Chapter 3

Analysis of infinite buffer $M/M^{(a,b)}/1$ queue with system size based balking

3.1 Introduction

Bulk service queues with impatient customers did not get much attention in literature. Few researchers (Tadj et al. (1998), Wang et al. (2014), Islam et al. (2014), etc.) studied the impatient phenomenon in bulk service queue. Tadj et al. (1998) considered the 'system size based' balking with a certain threshold policy in a bulk arrival bulk service queue under 'fixed batch size' service rule, and obtained the queue length distribution in steady state. Recently, the impatient behavior of the passengers in public transport is mathematically modeled as a bulk arrival bulk service queuing system by Wang et al. (2014). Further they analyzed the mathematical model with GBS rule for service, and obtained the mean and variance for the queue length. On public transportation problem, the situations in which passengers abandon the system after a certain amount of waiting time in a bulk arrival bulk service queuing model is analyzed by Islam et al. (2014) and studied the impact of headway variations and passenger waiting behavior on public transit performance. The recent literature on bulk service queueing models with impatient

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customers are focused only on obtaining queue length distribution. However, in recent studies on batch service queues authors already have established the fact that congestion in bulk service queues can be controlled by applying batch size dependent service policy in which joint distribution of the queue content and server content plays an important role (Banerjee et al. (2011), Banerjee and Gupta (2012), Banerjee et al. (2013), Banerjee et al. (2014), Maity and Gupta (2015), Banerjee et al. (2015), Yu and Alfa (2015), Pradhan and Gupta (2017a,b)).

In this chapter we have considered an infinite buffer bulk service Poisson queue with system size based balking. The general bulk service rule is considered for service rendered by a single server. Using probability generating function method we obtained the joint distribution of the queue content and the server content, in steady state, for two special cases involving two particular form of balking probabilities. A practical application of the model under consideration is as follows. Let us suppose that a manufacturer is providing service in terms of production, for the orders (customers) of a certain kind of commodity with FCFS discipline, in which the production does not start until some specified number of orders, say a, are accumulated during an idle period of the server. Once the number of the orders reaches to a, the production starts. At each busy period the server serves min(b, queue length $\geq a$) number of orders at a time and at the end of the production (or a busy period) the server checks for the number of orders waiting in the queue, if it is found to be less than a then the server will remain idle, otherwise, start production. Now during idle period of the server a joining order may get impatient and balk from the system with certain probability, depending on the number of orders that are already present ahead of him, because the server is not ready for the production. The orders may also balk during the busy period of the server with certain probability by looking into the queue length (number of orders ahead of him) and expecting some delay, at the joining time to the system. In the above described situation our model may be useful for the mathematical investigation of the system performance.

The outline of the chapter is as follows. In Section 3.2, we briefly described the model and obtained the steady state joint probabilities of queue content and server content in terms of the steady state probability of the queue content. Then in Section 3.2.1, we obtained the probabilities of queue content by considering special cases for two particular forms of the balking probability using the probability generating method which finally leads to the joint distribution of queue content and server content in completely known terms. Section 3.3 is assigned to present various important performance measures. Several numerical examples are presented in Section 3.4. Some conclusions are drawn in Section 3.5 followed by the references.

3.2 Model description and steady state analysis

In this chapter, an infinite buffer single server bulk service queue is considered where customers are arriving to the system following the Poisson process with rate λ, and are served in batches according to the general bulk service (GBS (a,b)) rule. In GBS rule, customers are served in batches of minimum size 'a' with a maximum threshold limit 'b'. That is, if server finds less than 'a' customers in the queue it will be idle until the queue length reaches to 'a' and immediately resumes its service of a batch of size 'a'. However, if the server finds that the queue length is greater than 'a', the server immediately renders service to a batch of size $min\{\text{queue length}, b\}$. The service time distribution is considered to be exponentially distributed with mean $\frac{1}{\mu}$. On arrival a customer first observes the state of the system, i.e., how many customers are waiting in the queue and whether server is idle or busy. When a customer finds that the server is idle and n $(0 \le n \le a-1)$ customers are waiting in queue, then either he decides to join the system with probability '$\beta_n^{'}$' or balk with probability $(1-\beta_n^{'})$. It is obvious that the balking probability of an arriving customer is always zero when he finds queue size is $a-1$ and the server is idle, i.e., $\beta_{a-1}^{'} = 1$ and $0 < \beta_n^{'} \le 1$ $(0 \le n \le a-1)$. Further, if an arrival finds that the server is busy and the queue size is n $(n \ge 0)$ then he/she decides to join the system with probability 'β_n' or balk with probability $(1-\beta_n)$, $0 < \beta_n \le 1$.

The stochastic process related to the queueing model under consideration are described here below

- $N(t) \equiv$ the number of customers present in the queue, at time t,

- $\xi(t) \equiv$ state of the server, i.e., whether idle or busy, at time t, hence,

$$\xi(t) = \begin{cases} 0, & \text{if server is idle,} \\ r, & \text{if server is busy in serving } r \text{ customers } (a \le r \le b). \end{cases}$$

Clearly, $\{N(t), \xi(t)\}$ constitute a two dimensional continuous time Markov chain with state space

$\Omega = \{(n,0) : 0 \le n \le a-1\} \cup \{(n,r) : 0 \le n \le N, a \le r \le b\}$.

Let us further define the following joint state possibilities, at time t, as follows :

- $P_{n,0}(t) \equiv prob.\{N(t) = n,\ \xi(t) = 0\}, \quad 0 \le n \le a-1,$
- $P_{n,r}(t) \equiv prob.\{N(t) = n,\ \xi(t) = r\}, \quad n \ge 0, a \le r \le b,$

i.e., $P_{n,0}(t)$ represents the probability that, at time t, there are n $(0 \le n \le a-1)$ customers are present in the queue and server is idle; $P_{n,r}(t)$ represents the probability that, at time t, there are n customers are present in the queue and server is busy in serving r $(a \le r \le b)$ customers.

Relating the state of the system at time t and $t+dt$ the Kolmogrov equations of the model under consideration is given by

$$\frac{d}{dt}P_{0,0}(t) = -\lambda\beta_0' P_{0,0}(t) + \mu\sum_{r=a}^{b} P_{0,r}(t), \tag{3.1}$$

$$\frac{d}{dt}P_{n,0}(t) = -\lambda\beta_n' P_{n,0}(t) + \lambda\beta_{n-1}' P_{n-1,0}(t) + \mu\sum_{r=a}^{b} P_{n,r}(t),\ 1 \le n \le a-2, \tag{3.2}$$

$$\frac{d}{dt}P_{a-1,0}(t) = -\lambda P_{a-1,0}(t) + \lambda\beta_{a-2}' P_{a-2,0}(t) + \mu\sum_{r=a}^{b} P_{a-1,r}(t), \tag{3.3}$$

$$\frac{d}{dt}P_{0,a}(t) = -(\lambda\beta_0 + \mu)P_{0,a}(t) + \lambda P_{a-1,0}(t) + \mu\sum_{r=a}^{b} P_{a,r}(t), \tag{3.4}$$

$$\frac{d}{dt}P_{0,n}(t) = -(\lambda\beta_0 + \mu)P_{0,n}(t) + \mu\sum_{r=a}^{b} P_{n,r}(t),\ a+1 \le n \le b, \tag{3.5}$$

$$\frac{d}{dt}P_{n,r}(t) = -(\lambda\beta_n + \mu)P_{n,r}(t) + \lambda\beta_{n-1} P_{n-1,r}(t), a \le r \le b-1, n \ge 1, \tag{3.6}$$

$$\frac{d}{dt}P_{n,b}(t) = -(\lambda\beta_n + \mu)P_{n,b}(t) + \lambda\beta_{n-1} P_{n-1,b}(t) + \mu\sum_{r=a}^{b} P_{n+b,r}(t), n \ge 1. \tag{3.7}$$

