.005655	0.005792	0.008934	0.032490
6	0.011371	0.002428	0.001494	0.003117	0.018411	0.005813	0.001629	0.001390	0.002144	0.010975
7	0.006425	0.001526	0.001015	0.002509	0.011476	0.002391	0.000391	0.000267	0.000386	0.003435
8	0.003573	0.000944	0.000679	0.002048	0.007245	0.000861	0.000080	0.000043	0.000056	0.001040
9	0.001963	0.000577	0.000449	0.001586	0.004575	0.000275	0.000014	0.000006	0.000007	0.000302
10	0.001067	0.000349	0.000293	0.001186	0.002896	0.000079	0.000002	0.000001	0.000001	0.000083
11	0.000576	0.000210	0.000190	0.000865	0.001841	0.000021	0.000000	0.000000	0.000000	0.000021
12	0.000308	0.000125	0.000123	0.000620	0.001176	0.000005	0.000000	0.000000	0.000000	0.000005
13	0.000164	0.000074	0.000079	0.000438	0.000754	0.000001	0.000000	0.000000	0.000000	0.000001
14	0.000087	0.000044	0.000050	0.000306	0.000486	0.000000	0.000000	0.000000	0.000000	0.000000
15	0.000046	0.000026	0.000032	0.000799	0.000902	0.000000	0.000000	0.000000	0.000000	0.000000
Total	0.435157	0.061854	0.029333	0.036380	0.562724	0.130652	0.133228	0.102588	0.070808	0.437276

Table 4.3: Joint distributions at arbitrary epoch for $M/G_r^{(4,7)}/1/15$ queue with SV, E_2 STD, deterministic VTD and $\lambda = 2.1$.

n	$P_{n,0}$	$P_{n,4}$	$P_{n,5}$	$P_{n,6}$	$P_{n,7}$	p_n^{busy}	$Q_n^{[0]}$	$Q_{n-1}^{[1]}$	$Q_{n-2}^{[2]}$	$Q_{n-3}^{[3]}$	$q_n^{[vac]}$	p_n^{queue}
0	0.002955	0.130520	0.019886	0.009894	0.005190	0.165491	0.049685				0.049685	0.218131
1	0.024183	0.086196	0.014332	0.007580	0.007043	0.115150	0.041175	0.040960			0.082135	0.221469
2	0.070578	0.053327	0.009749	0.005516	0.007190	0.075782	0.028920	0.022647	0.025507		0.077074	0.223434
3	0.124608	0.031660	0.006389	0.003880	0.006525	0.048454	0.017156	0.009461	0.010314	0.014642	0.051573	0.224634
4		0.018271	0.004079	0.002664	0.005542	0.030555	0.008685	0.003132	0.003021	0.004644	0.019482	0.050037
5		0.010327	0.002554	0.001796	0.004509	0.019186	0.003806	0.000853	0.000687	0.001045	0.006391	0.025577
6		0.005746	0.001575	0.001194	0.003557	0.012072	0.001464	0.000197	0.000127	0.000181	0.001969	0.014042
7		0.003157	0.000960	0.000785	0.002742	0.007645	0.000501	0.000039	0.000020	0.000025	0.000585	0.008230
8		0.001717	0.000580	0.000511	0.002281	0.005089	0.000154	0.000007	0.000003	0.000003	0.000166	0.005255
9		0.000926	0.000347	0.000331	0.001642	0.003246	0.000043	0.000001	0.000000	0.000000	0.000045	0.003290
10		0.000496	0.000207	0.000212	0.001164	0.002079	0.000011	0.000000	0.000000	0.000000	0.000011	0.002090
11		0.000264	0.000122	0.000136	0.000815	0.001337	0.000003	0.000000	0.000000	0.000000	0.000003	0.001340
12		0.000140	0.000072	0.000086	0.000565	0.000863	0.000001	0.000000	0.000000	0.000000	0.000001	0.000864
13		0.000074	0.000042	0.000055	0.000389	0.000559	0.000000	0.000000	0.000000	0.000000	0.000000	0.000560
14		0.000039	0.000025	0.000034	0.000266	0.000363	0.000000	0.000000	0.000000	0.000000	0.000000	0.000363
15		0.000042	0.000034	0.000058	0.000548	0.000682	0.000000	0.000000	0.000000	0.000000	0.000000	0.000682
Total	0.222324 (P_{dor})	0.342902	0.060954	0.034731	0.049969	0.488555 (P_{busy})	0.151604	0.077297	0.039680	0.020541	0.289121 (Q_{vac})	1.000000