Now in steady state, as $t \to \infty$, we define the steady state joint probabilities as

$$\lim_{t\to\infty} P_{n,0}(t) = P_{n,0},\ 0 \le n \le a-1,$$
$$\lim_{t\to\infty} P_{n,r}(t) = P_{n,r},\ n \ge 0,\ a \le r \le b.$$

The steady state equations of the model is obtained from (3.1)-(3.7) as follows.

$$0 = -\lambda\beta_0' P_{0,0} + \mu\sum_{r=a}^{b} P_{0,r}, \tag{3.8}$$
$$0 = -\lambda\beta_n' P_{n,0} + \lambda\beta_{n-1}' P_{n-1,0} + \mu\sum_{r=a}^{b} P_{n,r},\ \ 1 \le n \le a-1, \tag{3.9}$$
$$0 = -(\lambda\beta_0+\mu)P_{0,a} + \lambda\beta_{a-1}' P_{a-1,0} + \mu\sum_{r=a}^{b} P_{a,r}, \tag{3.10}$$
$$0 = -(\lambda\beta_0+\mu)P_{0,n} + \mu\sum_{r=a}^{b} P_{n,r},\ \ a+1 \le n \le b, \tag{3.11}$$
$$0 = -(\lambda\beta_n+\mu)P_{n,r} + \lambda\beta_{n-1}P_{n-1,r},\ \ a \le r \le b-1,\ n \ge 1, \tag{3.12}$$
$$0 = -(\lambda\beta_n+\mu)P_{n,b} + \lambda\beta_{n-1}P_{n-1,b} + \mu\sum_{r=a}^{b} P_{n+b,r},\ n \ge 1. \tag{3.13}$$

For use in sequel, let us now denote the arbitrary epoch queue length distribution when server is busy, by P_n $(n \ge 0)$, which is given by

$$P_n = \sum_{r=a}^{b} P_{n,r}, n \ge 0. \tag{3.14}$$

Equations (3.8)-(3.13) yield recursive relation to obtain $P_{n,0}$ and $P_{n,r}$ in terms of P_n which is demonstrated in Theorem 3.1, here below.

Theorem 3.1. The joint distributions $P_{n,0}$ $(0 \le n \le a-1)$ and $P_{n,r}$ $(n \ge 0, a \le r \le b)$ can be obtained in terms of $P_n(n \ge 0)$ as follows

$$P_{n,0} = \frac{\mu}{\lambda\beta_n'}\sum_{k=0}^{n} P_k,\ \ 0 \le n \le a-1, \tag{3.15}$$
$$P_{n,a} = \lambda^n \mu\delta_{n,0}\sum_{k=0}^{a} P_k,\ n \ge 0, \tag{3.16}$$
$$P_{n.r} = \lambda^n \mu\delta_{n,0} P_r,\ n \ge 0,\ a+1 \le r \le b-1, \tag{3.17}$$
$$P_{n,b} = \sum_{k=0}^{n} \lambda^{n-k}\mu\delta_{n,k}P_{b+k},\ n \ge 0, \tag{3.18}$$

where, $\delta_{n,k} = \begin{cases} \frac{1}{(\lambda\beta_0+\mu)} & n=0, k\geq 0, \\ \frac{\prod_{i=k}^{n-1}\beta_i}{\prod_{i=k}^{n}(\lambda\beta_i+\mu)} & n\geq 1, k\geq 0. \end{cases}$

Proof: Using (3.14) in (3.8)-(3.13) we obtain desired results (3.15)-(3.18), recursively.

It is clearly evident from (3.15) to (3.18) that $P_{n,0}$ $(0\leq n\leq a-1)$ and $P_{n,r}$ $(n\geq 0,\ a\leq r\leq b)$ can be written in terms of $P_n(n\geq 0)$. Hence, to obtain $P_{n,0}$ and $P_{n,r}$ we first need to obtain P_n explicitly. Here we employed probability generating function (pgf) method to derive P_n. Towards this end, let us define the partial pgf as

$$G(z) = \sum_{n=0}^{\infty} P_n z^n\ , \quad |\, z\,|\leq 1. \tag{3.19}$$

Multiplying equations (3.10)-(3.13) by z^n, ($n=0$, for equation (3.10)-(3.11); and $n\geq 1$ for equation (3.12)-(3.13)) using (3.14), and summing over the range of n, after little algebraic manipulations, we obtain,

$$0 \ = \ \lambda(z-1)\sum_{n=0}^{\infty}\beta_n P_n z^{n+b} + \mu\sum_{n=b}^{\infty}P_n z^n - \mu\sum_{n=0}^{\infty}P_n z^{n+b} + \mu\sum_{n=0}^{b-1}P_n z^b. \tag{3.20}$$

To carry out the further analysis we need to know the probabilities β_n $(n\geq 0)$ completely. Therefore, we consider the following cases.

3.2.1 Special cases

In this section, we consider the following two special cases for computational purpose.

Case 1:

Let us consider $\beta_n = \beta$ for $n\geq 0$, (where β is a constant probability with $0<\beta\leq 1$). Then, under this consideration and using (3.19) in (3.20) we obtain

$$G(z) = \frac{\mu\sum_{n=0}^{b-1}P_n\left(z^n - z^b\right)}{\lambda\beta z^{b+1} - (\mu+\lambda\beta)z^b + \mu}\ , \quad |\, z\,|\leq 1. \tag{3.21}$$

Our objective is to obtain the closed form expression for the steady state probabilities P_n $(n\geq$

0), from (3.21). However, this is not straight forward as right hand side (RHS) of (3.21) is a rational polynomial containing b unknown terms P_n $(0 \leq n \leq b-1)$. To resolve this, we denote the numerator of $G(z)$ by $f_1(z)$ and denominator by $g(z)$, i.e.,

$$f_1(z) = \mu \sum_{n=0}^{b-1} P_n \left(z^n - z^b\right) \tag{3.22}$$

$$g(z) = \lambda\beta z^{b+1} - (\mu + \lambda\beta)z^b + \mu. \tag{3.23}$$

Case 2:

In this case, we consider $\beta_n = \beta$ for $n \geq b$ (where β is a constant probability with $0 < \beta \leq 1$). Then, under this consideration and using (3.19) in (3.20) we obtain

$$G(z) = \frac{\lambda(1-z)z^b \sum_{n=0}^{b-1}(\beta_n - \beta)P_n z^n + \mu \sum_{n=0}^{b-1} P_n(z^n - z^b)}{\lambda\beta z^{b+1} - (\mu + \lambda\beta)z^b + \mu}, \quad |z| \leq 1. \tag{3.24}$$

Again we denote the numerator and the denominator of $G(z)$ by $f_2(z)$ and $g(z)$ respectively,

i.e.,

$$f_2(z) = \lambda(1-z)z^b \sum_{n=0}^{b-1}(\beta_n - \beta)P_n z^n + \mu \sum_{n=0}^{b-1} P_n(z^n - z^b), \tag{3.25}$$

$$g(z) = \lambda\beta z^{b+1} - (\mu + \lambda\beta)z^b + \mu. \tag{3.26}$$

It should be noted here that for both the cases, as discussed above, $g(z)$ is the same polynomial of degree $(b+1)$, where as $f_1(z)$ is a polynomial of degree b for Case 1 and $f_2(z)$ is a polynomial of degree $2b$ for Case 2. Also it is clearly evident that $z = 1$ is an obvious zero of the polynomials $f_1(z)$, $f_2(z)$ and $g(z)$. Therefore, analysis of Case 1 and Case 2 may be carried out using similar logic and is elaborated in the following section.