$L = 4.186942$, $W = 1.995143$, $L_q = 1.952399$, $W_q = 0.930349$, $P_{Block} = 0.000682$,
$L_s = 4.573782$, $\zeta = 0.754975$, $L_q^{dor} = 2.425118$, $L_q^{busy} = 1.831020$, $L_q^{vac} = 1.793999$

Table 4.4: Joint distributions at departure epoch for $M/G_r^{(4,7)}/1/15$ queue with MV, E_2 STD, deterministic VTD and $\lambda = 2.1$.

n	$p_{n,4}^{+}$	$p_{n,5}^{+}$	$p_{n,6}^{+}$	$p_{n,7}^{+}$	p_n^{+}	$q_n^{[0]+}$	$q_{n-1}^{[1]+}$	$q_{n-2}^{[2]+}$	$q_{n-3}^{[3]+}$	q_n^{+}
0	0.048989	0.015896	0.004625	0.001550	0.071060	0.004226				0.004226
1	0.048439	0.017486	0.005500	0.002744	0.074169	0.012171	0.026808			0.038979
2	0.035921	0.014426	0.004905	0.003308	0.058560	0.017527	0.038603	0.071161		0.127291
3	0.023678	0.010579	0.003889	0.003370	0.041516	0.016826	0.027794	0.068315	0.146478	0.259413
4	0.014633	0.007273	0.002890	0.003120	0.027916	0.012115	0.013341	0.032791	0.105464	0.163711
5	0.008681	0.004800	0.002062	0.002714	0.018257	0.006978	0.004803	0.010493	0.037967	0.060241
6	0.005007	0.003080	0.001431	0.002259	0.011777	0.003349	0.001383	0.002518	0.009112	0.016363
7	0.002829	0.001936	0.000972	0.001821	0.007558	0.001378	0.000332	0.000484	0.001640	0.003834
8	0.001573	0.001198	0.000650	0.001492	0.004913	0.000496	0.000068	0.000077	0.000236	0.000878
9	0.000864	0.000732	0.000430	0.001158	0.003184	0.000159	0.000012	0.000011	0.000028	0.000210
10	0.000470	0.000443	0.000281	0.000867	0.002061	0.000046	0.000002	0.000001	0.000003	0.000052
11	0.000253	0.000266	0.000182	0.000633	0.001335	0.000012	0.000000	0.000000	0.000000	0.000013
12	0.000136	0.000158	0.000117	0.000454	0.000866	0.000003	0.000000	0.000000	0.000000	0.000003
13	0.000072	0.000094	0.000075	0.000321	0.000562	0.000001	0.000000	0.000000	0.000000	0.000001
14	0.000038	0.000055	0.000048	0.000224	0.000366	0.000000	0.000000	0.000000	0.000000	0.000000
15	0.000020	0.000032	0.000030	0.000603	0.000686	0.000000	0.000000	0.000000	0.000000	0.000000
Total	0.191605	0.078453	0.028089	0.026638	0.324785	0.075286	0.113148	0.185852	0.300929	0.675215

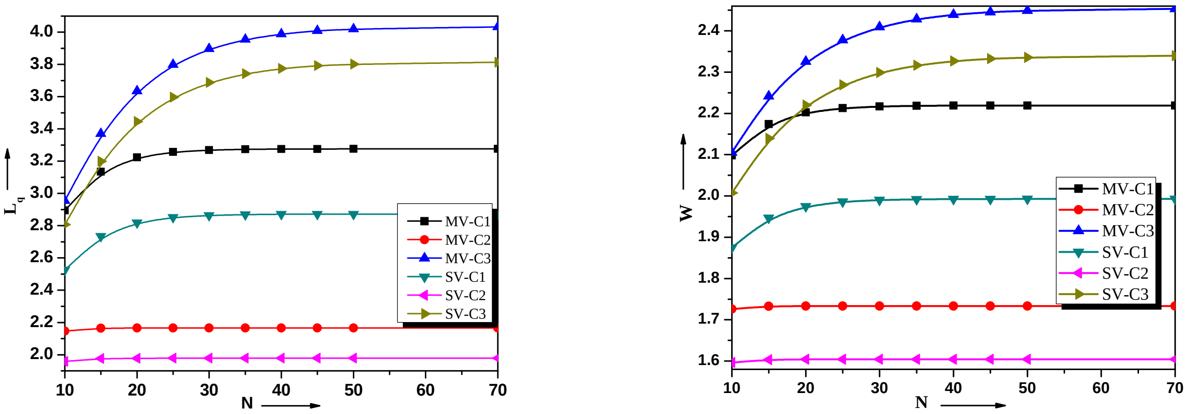
Table 4.5: Joint distributions at arbitrary epoch for $M/G_r^{(4,7)}/1/15$ queue with MV, E_2 STD, deterministic VTD and $\lambda = 2.1$.

n	$P_{n,4}$	$P_{n,5}$	$P_{n,6}$	$P_{n,7}$	p_n^{busy}	$Q_n^{[0]}$	$Q_{n-1}^{[1]}$	$Q_{n-2}^{[2]}$	$Q_{n-3}^{[3]}$	$q_n^{[vac]}$	p_n^{queue}
0	0.094165	0.041328	0.015524	0.006497	0.157514	0.046912				0.046912	0.204426
1	0.062187	0.029785	0.011893	0.008509	0.112374	0.038876	0.056999			0.095875	0.208249
2	0.038473	0.020261	0.008655	0.008565	0.075954	0.027306	0.031514	0.075715		0.134535	0.210489
3	0.022841	0.013278	0.006087	0.007735	0.049941	0.016198	0.013165	0.030616	0.101964	0.161943	0.211884
4	0.013181	0.008476	0.004179	0.006564	0.032402	0.008200	0.004358	0.008968	0.032340	0.053866	0.086268
5	0.007451	0.005308	0.002818	0.005346	0.020922	0.003594	0.001187	0.002041	0.007275	0.014097	0.035019
6	0.004145	0.003274	0.001873	0.004226	0.013519	0.001383	0.000274	0.000378	0.001260	0.003294	0.016813
7	0.002278	0.001996	0.001232	0.003266	0.008771	0.000473	0.000055	0.000059	0.000177	0.000763	0.009534
8	0.001239	0.001205	0.000802	0.002734	0.005980	0.000145	0.000010	0.000008	0.000021	0.000184	0.006164
9	0.000668	0.000722	0.000519	0.001969	0.003878	0.000041	0.000002	0.000001	0.000002	0.000045	0.003923
10	0.000358	0.000430	0.000333	0.001397	0.002518	0.000010	0.000000	0.000000	0.000000	0.000011	0.002529
11	0.000191	0.000254	0.000213	0.000979	0.001637	0.000002	0.000000	0.000000	0.000000	0.000002	0.001639
12	0.000101	0.000150	0.000135	0.000679	0.001065	0.000001	0.000000	0.000000	0.000000	0.000001	0.001066
13	0.000053	0.000088	0.000086	0.000467	0.000694	0.000000	0.000000	0.000000	0.000000	0.000000	0.000694
14	0.000028	0.000051	0.000054	0.000319	0.000453	0.000000	0.000000	0.000000	0.000000	0.000000	0.000453
15	0.000031	0.000071	0.000090	0.000660	0.000852	0.000000	0.000000	0.000000	0.000000	0.000000	0.000852
Total	0.247390	0.126676	0.054493	0.059913	0.488473 (P_{busy})	0.143141	0.107563	0.117786	0.143038	0.511527 Q_{vac}	1.000000

$L = 4.490809$, $W = 2.140303$, $L_q = 2.121516$, $W_q = 1.011107$, $P_{Block} = 0.000852$,
$L_s = 4.850407$, $\zeta = 1.509693$, $L_q^{busy} = 1.960536$, $L_q^{vac} = 2.275242$,

Table 4.6: Service and vacation rates for Figure 4.1-4.4.

Service rate		Vacation rate	
batch size (r)	IP	queue length (k)	DP
4	1.250	0	0.7500
5	1.0000	1	1.5000
6	0.83333	2	2.2500
7	0.714286	3	3.0000