Remark : Using the results given in Neuts (1967) one can conclude here that the states of the Markov chain of the considered model will be positive recurrent if and only if $\frac{\lambda\beta}{b\mu} < 1$, which ensures the existence of steady state solution, i.e., $P_{n,0}$ $(0 \leq n \leq a-1)$ and $P_{n,r}$ $(n \geq 0,\, a \leq r \leq b)$.

3.2.2 Analysis of the characteristic roots for Case 1 and Case 2

Let us consider a closed contour C defined by $|z| = 1+\delta$, where δ is small positive real number. By using Rouche's theorem, we can conclude that $g(z)$ must have b zeros within the contour C. Now since $z=1$ is one zero of $g(z)$, the remaining $(b-1)$ zeros must lie inside the unit circle $|z|=1$, if and only if $\frac{\lambda\beta}{b\mu} < 1$. Therefore, $g(z)$ has only one zero outside the unit circle $|z|=1$ and let us denote it by z_0.
Since $G(z)$ is analytic within and on C, $f_l(z); l=1,2$ must have b common zeros as that of $g(z)$ within and on C. Therefore,

$$f_l(z_i) = 0;\ \{z_i : |z_i| \leq 1, g(z_i) = 0,\ 1 \leq i \leq b\} \qquad l = 1,2. \tag{3.27}$$

One may note here that these common zeros may be all distinct or few of them may be repeated. Therefore depending on the nature of the zeros we now discuss following two scenario.

Scenario I. **When the zeros of g(z) in $|z| \leq 1$ are all distinct**

From (3.27) one can derive $(b-1)$ simultaneous linear equations in b unknowns P_n $(0 \leq n \leq b-1)$, which may results in

$$P_n = \zeta_n P_0\ ,\ 1 \leq n \leq b-1. \tag{3.28}$$

where ζ_n's are known constants.

Scenario II. **When few zeros of g(z) in $|z| \leq 1$ are repeated**

Let us assume that $g(z)$ has few multiple roots inside the closed complex unit disk. Those multiple roots of $g(z)$ are denoted as $x_1, x_2, ..., x_f$ with multiplicity $r_1, r_2, ..., r_f$, hence, $m = \sum_{i=1}^{f} r_i$. The other distinct roots are denoted by $x_{m+1}, x_{m+2}, ..., x_b$ in $|z| \leq 1$ with $x_b = 1$. Analyticity of $G(z)$ in $|z| \leq 1$ implies that, for $l = 1,2$,

$$\begin{aligned} f_l^{(i-1)}(x_j) &= 0 \quad , j = 1,2,...,f,\ i = 1,2,...,r_j, \\ f_l(x_i) &= 0 \quad , i = m+1, m+2, ..., b-1, \end{aligned}$$

where $f^{(i)}(x)$ is the i^{th}derivative of $f(z)$ at $z = x$. This yields total $b-1$ linearly independent simultaneous equations in b unknowns, which results in

$$P_n = \zeta_n' P_0 \ , \ 1 \leq n \leq b-1, \tag{3.29}$$

where ζ_n''s are known constants.

Therefore, from (3.28) and (3.29) we obtain , P_n $(1 \leq n \leq b-1)$ in terms of P_0, for Case1 and Case 2, depending on the nature of the roots of $g(z)$ in $|\, z \,| \leq 1$ as discussed in *Scenario I and II.*

Now corresponding to each zero z_i $(z_i : |\, z_i \,| \leq 1;\ 1 \leq i \leq b)$ of $g(z)$, each pair of functions $\{f_1(z), g(z)\}$ and $\{f_2(z), g(z)\}$ have common factors $(z - z_i)$, $1 \leq i \leq b$. On canceling the common factors and writing P_n $(1 \leq n \leq b-1)$ in terms of P_0 we reduce $G(z)$ for Case1 and Case 2 as follows for further investigation.

Case 1:

As in this case $f_1(z)$ is a polynomial of degree b, using $P_n = \zeta_n P_0$ $(1 \leq n \leq b-1)$ in (3.21) we get

$$G(z) = \frac{\eta_1 \mu P_0}{(z_0 - z)}, \quad |\, z \,| \leq 1, \tag{3.30}$$

where η_1 is a constant and is obtained in Lemma 3.1.

Lemma 3.1. The constant η_1 as appeared in (3.30) is given by

$$\eta_1 = \frac{z_0}{\mu}. \tag{3.31}$$

Proof: Using Binomial expansion in (3.30) we get

$$G(z) = \eta_1 \mu P_0 \sum_{n=0}^{\infty} \frac{z^n}{z_0^{n+1}}, \quad |\, z \,| \leq 1. \tag{3.32}$$

then using the result $G(0) = P_0$, from (3.32) after little algebraic manipulation we obtain the result (3.31).

Theorem 3.2. The Steady state probabilities $P_n\,(n \geq 0)$, for Case 1, are given as follows

$$P_n = \begin{cases} \frac{(z_0-1)}{z_0+d_1(z_0-1)}, & n=0, \\ \frac{(z_0-1)}{z_0+d_1(z_0-1)}\left(\frac{1}{z_0}\right)^n, & n>0, \end{cases} \tag{3.33}$$

Proof: Using the normalization condition $\sum_{n=0}^{a-1} P_{n,0} + \sum_{n=0}^{\infty}\sum_{r=a}^{b} P_{n,r} = 1$ in (3.19) we obtain

$$G(1) = 1 - \sum_{n=0}^{a-1} P_{n,0}\,, \tag{3.34}$$

Substituting $z = 1$ in (3.32) and with the help of Lemma 3.1 and (3.34) we get

$$1 - \sum_{n=0}^{a-1} P_{n,0} = \frac{P_0 z_0}{(z_0-1)}\,. \tag{3.35}$$

Now using (3.28) in (3.15) in (3.35) and after little algebraic manipulation we get

$$P_0 = \frac{(z_0-1)}{z_0 + d_1\,(z_0-1)}, \tag{3.36}$$

where $d_1 = \frac{\mu}{\lambda}\left(\sum_{n=0}^{a-2}\sum_{k=n}^{a-2}\frac{\zeta_n}{\beta_k'} + \sum_{n=0}^{a-1}\zeta_n\right)$.
With the help of Lemma 3.1 and (3.36), (3.32) reduces to

$$G(z) = \left(\frac{(z_0-1)}{z_0 + d_1\,(z_0-1)}\right)\sum_{n=0}^{\infty}\left(\frac{z}{z_0}\right)^n, \quad |\, z\,| \leq 1. \tag{3.37}$$

Expression (3.37) will generate the steady state queue length distribution when server is busy. Comparing the coefficients, corresponding to power of z^n, of RHS and LHS of the expression (3.37), we obtain the steady state probabilities in our desired form (3.33).

Case 2:

As $f_2(z)$ is a polynomial of degree $2b$ for Case 2, using $P_n = \zeta_n P_0$ $(1 \leq n \leq b-1)$ in (3.22) we get

$$G(z) = \frac{\eta_2 P_0 A(z)}{(z_0 - z)}, \quad |\, z\,| \leq 1, \tag{3.38}$$

where, η_2 is a constant, and is obtained in Lemma 3.2. Since $A(z)$, as appeared in (3.38) is a monic polynomial of degree b, it can be rewritten as

$$A(z) = \prod_{i=1}^{b} (z - \alpha_i), \quad |z| \leq 1, \tag{3.39}$$

where α_i's, are those zeros of $f_2(z)$ which are not a zero of $g(z)$. $A(z)$ can be further modified as follows.

$$A(z) = \sum_{r=0}^{b} (-1)^{b-r} S_{b-r} z^r, \tag{3.40}$$

where $S_0 = 1$, $S_r = \sum_{\substack{i_1,i_2,i_3,..,i_k,.,i_r=1 \\ i_1<i_2<i_3<...<i_k<...<i_r}}^{b} \left(\prod_{k=1}^{r} \alpha_{i_k} \right)$, $1 \leq r \leq b$, and each α_{i_k} is obtain from the relation $\{f_2(\alpha_{i_k}) = 0 \text{ and } g(\alpha_{i_k}) \neq 0\}$.