(a) L_q vs N (b) W vs N

Figure 4.1: Effect of N on L_q and W

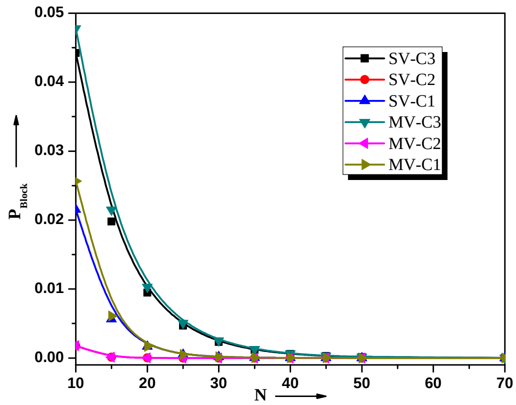


Figure 4.2: Effect of N on P_{Block}

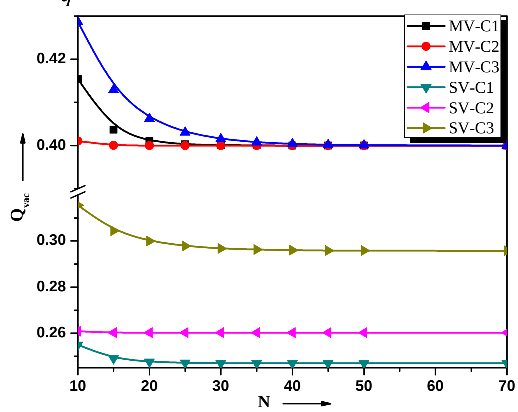


Figure 4.3: Effect of N on Q_{vac}

In Figures 4.1-4.4 we have considered the following three combinations of STD and VTD.

C1 - STD : E_4 (ERS) and VTD : exponential (EXV) ;

C2 - STD : deterministic (DTS) and VTD : E_4 (ERV) ;

C3 - STD : exponential (EXS) and VTD : deterministic (DTV).

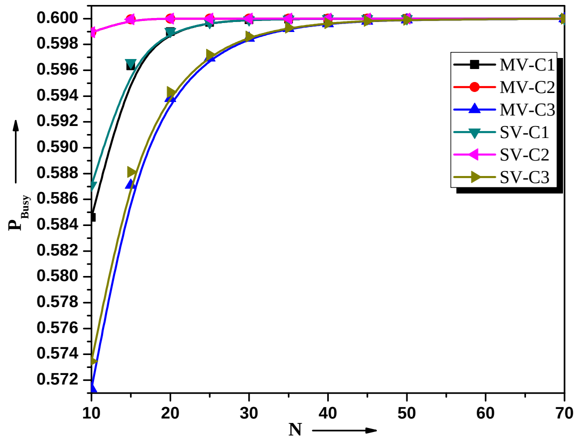


Figure 4.4: Effect of N on P_{Busy}

The effect of N on some important performance measures, viz., L_q, W, P_{Block}, Q_{vac}, P_{Busy} are studied in Figures 4.1-4.4. It is clearly observed from Figures 4.1a and 4.1b that as N increases, L_q and W both increases initially, however, for large N no significant changes in the values of L_q are observed, i.e., the values are almost constant. Also Figure 4.2 reveals that as N increases P_{Block} initially decreases rapidly, however, for large N (i.e., $N \geq 45$) the values of P_{Block}, for all combinations of STD and VTD and for SV as well as MV, merged to each other and are almost zero. These observations help us in concluding that for large values of N, finite buffer model is behaves like an infinite buffer queue, which is quite obvious. It is also observed from Figure 4.1a that L_q and W are less for SV in comparison to MV, for all considered combinations of STD and VTD, viz., C1, C2 and C3. The similar behavior is also observed for P_{Block} and Q_{vac} from Figures 4.2 and 4.3, respectively. Such behavior is quite obvious, as in SV the availability of the server to the system is more in comparison to MV. Another observation can be made from Figures 4.1, 4.2 and 4.3 that the combination C2, among the other considered STD and VTD combinations, gives the lowest value for L_q, W, P_{Block} and Q_{vac}.

Figure 4.4 presents the effect of N on P_{Busy}. It is observed that as N increases P_{Busy} initially increases rapidly but for large N the values of P_{Busy}, for all considered combinations (i.e., C1, C2 and C3) and SV as well as MV, merged together and attains a constant value, which is quite obvious. As it is already evident from Figure 4.2 that for $N < 30$, P_{Block} decreases rapidly, hence, it is expected that for $N < 30$ there will be more customers in the queue to be served, which will eventually increase P_{Busy}. However, for large N no effect on P_{Busy} is seen

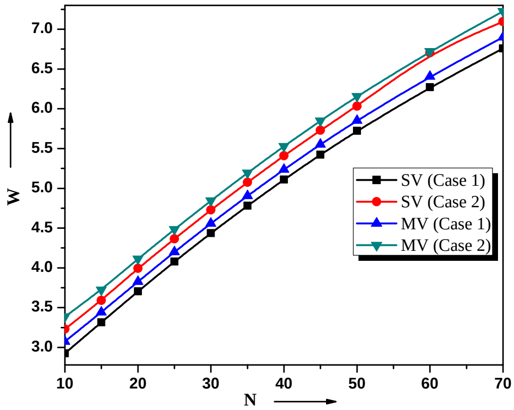


Figure 4.5: Effect of N on W for Case 1 and Case 2

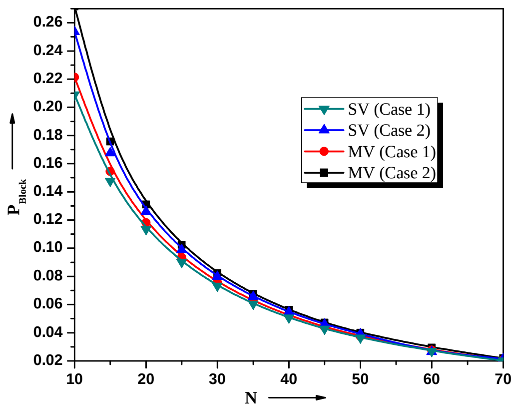


Figure 4.6: Effect of N on P_{Block} for Case 1 and Case 2

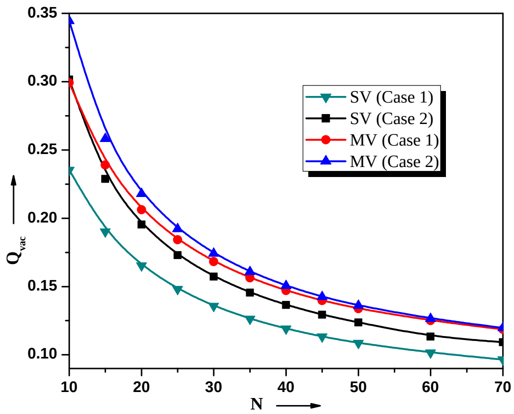


Figure 4.7: Effect of N on Q_{vac} for Case 1 and Case 2

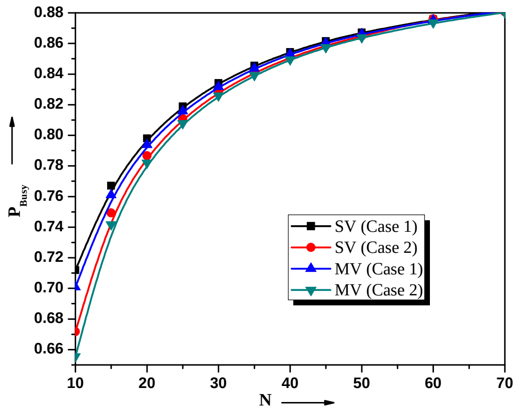


Figure 4.8: Effect of N on P_{Busy} for Case 1 and Case 2

as the system behaves like an infinite buffer queue.

To justify the applicability of our considered model, in Figures 4.5-4.8, we present a comparison between queue length dependent vacation with queue length independent vacation, by considering the batch size dependent service for both the scenario. For this purpose we consider $M/G_r^{(6,8)}/1/N$ queueing system with SV and MV and STD is taken as EXS, and VTD as ERS (E_4 distribution). The batch size dependent service rates are considered as $\mu_r = \mu/r$ $(a \le r \le b)$ with $\mu = 5.0$ and the arrival rate is taken as $\lambda = 4.5$.

For the comparison purpose we have considered the following two cases:

Case 1. The queue length dependent vacation rates are taken as $\nu_k = (\frac{k}{4}+1)\nu$ $(0 \le k \le a-1)$ with $\nu = 0.75$.

Case 2. The queue length independent vacation rate is considered as $\nu_k = \nu_0$ $(0 \le k \le a-1)$.