Lemma 3.2. The constant η_2 as appeared in (3.38) is given by

$$\eta_2 = \frac{z_0}{(-1)^b S_b}. \tag{3.41}$$

Proof: Using Binomial expansion in (3.38) we obtain

$$G(z) = \eta_2 P_0 A(z) \sum_{n=0}^{\infty} \frac{z^n}{z_0^{n+1}}, \quad |z| \leq 1. \tag{3.42}$$

Next using the result $G(0) = P_0$ and $A(0) = (-1)^b S_b$ from (3.42) after little algebraic manipulations we obtain the desired result (3.41).

Theorem 3.3. The Steady state probabilities $P_n\,(n \geq 0)$, for Case 2, are given by

$$P_n = \begin{cases} \left(\frac{(z_0-1)z_0}{z_0\sum_{r=0}^{b} y_r + d_2(z_0-1)y_0} \right) \sum_{i=0}^{n} x_{n-i} y_i, & 0 \leq n \leq b-1, \\ \left(\frac{(z_0-1)z_0}{z_0\sum_{r=0}^{b} y_r + d_2(z_0-1)y_0} \right) \sum_{i=0}^{b} x_{n-i} y_i, & n \geq b, \end{cases} \tag{3.43}$$

Proof: Substituting $z = 1$ in (3.42) and with the help of Lemma 3.2 and (3.34) we get

$$1 - \sum_{n=0}^{a-1} P_{n,0} = \frac{P_0 z_0 A(1)}{(z_0 - 1) A(0)}, \tag{3.44}$$

Now using (3.28) in (3.15) in (3.44) and after little algebraic manipulation we get

$$P_0 = \frac{(z_0-1)A(0)}{z_0A(1)+d_2(z_0-1)A(0)}, \tag{3.45}$$

where $d_2 = \frac{\mu}{\lambda}\left(\sum_{n=0}^{a-2}\sum_{k=n}^{a-2}\frac{\zeta_n}{\beta_k'}+\sum_{n=0}^{a-1}\zeta_n\right)$.
With the help of Lemma 3.2 and (3.45), (3.42) reduces to

$$G(z) = \left(\frac{(z_0-1)z_0}{z_0A(1)+d_2(z_0-1)A(0)}\right)A(z)\sum_{n=0}^{\infty}\frac{z^n}{z_0^{n+1}}, \tag{3.46}$$

Using (3.40), (3.46) can be rewritten as follows

$$G(z) = \Gamma\left[\sum_{n=0}^{b-1}\sum_{i=0}^{n}(-1)^{b-i}\frac{1}{z_0^{n-i+1}}S_{b-i}z^n+\sum_{n=0}^{\infty}\sum_{i=0}^{b}(-1)^{b-i}\frac{1}{z_0^{n+b-i+1}}S_{b-i}z^{n+b}\right], \tag{3.47}$$

where, $\Gamma = \left(\frac{(z_0-1)z_0}{z_0\sum_{r=0}^{b}(-1)^{b-r}S_{b-r}+d_2(z_0-1)(-1)^bS_b}\right)$.
It may rewritten as

$$G(z) = \left(\frac{(z_0-1)z_0}{z_0\sum_{r=0}^{b}y_r+d_2(z_0-1)y_0}\right)\left[\sum_{n=0}^{b-1}\sum_{i=0}^{n}x_{n-i}y_iz^n+\sum_{n=0}^{\infty}\sum_{i=0}^{b}x_{n+b-i}y_iz^{n+b}\right], \tag{3.48}$$

where, $x_n = \frac{1}{z_0^{n+1}}$; $n \geq 0$ and $y_n = (-1)^{b-n}S_{b-n}; 0 \leq n \leq b$.

Expression (3.48) will generate the steady state probabilities. Comparing the coefficients corresponding to power of z^n of RHS and LHS of the expression (3.48), we get the steady state probabilities in our desired result form (3.43).

Hence, Theorem 3.2 and Theorem 3.3, gives us the closed form expression for P_n $(n \geq 0)$ in terms of roots of characteristic equations of corresponding pgf as appeared in for Case 1 and Case 2. Once all P_n's are obtained, the required joint probabilities $P_{n,0}(0 \leq n \leq a-1)$ and $P_{n,r}(n \geq 0,\, a \leq r \leq b)$ can be obtained for Case 1 and Case 2 completely from Theorem 3.1.

For better understanding, here below we are providing a step wise algorithm for calculating $P_{n,r}(n \geq 0,\, a \leq r \leq b)$.

Step-wise Algorithm

Step 1. Input data :

Lower threshold limit (a), upper threshold limit (b), arrival-rate (λ), service rate (μ), joining probabilities $(\beta_n, 0 < \beta_n \leq 1)$.

Step 2. Calculate the roots of eq. (3.23) of Case 1 (or eq. (3.26) for Case 2) and identify the root that is strictly greater than one.

Step 3. Using eq. (3.28) or (3.29), depending on the nature of the roots, calculate P_n $(1 \leq n \leq b-1)$.

Step 4. Calculate η_1 from Lemma 3.1 for Case 1 (η_2 from Lemma 3.2 for Case 2).

Step 5. Calculate P_n $(n \geq 0)$ completely using the results of Theorem 3.2 for Case 1 (from Theorem 3.3 for Case 2).

Step 6. Finally calculate $P_{n,r}$ $(n \geq 0, a \leq r \leq b)$ using the result from Theorem 3.1 for both the Cases.

As the joint distribution of queue and server content are known, we can now obtain other significant distributions

- The probability that the server is idle $(i.e., P_{idle})$ and busy $\left(i.e., P_{busy}\right)$ are given by $P_{idle} = \sum\limits_{n=0}^{a-1} P_{n,0}$ and $P_{busy} = \sum\limits_{r=a}^{b} \sum\limits_{n=0}^{N} P_{n,r}$, respectively.

- The distribution of queue content is given by

$$p_n^{queue} = \begin{cases} P_{n,0} + \sum_{r=a}^{b} P_{n,r}, & 0 \leq n \leq a-1, \\ \sum\limits_{r=a}^{b} P_{n,r}, & n \geq a. \end{cases}$$

- The conditional probability distribution of the server content given that the server is busy is given by

$$p_r^{ser} = \sum_{n=0}^{N} P_{n,r}/P_{busy}\,(a \leq r \leq b).$$

- The distribution of the system content (including number of customers with the server) is given by

$$p_n^{sys} = \begin{cases} P_{n,0}, & 0 \le n \le a-1, \\ \sum\limits_{r=a}^{min(b,n)} P_{n-r,r}, & a \le n \le b, \\ \sum\limits_{r=a}^{b} P_{n-r,r}, & n \ge b+1. \end{cases}$$

3.3 Performance measures

The performance measures of the model presented in this chapter is given here below

- Average number of customers in the queue, $L_q = \sum\limits_{n=0}^{\infty} np_n^{queue}$.
- Average number of customers in the system, $L = \sum\limits_{n=0}^{\infty} np_n^{sys}$.
- Average number of customers with the server given that the server is busy, $L_s = \sum\limits_{r=a}^{b} rp_r^{ser}$.
- Average number of customers in the queue when server is idle, $L^{idle} = \sum\limits_{n=0}^{a-1} nP_{n,0}/P_{idle}$ and when server is busy, $L_q^{busy} = \sum\limits_{n=0}^{\infty} nP_n/P_{busy}$.
- Average number of customers in the queue when server is busy with $r\,(a \le r \le b)$ customers $L_r^{busy} = \sum\limits_{n=0}^{\infty} nP_{n,r}/P_{busy}$.
- Using Little's law the average waiting time of a customer in the queue is given by $W_q = L_q/\bar{\lambda}$ and in the system is given by $W = L/\bar{\lambda}$, where $\bar{\lambda}$ is the effective arrival rate of the system and is given by $\bar{\lambda} = \sum\limits_{n=0}^{a-1} \lambda \beta_n' P_{n,0} + \sum\limits_{n=0}^{\infty} \sum\limits_{r=a}^{b} \lambda \beta_n P_{n,r}$.