The assumptions for vacation rates, in Case 1 and Case 2, are made in such a way that for Case 2 the server always takes a vacation with constant vacation rate v_0 irrespective of the queue length at vacation initiation epoch, and for Case 1 the server starts a vacation with vacation rate v_0, when it finds an empty queue and starts a vacation with higher vacation rate (i.e., $v_k > v_{k-1}$, $k = 1, 2, ...a-1$.) depending on queue length. These assumptions ensure us that due to queue length dependent vacation (Case 1) the server is modulating the mean length of the vacation in such a way that the server takes a longer vacation when queue is empty and shorter vacation when queue is non empty. It is observed from Figures 4.5-4.8 that the behavior of the performance measures considered in the respective figures, i.e., W, P_{Block}, Q_{vac} and P_{Busy}, is almost similar to that of Figures 4.1b, 4.2, 4.3 and 4.4. Figures 4.5, 4.6 and 4.7 reveal that the values of W, P_{Block} and Q_{vac} are less for Case 1 in comparison to Case 2, and Figure 4.8 reveals that the values of P_{Busy} is more for Case 1 in comparison to Case 2. This study leads to the fact that the inclusion of queue length dependent vacation helps in reducing congestion at the batch size dependent bulk service queues.

Table 4.7: Performance measures corresponding to different values of λ for $M/G_r^{(10,16)}/1/20$ with SV, $STD \sim Exponential$ and $VTD \sim E_4$ for Case 1 and Case 2

	Case 1				Case 2			
λ	4.0	6.0	8.0	10.0	4.0	6.0	8.0	10.0
L	16.88119	23.18483	26.92517	29.14286	17.94512	24.23872	27.62527	29.54322
W	4.769535	5.424648	5.881799	6.176653	5.149569	5.845646	6.24375	6.471861
L_q	8.142614	11.48849	13.68984	15.07414	8.807857	12.23136	14.27321	15.5013
W_q	2.300577	2.688008	2.990544	3.194873	2.527521	2.949835	3.225971	3.395781
L_s	12.34479	13.6832	14.45627	14.9089	13.11024	14.47905	15.08889	15.3804
P_{dor}	0.1495326	0.04647087	0.01634014	0.006625155	0.08720	0.019198	0.0051995	0.001730
P_{Busy}	0.7078757	0.8547959	0.9155419	0.9436456	0.6969561	0.8292912	0.8848935	0.9129743
Q_{vac}	0.1425917	0.09873322	0.06811792	0.0497292	0.2158422	0.1515099	0.1099069	0.08529525
P_{Block}	0.1151554	0.2876701	0.4277863	0.5281772	0.1288048	0.308924	0.4469415	0.5435128

Tables 4.7 and 4.8 present various performance measures corresponding to different values of λ for both Case 1 and Case 2 and SV and MV, respectively. It is observed from Tables 4.7 and 4.8 that L, W, L_q, W_q and L_s increases whereas Q_{vac} decreases with increase in λ. This behavior is quite obvious as increase in arrival rate will increase queue (system) length, waiting time etc., which eventually decreases the chance that the server takes vacation. Also it is observed from the tables that L, W, L_q, W_q and L_s are less for Case 1 in comparison to Case 2 where as P_{Busy} is more for Case 1 in comparison to Case 2, which leads to the conclusion

Table 4.8: Performance measures corresponding to different values of λ for $M/G_r^{(10,16)}/1/20$ with MV, $STD \sim Exponential$ and $VTD \sim E_4$ for Case 1 and Case 2

	Case 1				Case 2			
λ	4.0	6.0	8.0	10.0	4.0	6.0	8.0	10.0
L	17.59321	23.63894	27.18162	29.29411	19.20386	24.73066	27.79963	29.60854
W	5.016344	5.585923	5.980266	6.23793	5.61462	6.049122	6.332886	6.513986
L_q	8.535636	11.72838	13.81454	15.14089	9.591031	12.56753	14.4048	15.55613
W_q	2.433762	2.771437	3.039356	3.224123	2.804124	3.074019	3.281481	3.422405
L_s	12.91291	14.07243	14.70455	15.06904	14.05248	14.87551	15.25702	15.45791
P_{Busy}	0.7014355	0.8463755	0.9090438	0.9392253	0.6840662	0.8176612	0.8779449	0.9090758
Q_{vac}	0.2985645	0.1536245	0.09095623	0.06077474	0.3159338	0.1823388	0.1220551	0.09092418
P_{Block}	0.1232057	0.2946871	0.4318476	0.5303874	0.1449173	0.3186157	0.4512844	0.5454621

that queue length dependent vacation helps in reducing congestion.

To further justify our model we present here a numerical cost optimization problem. For this purpose we consider the bus depot example as presented in Section 4.1.

For constructing the cost model we define :

$C_h \equiv$ Holding cost per customer per unit time in the queue.

$C_I \equiv$ Idleness cost of the server per unit time.

$C_o \equiv$Operating (serving) cost per customer per unit time.

$C_r \equiv$ Rejection cost per unit time.