The effective arrival rate, i.e., $\bar{\lambda}$, for Case 1 and Case 2 will take the following forms.

- For Case 1 : $\bar{\lambda} = \sum\limits_{n=0}^{a-2} \lambda \left(\beta_n' - \beta\right) P_{n,0} + \lambda\,(1-\beta)\,P_{a-1,0} + \lambda\beta$.
- For Case 2 : $\bar{\lambda} = \sum\limits_{n=0}^{a-2} \lambda(\beta_n' - \beta)P_{n,0} + \lambda(1-\beta)P_{a-1,0} + \sum\limits_{n=0}^{b-1} \lambda(\beta_n - \beta)P_n + \lambda\beta$.
- Balking Rate : The instantaneous balking rate of the states of the queueing model is important feature of the model and is given by the product of arrival rate with its balking

probability. For more detail in this topic readers are refered to Ancker and Gafarian (1963a,b). The average balking rate is given by

$$BR = \sum_{n=0}^{a-2} \lambda(1-\beta_n^{'})P_{n,0} + \sum_{n=0}^{\infty}\sum_{r=a}^{b} \lambda(1-\beta_n)P_{n,r}.$$

More precisely,

. for Case 1, $BR = \sum_{n=0}^{a-2} \lambda(\beta-\beta_n^{'})P_{n,0} + \lambda\,(1-\beta)\,(1-P_{a-1,0}),$

. and for Case 2, $BR = \sum_{n=0}^{a-2} \lambda(\beta-\beta_n^{'})P_{n,0} + \lambda\,(1-\beta)\,(1-P_{a-1,0}) + \sum_{n=0}^{b-1} \lambda(\beta-\beta_n)P_n.$

3.3.1 Closed form expressions of L_q:

In this section we obtained the closed form expression for expected queue length L_q. As $L_q = \sum_{n=0}^{\infty} np_n^{queue} = \sum_{n=0}^{a-1} nP_{n,0} + G'(1)$, after obtaining the expression for $G'(1)$ by differentiating (3.32) and (3.42), for Case 1 and Case 2 respectively, the final expression for L_q is given as follows

For Case 1 : $L_q = \sum_{n=0}^{a-1} nP_{n,0} + \dfrac{P_0 z_0}{(z_0-1)^2},$

For Case 2 : $L_q = \sum_{n=0}^{a-1} nP_{n,0} + \dfrac{P_0 z_0}{(z_0-1)^2 y_0}\left[\sum_{r=0}^{b} y_r + (z_0-1)\sum_{r=0}^{b} r y_r\right].$

3.4 Numerical results

This section illustrates several numerical results in the form of the tables and graphs and discuss the effect of key parameters on the performance indices. Table 3.1 presents the steady state joint probability distribution of the queue content and server content of $M/M^{(5,10)}/1$ queue for Case 1. The joint distribution of the queue content and server content of $M/M^{(17,25)}/1$ queue for Case 2 is presented in Table 3.2. The input parameters for both the tables are indicated at top of the tables while the key performance measures are presented at the bottom of the respective tables. Table 3.3 presents the several performance measures of $M/M^{(a,b)}/1$ queue with balking for Case 2 and for different values of a, by considering fixed service capacity at 5 (i.e., $b-a=5$). The values of a varies from 5 to 20, and as a result the values of b varies from 10 to 25. The other input parameters are indicated at the top of the table. An interesting observation is clearly seen from Table 3.3 is that the values of W and W_q attain their minimum value when $a = 15$. Therefore, the threshold limit $(15,20)$ is minimizing the waiting time of the customer and hence $(15,20)$ may be considered as optimal value of the threshold limits for the considered model.

After representing the data in tabular form we move to see the influence of different system parameters on several performance measures in the form of self explanatory graphs. Towards this end, we have considered the following four sets of input parameters.

Set I : Case 1 and the input parameters are $\mu = 0.8$, $\beta = 0.75$, $\beta_i^{'} = 1.1 - 0.1(5-i); 0 \leq i \leq 3$.

Set II : Case 1 and the input parameters are $\lambda = 5.5$, $\beta = 0.65$, $\beta_i^{'} = 1.1 - 0.1(5-i); 0 \leq i \leq 3$.

Set III : Case 2 and the input parameters are $\mu = 0.8$, $\beta = 0.75$, $\beta_i^{'} = 1.1 - 0.1(5-i); 0 \leq i \leq 3$, $\beta_i = 1.0 - 0.05(10-i); 0 \leq i \leq 9$.

Set IV : Case 2 and the input parameters are $\lambda = 5.25$, $\beta = 0.75$, $\beta_i^{'} = 1.1 - 0.1(5-i); 0 \leq i \leq 3$, $\beta_i = 1.0 - 0.05(10-i); 0 \leq i \leq 9$.

Now in Figure (3.1) to Figure (3.3) we have considered $M/M^{(5,10)}/1$ queuing model for investigation. The input parameters for Figure (3.1a) and (3.2a) have been considered as presented in Set I and Set II, respectively, and for Figure (3.1b) and (3.2b) have been considered as presented in Set III and Set IV, respectively. Figure (3.1) depicts the impact of λ on performance measures and Figure (3.2) depicts the impact of μ on the performance measures. From Figure (3.1a) it is observed, for Case 1, that with the increasing values of the parameter λ, the values of L, L_q, W, W_q and BR are also increasing, and the same results are verified, for Case 2, from Figure (3.1b). Figure (3.2a) exhibits that with the increase in the values of the parameter μ, the values of L, L_q, W, W_q and BR decreases, for Case 1, and the same result is validated from Figure (3.2b) for Case 2. These behavior of the performance measures with respect to λ and μ are quite obvious.

Next we concentrate our attention on the behavior of one of the important performance measures L_s with respect to the input parameters λ and μ in Figure (3.3a) and (3.3b), respectively. The input parameters for Figure (3.3a) and (3.3b) are considered as presented in Set III and Set IV, respectively. It is clearly observed from Figure (3.3a) that for fixed μ(=0.8), as λ increases L_s increases and approaches to the maximum threshold limit 10, and exactly the reverse behavior is observed from Figure (3.3b), i.e., for fixed λ (=5.25) as μ increases L_s decreases gradually and almost tend to 5 (the minimum threshold limit), for large μ ($\mu > 3.8$). It is to be noted here that for Figure (3.3a) and (3.3b) the values of the parameter $\rho = \frac{\lambda\beta}{b\mu}$ varies from 0.234 to 0.797 and 0.875 to 0.1009, respectively. This behavior of L_s is quite expected.

The balking phenomenon, studied in this chapter, can be characterized by the balking rate (BR) which is calculated using the formula $BR = \sum_{n=0}^{a-2} \lambda(\beta - \beta_n^{'})P_{n,0} + \lambda\,(1-\beta)\,(1-P_{a-1,0})$. To illustrate the effect of BR on important performance measures L_q and L_s we have presented Figure (3.4a) and (3.4b), respectively, for $M/M^{(8,20)}/1$ queueing model and Case 1, with input parameters $\lambda = 19.5$, $\mu = 1.3$, $\beta_i^{'} = 1.1 - 0.1(8-i)$; $0 \leq i \leq 6$. We obtained the values of BR for Figure (3.4a) and (3.4b) by varying β in such a way that as β increases BR decreases. From Figure (3.4a) and (3.4b) it is clearly observed that with the increase in BR, L_q and L_s decreases. Now as β increases the parameter $\rho = \frac{\lambda\beta}{b\mu}$ will also increase and hence with the increase in the value of β, L_q and L_s is also expected to increase. However, as β and BR are inversely related, the reverse behavior is expected to observe and is clearly evident from Figure (3.4a) and (3.4b).

In Figure (3.5) we represented the effect of the values of minimum threshold limit of the serving capacity of the server, i.e., a, for fixed values of the maximum serving capacity of the server, i.e., b (=15), on BR of the system, for Case 1 and Case 2. The input parameters are taken as follow

- a varies from 2 to 10.
- $\lambda = 15.0$, $\mu = 1.3$, $\beta = 0.7$ and hence $\rho = \frac{\lambda\beta}{b\mu} = 0.538$, (for Case 1 and Case 2)
- $\beta_i^{'} = 1.1 - 0.1(a-i)$; $0 \leq i \leq a-2$ (for Case 1 and Case 2)
- $\beta_i = 0.95 - 0.03(b-i)$; $0 \leq i \leq b-1$ (for Case 2)

It is clearly observed from Figure (3.5) that, for fixed b, as a increases BR decreases slowly initially and then start increasing as $a > 5$ for Case 1, and $a > 6$ for Case 2. Hence, one can conclude here that for the current considered model and for fixed $b = 15$, the BR of the system is minimum when $a = 5$ for Case 1, and $a = 6$ for Case 2.

Table 3.1: Joint distribution of queue and server content of $M/M^{(5,10)}/1$ queue with balking for Case 1 with parameters $\lambda = 5.25$, $\mu = 0.5$, $\beta = 0.75$, $\beta_i^{'} = 1.1-(5-i)*0.1; 0 \le i \le 3$, $\rho = 0.7875$.

n	$P_{n,0}$	$P_{n,5}$	$P_{n,6}$	$P_{n,7}$	$P_{n,8}$	$P_{n,9}$	$P_{n,10}$	p_n^{queue}
0	0.00652	0.02489	0.00353	0.00338	0.00323	0.00309	0.00295	0.04758
1	0.01093	0.02208	0.00314	0.00300	0.00287	0.00274	0.00544	0.05019
2	0.01403	0.01959	0.00278	0.00266	0.00254	0.00243	0.00753	0.05157
3	0.01627	0.01739	0.00247	0.00236	0.00226	0.00216	0.00926	0.05216
4	0.01791	0.01543	0.00219	0.00209	0.00200	0.00191	0.01068	0.05222
5		0.01369	0.00194	0.00186	0.00178	0.00170	0.01184	0.03280
6		0.01215	0.00172	0.00165	0.00158	0.00151	0.01276	0.03136
7		0.01078	0.00153	0.00146	0.00140	0.00134	0.01347	0.02998
8		0.00956	0.00136	0.00130	0.00124	0.00119	0.01402	0.02866
9		0.00849	0.00120	0.00115	0.00110	0.00105	0.01441	0.02740
10		0.00753	0.00107	0.00102	0.00098	0.00093	0.01467	0.02620
11		0.00668	0.00095	0.00091	0.00087	0.00083	0.01481	0.02505
12		0.00593	0.00084	0.00080	0.00077	0.00074	0.01487	0.02395
13		0.00526	0.00075	0.00071	0.00068	0.00065	0.01484	0.02289
14		0.00467	0.00066	0.00063	0.00061	0.00058	0.01474	0.02189
15		0.00414	0.00059	0.00056	0.00054	0.00051	0.01458	0.02093
⋮		⋮	⋮	⋮	⋮	⋮	⋮	⋮
28		0.00088	0.00012	0.00012	0.00011	0.00011	0.01032	0.01167
29		0.00078	0.00011	0.00011	0.00010	0.00010	0.00996	0.01115
30		0.00069	0.00010	0.00009	0.00009	0.00009	0.00961	0.01066
⋮		⋮	⋮	⋮	⋮	⋮	⋮	⋮
50		0.00006	0.00001	0.00001	0.00001	0.00001	0.00424	0.00434
51		0.00006	0.00001	0.00001	0.00001	0.00001	0.00406	0.00415
52		0.00005	0.00001	0.00001	0.00001	0.00001	0.00389	0.00397
53		0.00004	0.00001	0.00001	0.00001	0.00001	0.00372	0.00379
54		0.00004	0.00001	0.00001	0.00001	0.00000	0.00357	0.00363
55		0.00003	0.00000	0.00000	0.00000	0.00000	0.00341	0.00347
⋮		⋮	⋮	⋮	⋮	⋮	⋮	⋮
98		0.00000	0.00000	0.00000	0.00000	0.00000	0.00050	0.00050
99		0.00000	0.00000	0.00000	0.00000	0.00000	0.00048	0.00048
100		0.00000	0.00000	0.00000	0.00000	0.00000	0.00046	0.00046
⋮		⋮	⋮	⋮	⋮	⋮	⋮	⋮
Total	0.06566	0.22086	0.03136	0.02998	0.02866	0.02740	0.59607	0.99999

$L = 28.42200, W = 7.16010, L_q = 20.41100, W_q = 5.14210, L_s = 8.4969,$
$BR = 1.28050, L^{idle} = 0.08780, L_q^{busy} = 20.32400, L_5^{busy} = 1.73930, L_6^{busy} = 0.24695,$
$L_7^{busy} = 0.23610, L_8^{busy} = 0.22572, L_9^{busy} = 0.21580, L_{10}^{busy} = 17.660.$

Table 3.2: Joint distribution of queue and server content of $M/M^{(17,25)}/1$ queue with balking for Case 2 with parameters $\lambda = 11.5$, $\mu = 0.5$, $\beta = 0.75$, $\beta_i^{'} = 1.05-(17-i)*0.05;\ 0 \le i \le 15$, $\beta_i = 0.95-(25-i)*0.03;\ 0 \le i \le 24$, $\rho = 0.69$.