Thus in long run the average holding cost, idleness cost, operating cost, and rejection cost are given by C_hL_q, $C_I(\delta_sP_{dor}+Q_{vac})$, C_oL_s and λC_rP_{Block}, respectively.

Therefore, in long run, the total system cost is given by

$$TSC = C_hL_q + C_I(\delta_sP_{dor}+Q_{vac}) + C_oL_s + \lambda C_rP_{Block}.$$

Now the penalty cost of the server (driver), which is incurred to the server when enough number of passengers (to start a bus) are waiting at the bus stop while driver is on vacation. If C_p be the penalty cost per unit time, then in long run the penalty cost to the server (driver) is given by

$$PCS = \sum_{n=a}^{N} \sum_{k=0}^{a-1} C_pQ_{n-k}^{[k]}.$$

To demonstrate the example numerically (Figures 4.9-4.14) we consider the cost parameters as C_h= 0.5 unit, C_I = 3 unit, C_o = 5.5 unit, C_r = 2.5 unit C_p = 100 unit.

The other system parameters are considered as follows :

- vacation rates (taken by the driver) are considered same as discussed above in **Case 1** and **Case 2**.
- batch size dependent service rates are considered as $\mu_r = \mu/r$ $(a \leq r \leq b)$ with $\mu = 5.0$.
- service and vacation time distributions are taken as exponential and E_4 distributions, respectively.

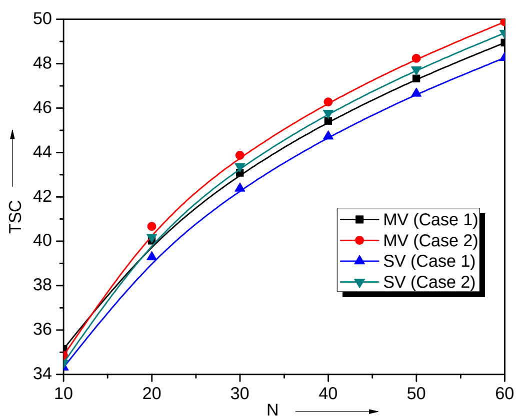


Figure 4.9: Effect of N on TSC

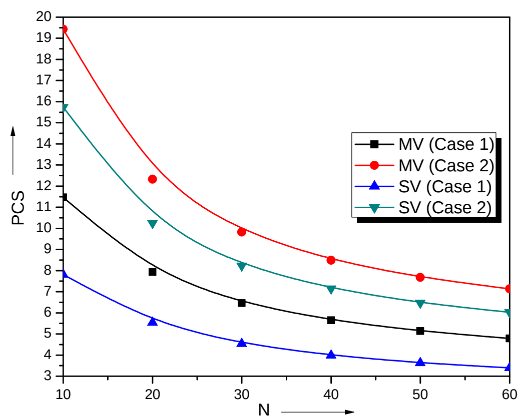


Figure 4.10: Effect of N on PCS.