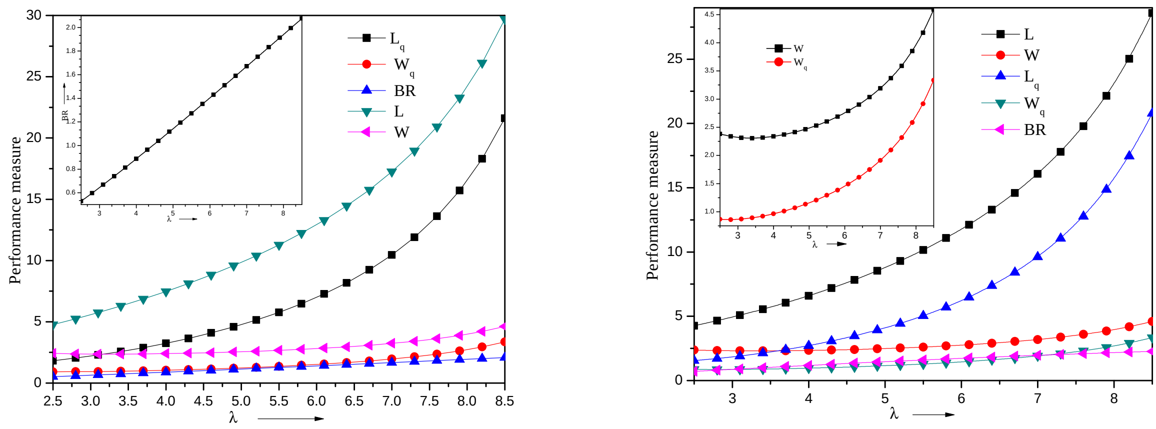
n	$P_{n,0}$	$P_{n,17}$	$P_{n,18}$	$P_{n,19}$	$P_{n,20}$	$P_{n,21}$	$P_{n,22}$	$P_{n,23}$	$P_{n,24}$	$P_{n,25}$	p_n^{queue}
0	**0.017622**	**0.074091**	**0.001043**	**0.000975**	**0.000914**	**0.000860**	**0.000812**	**0.000768**	**0.000728**	**0.000866**	**0.09868**
1	**0.024539**	**0.054184**	**0.000763**	**0.000713**	**0.000669**	**0.000629**	**0.000594**	**0.000562**	**0.000532**	**0.001382**	**0.08457**
2	**0.027136**	**0.041065**	**0.000578**	**0.000540**	**0.000507**	**0.000477**	**0.000450**	**0.000426**	**0.000404**	**0.001702**	**0.07328**
3	**0.027801**	**0.032017**	**0.000451**	**0.000421**	**0.000395**	**0.000372**	**0.000351**	**0.000332**	**0.000315**	**0.001904**	**0.06436**
4	**0.027552**	**0.025544**	**0.000360**	**0.000336**	**0.000315**	**0.000297**	**0.000280**	**0.000265**	**0.000251**	**0.002033**	**0.05723**
5	**0.026868**	**0.020774**	**0.000292**	**0.000273**	**0.000256**	**0.000241**	**0.000228**	**0.000215**	**0.000204**	**0.002114**	**0.05147**
6	**0.025985**	**0.017170**	**0.000242**	**0.000226**	**0.000212**	**0.000199**	**0.000188**	**0.000178**	**0.000169**	**0.002162**	**0.04673**
7	**0.025026**	**0.014388**	**0.000203**	**0.000189**	**0.000178**	**0.000167**	**0.000158**	**0.000149**	**0.000141**	**0.002188**	**0.04279**
8	**0.024057**	**0.012201**	**0.000172**	**0.000161**	**0.000151**	**0.000142**	**0.000134**	**0.000126**	**0.000120**	**0.002198**	**0.03946**
9	**0.023110**	**0.010455**	**0.000147**	**0.000138**	**0.000129**	**0.000121**	**0.000115**	**0.000108**	**0.000103**	**0.002196**	**0.03662**
10	**0.022203**	**0.009042**	**0.000127**	**0.000119**	**0.000112**	**0.000105**	**0.000099**	**0.000094**	**0.000089**	**0.002185**	**0.03417**
11	**0.021343**	**0.007883**	**0.000111**	**0.000104**	**0.000097**	**0.000092**	**0.000086**	**0.000082**	**0.000077**	**0.002169**	**0.03204**
12	**0.020533**	**0.006923**	**0.000097**	**0.000091**	**0.000085**	**0.000080**	**0.000076**	**0.000072**	**0.000068**	**0.002148**	**0.03017**
13	**0.019772**	**0.006120**	**0.000086**	**0.000081**	**0.000076**	**0.000071**	**0.000067**	**0.000063**	**0.000060**	**0.002123**	**0.02852**
14	**0.019060**	**0.005442**	**0.000077**	**0.000072**	**0.000067**	**0.000063**	**0.000060**	**0.000056**	**0.000053**	**0.002096**	**0.02705**
15	**0.018392**	**0.004866**	**0.000069**	**0.000064**	**0.000060**	**0.000056**	**0.000053**	**0.000050**	**0.000048**	**0.002066**	**0.02572**
16	**0.017767**	**0.004372**	**0.000062**	**0.000058**	**0.000054**	**0.000051**	**0.000048**	**0.000045**	**0.000043**	**0.002036**	**0.02453**
⋮		⋮	⋮				⋮	⋮	⋮	⋮	⋮
48		**0.000674**	**0.000009**	**0.000009**	**0.000008**	**0.000008**	**0.000007**	**0.000007**	**0.000007**	**0.001683**	**0.002412**
49		**0.000637**	**0.000009**	**0.000008**	**0.000008**	**0.000007**	**0.000007**	**0.000007**	**0.000006**	**0.001650**	**0.002340**
50		**0.000603**	**0.000008**	**0.000008**	**0.000007**	**0.000007**	**0.000007**	**0.000006**	**0.000006**	**0.001618**	**0.002270**
⋮		⋮	⋮				⋮	⋮	⋮	⋮	⋮
95		**0.000048**	**0.000001**	**0.000001**	**0.000001**	**0.000001**	**0.000001**	**0.000000**	**0.000000**	**0.000527**	**0.000579**
96		**0.000045**	**0.000001**	**0.000001**	**0.000001**	**0.000001**	**0.000000**	**0.000000**	**0.000000**	**0.000512**	**0.000561**
97		**0.000043**	**0.000001**	**0.000001**	**0.000001**	**0.000000**	**0.000000**	**0.000000**	**0.000000**	**0.000498**	**0.000544**
98		**0.000040**	**0.000001**	**0.000001**	**0.000000**	**0.000000**	**0.000000**	**0.000000**	**0.000000**	**0.000485**	**0.000528**
99		**0.000038**	**0.000001**	**0.000001**	**0.000000**	**0.000000**	**0.000000**	**0.000000**	**0.000000**	**0.000471**	**0.000512**
100		**0.000036**	**0.000001**	**0.000000**	**0.000000**	**0.000000**	**0.000000**	**0.000000**	**0.000000**	**0.000458**	**0.000497**
⋮		⋮	⋮				⋮	⋮	⋮	⋮	⋮
174		**0.000001**	**0.000000**	**0.000000**	**0.000000**	**0.000000**	**0.000000**	**0.000000**	**0.000000**	**0.000052**	**0.000052**
175		**0.000001**	**0.000000**	**0.000000**	**0.000000**	**0.000000**	**0.000000**	**0.000000**	**0.000000**	**0.000050**	**0.000051**
176		**0.000000**	**0.000000**	**0.000000**	**0.000000**	**0.000000**	**0.000000**	**0.000000**	**0.000000**	**0.000049**	**0.000049**
⋮		⋮	⋮				⋮	⋮	⋮	⋮	⋮
$\geq$ 382		**0.000000**	**0.000000**	**0.000000**	**0.000000**	**0.000000**	**0.000000**	**0.000000**	**0.000000**	**0.000000**	**0.000000**
Total	**0.388765**	**0.414908**	**0.005841**	**0.005460**	**0.005120**	**0.004817**	**0.004545**	**0.004300**	**0.004078**	**0.162152**	**0.999999**

$L_{17}^{busy} = 3.7620, L_{18}^{busy} = 0.05297, L_{19}^{busy} = 0.04951, L_{20}^{busy} = 0.04643,$
$L_{21}^{busy} = 0.04368, L_{22}^{busy} = 0.04121\, L_{23}^{busy} = 0.03899, L_{24}^{busy} = 0.03697, L_{25}^{busy} = 7.8487,$

$L = 26.6620, W = 4.5127, L_q = 14.5620, W_q = 2.4646, L_s = 19.3314, .$
$BR = 5.5917, L^{idle} = 2.6412, L_q^{busy} = 11.9200$