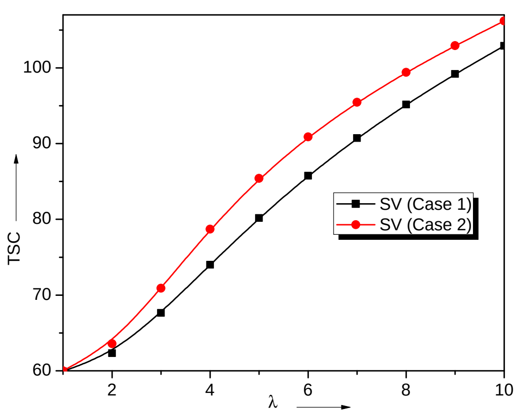


Figure 4.11: Effect of λ on TSC for SV

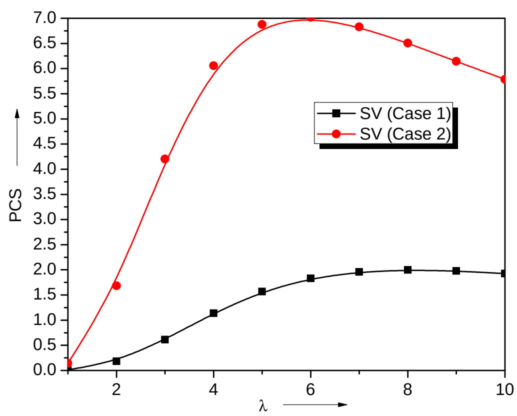


Figure 4.12: Effect of λ on PCS for SV

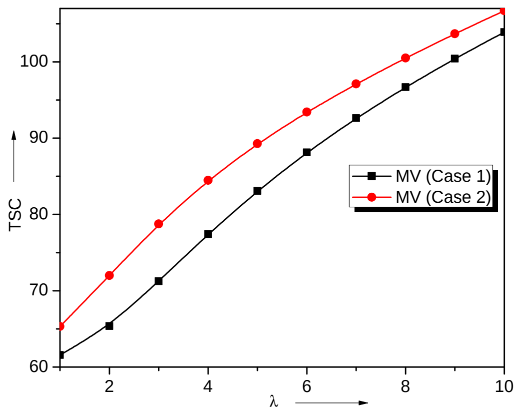


Figure 4.13: Effect of λ on TSC for MV

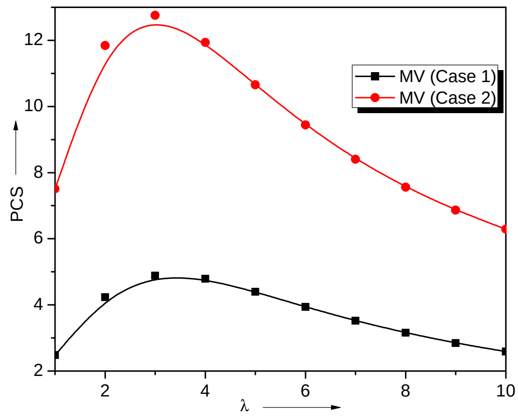


Figure 4.14: Effect of λ on PCS for MV

Figures 4.9 - 4.10 present the effect of increase in N on TSC and PCS by keeping λ, a and b fixed at $\lambda = 4.5$, $a = 6$ and $b = 8$, respectively, and varying N from 10 to 60. Figures 4.11 - 4.14, present the effect of increase in λ on TSC and PCS by keeping a, b and N fixed at $a = 10$,$b = 16$ and $N = 20$, respectively, and varying λ from 1.0 to 10.0.

It is observed from Figures 4.9 - 4.14 that TSC and PCS both are less for Case 1 in comparison to Case 2. Hence, one can conclude that while driver is taking shorter vacation by observing increased queue length at vacation initiation epoch, then TSC and PCS are less in comparison to the situation when driver takes vacation of constant duration, for both SV and MV. It is also evident from Figure 4.9 that as N increases TSC increases and this behavior is because of the increase in P_{Busy}, L (as L is proportional to W) and decrease in P_{Block} and Q_{vac} with increase in N, as demonstrated in Figures 4.5-4.8. Figure 4.10 demonstrates that the increase in N, decreases PCS and this is due to decrease in Q_{vac} with increase in N as already displayed in Figure 4.7. Figures 4.11 and 4.13 present that as λ increases, TSC increases, for SV and MV, this is because it is evident from Tables 4.7 and 4.8 that L_q, L_s, P_{Block} increases with increase in λ, only Q_{vac} (and P_{dor} for SV) decreases with increase in λ. It is expected that as λ increases the probability that during server's vacation queue length is greater than or equal to a will initially increase, however, after certain time this probability will start decreasing, as chances that the server is in vacation decreases with increase in λ (which is clearly evident from Tables 4.7-4.8 for SV and MV). Because of this it is observed from Figures 4.12 and 4.14 that, as λ increases PCS increases initially and then started decreasing for SV and

MV, and for Case 1 and 2, respectively.

4.5 Concluding remarks

In this chapter, we have studied a finite capacity bulk service queue with two types of vacation rules, viz., single vacation and multiple vacation. The service time, which depends on the size of the batches under service, is considered to be generally distributed. The vacation time, which depends on the queue length at vacation initiation epoch, is also considered to be generally distributed. We have analytically studied the model and obtained the steady state joint probabilities at various epochs. Several illustrative numerical studies are also presented here to show the impact of the various combinations of service time and vacation time distributions on selected system performance measures. We have also established the fact that the implementation of queue length dependent vacation in batch size dependent bulk service queue further reduces the congestion, which is measured in terms of mean waiting time, and blocking probability, etc. The analysis presented in this chapter can be extended to analyze more complex queuing models involving bulk arrival following compound Poison process, correlated arrival and/or service processes, etc. In the next chapter we have studied bulk arrival bulk service queue with batch size dependent service and queue length dependent single and multiple vacation.