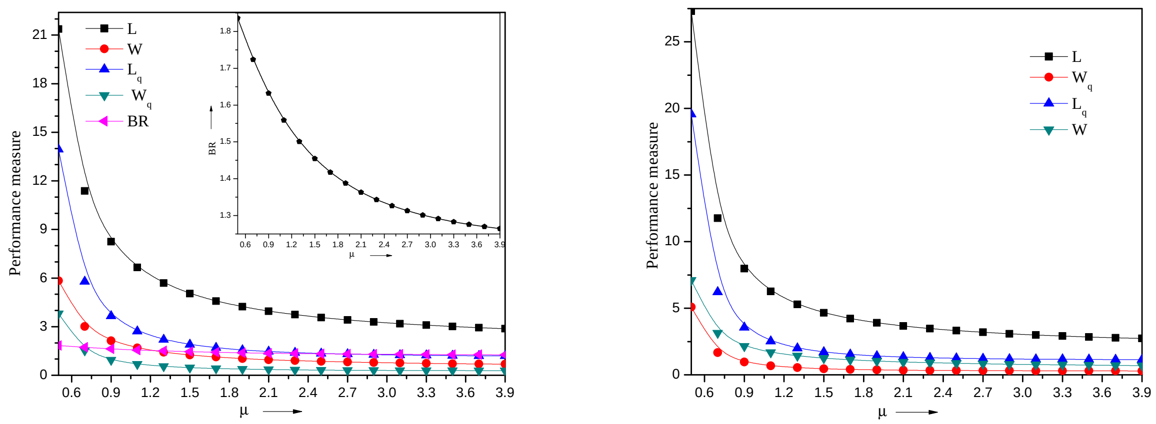
Table 3.3: Performance measures of $M/M^{(a,b)}/1$ queue with balking for Case 2 with parameters $\lambda = 6.5$, $\mu = 0.5$, $\beta = 0.75$, $\beta_i' = 1.05 - (a-i) * 0.05; 0 \leq i \leq a-2$, $\beta_i = 0.95 - (b-i) * 0.03; 0 \leq i \leq b-1$

ρ	0.975	0.75	0.65	0.542	0.513	0.488	0.464	0.423	0.39
(a,b)	(5,10)	(8,13)	(10,15)	(13,18)	(14,19)	(15,20)	(16,21)	(18,23)	(20,25)
L	151.362	29.975	22.5322	17.4491	16.3891	15.4934	14.6948	13.1396	10.7533
W	30.9618	6.17364	4.82444	4.12484	4.04764	**4.02308**	4.04329	4.20861	4.56473
L_q	215.708	20.1164	12.9292	8.60055	7.87977	7.36583	6.99698	6.48935	5.71627
W_q	44.124	4.14316	2.76831	2.03311	1.94608	**1.91264**	1.92523	2.07854	2.42652
BR	1.61133	1.64468	1.82956	2.26976	2.45095	2.64887	2.86564	3.37793	4.14426
L_s	8.7901	9.71048	9.34066	8.46022	8.09718	7.70232	7.269	6.2447	4.7112
L^{idle}	1.342	3.02483	4.09702	5.50054	5.88986	6.22685	6.49851	6.75412	6.09586
L_q^{busy}	240.824	22.5098	15.6118	10.7099	9.57193	8.56012	7.63728	5.97548	4.47033



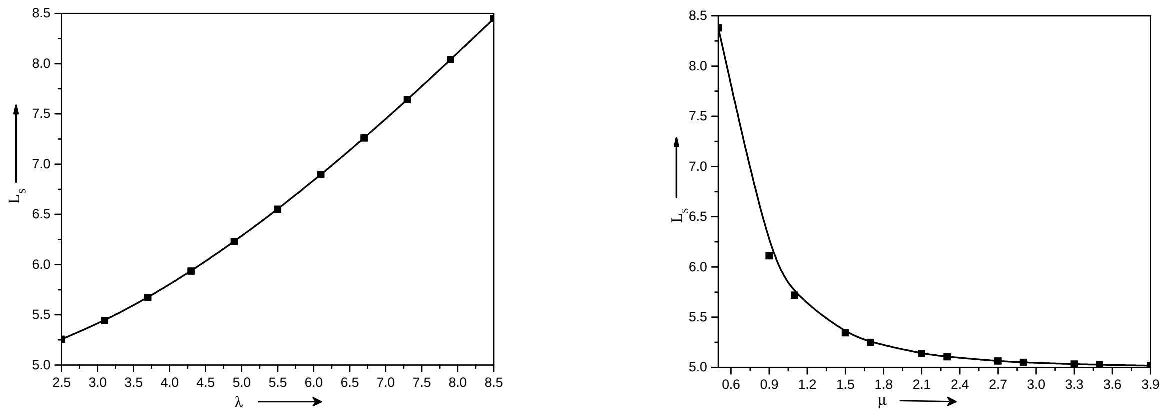
(a) (b)

Figure 3.1: Effect of λ on few performance measures for Case 1 and Case 2



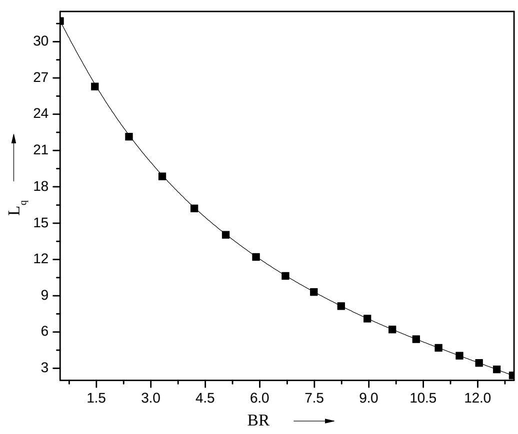
(a) (b)

Figure 3.2: Effect of μ on few performance measures for Case 1 and Case 2

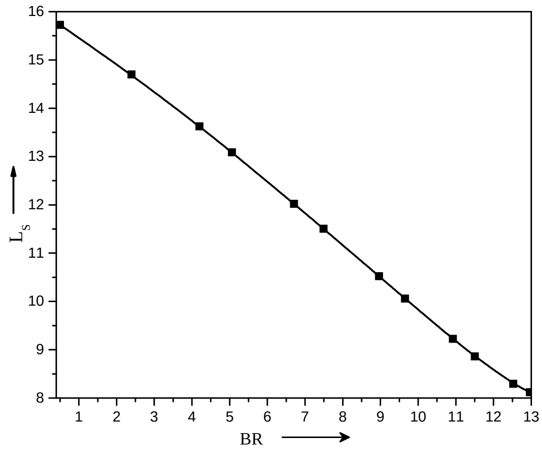


(a) Effect of λ on L_s (b) Effect of μ on L_s

Figure 3.3: Effect of system parameters on L_s for Case 2



(a) Effect of BR on L_q



(b) Effect of BR on L_s

Figure 3.4: Effect of BR on performance measures for Case 1

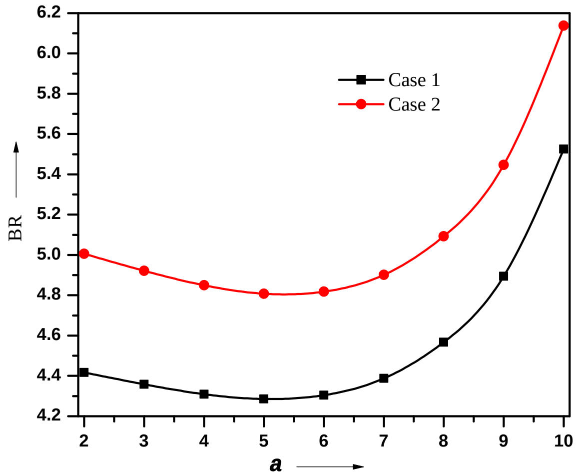


Figure 3.5: Effect of a on BR for Case 1 and Case 2

3.5 Concluding remarks

In this chapter, we studied a general bulk service queue with balking property of the arriving customers. The balking probability of the joining customers are considered to be dependent on the system states which is measured in terms of the queue length and state of the server, i.e., busy/idle. We analytically obtained the closed form expression for the joint probability distribution of the queue length as well as serving batch size, for two special cases, involving two particular known form of the balking probabilities, using the probability generating function method. We obtained various useful performance measures. We investigate the impact of the key parameters on the system performance measures through few numerical examples. The study of the current chapter may be a motivation for the researchers to further investigate the impact of impatient behavior of the joining customer to bulk queueing models. Further study of bulk service queue with balking and reneging seems to be more complex and left for the further study.