Chapter 2

Analysis of infinite buffer $M/M^b/1$ queue with system size based balking

2.1 Introduction

Numerous studies on queueing model with impatience phenomena of the customers have been found in literature, where server serves single customer at a time. A comprehensive review till 2010 on the study of queuing models with impatient customer is discussed by Wang et al. (2010). For recent development in queueing models with impatient customers, the readers are suggested to go through the papers, Laxmi and Jyothsna (2014), Singh et al. (2014a), Laxmi and Jyothsna (2015), Goswami (2015), Saffer and Yue (2015), Guha et al. (2016) and the references therein. In the mentioned papers the impatient behavior of the customers has been studied for $M/G/1$ or $GI/M/1$ queues with finite or infinite buffer and with or without vacation.

The bulk service queueing system is an important phenomenon in our day to day life and hence gained huge importance amongst researchers. The bulk service queues have wide range of applications in telecommunication network, computer communication network (see,

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Banerjee et al. (2015, 2014)), group testing (see, Bar-Lev et al. (2007), Claeys et al. (2010)) etc. Literature survey on bulk service queues reveals the fact that the system size based balking phenomena of the joining customers has not been explored much with the bulk service queues in literature (Jain and Pandey (2009), Wang et al. (2014), Islam et al. (2014)).

In this chapter, we consider a single server queueing system where customers, who join the system, are served in batches of fixed batch size 'b', i.e., server remains idle till queue length reaches to 'b', and as soon as the queue length attains the limit 'b' server initiates service. If queue length is greater than the threshold value 'b' then server initiates server with a batch of first 'b' customers and rest will wait for next round of service. The customers are arriving to the system according to the Poisson process. An arriving customer on arrival will decide whether to join the system or not, upon looking at the system length. That is an arriving customer will join the system with probability β_n or balk with probability $1-\beta_n$ where n is the number of customers present in the system ahead of him. The capacity of the system is considered to be infinite.

The queueing model considered in this chapter has potential application in practical system, viz., production line systems, machine operating or repairing systems, inventory systems etc. For example, let us consider a machine repairing facility system where each machine requires service from efficient and high skilled repairman (server). In such situations the shop keeper may decide to provide the services in groups/batches due to the high wages of the efficient server and will not be ready to provide services unless otherwise a minimum number of service request is accumulated to the service desk. As a result the order fulfillment time (the time between the service orders is placed and the service is actually received) or the waiting time of a customer may increase. Due to the increasing order fulfillment time the customers will become impatient and decide not to join this system and will go to the competitor and will never come back (i.e., the customers will balk). Hence, this may results in a loss to the shopkeeper. Therefore to optimize the cost effectiveness of such system (e.g. profit of the shopkeeper), the study of this chapter will be beneficial to the system analyst.

The rest of the chapter is organized as follows. In section 2.2, the model description and steady state analysis, using probability generating function method, is described for three different cases. The key performance measures are presented in section 2.3. The numerical results are discussed in section 2.4 and the chapter ends with concluding remarks in section 2.5.

2.2 Model description

We consider an infinite buffer single server bulk service queue where customers are served in batches of fixed size 'b', i.e., server remains idle if queue length is less than the threshold limit 'b', and as soon as, the queue length attains the limit 'b' server initiates service with a batch of size 'b' and if queue length is greater than the threshold limit 'b' then server serves first 'b' customers and rest will wait for next round of service. The service time distribution is considered to be exponentially distributed with mean $\frac{1}{\mu}$. The customers are arriving to the system according to the Poisson process with arrival rate λ. On arrival a customer either decide to join the system with probability 'β_n', $0 < \beta_n \leq 1$, or balk with probability $(1-\beta_n)$ where 'n' $(n \geq 0)$, is the number of customers present in the system ahead of him. Therefore, on arrival if a customer finds that the server is idle and queue length is $b-1$, then he certainly join the system, i.e., $\beta_{b-1} = 1$.

2.2.1 Steady State Analysis

In this section, we obtain the steady state probabilities of the system size at an arbitrary epoch. By system size here we mean the total number of customers present in the queue and the number with the server. To this end, let us define the following notations for use in sequel.

- $N(t) \equiv$ number of customers present in the queue, at time t.
- $\xi(t) \equiv$ state of the server, i.e., whether busy or idle, at time t, i.e.,

$$\xi(t) = \begin{cases} 0, & \text{if server is idle,} \\ 1, & \text{if server is busy in serving '}b\text{' customers.} \end{cases}$$

Clearly, $\{N(t),\xi(t)\}$ constitute a two dimensional continuous time Markov chain with state space $\Omega=\{(n,0):0\le n\le b-1\}\cup\{(n,1):n\ge 0\}$. Let us further define the following state possibilities, at time t, as

- $P_{n,0}(t)\equiv prob.\{N(t)=n,\ \xi(t)=0\};\quad 0\le n\le b-1,$
- $P_{n,1}(t)\equiv prob.\{N(t)=n,\ \xi(t)=1\};\quad n\ge 0,$

It should be noted here that $P_{n,0}(t)$ represents the probability that, at time t, there are n $(0\le n\le b-1)$ customers are present in the queue and server is idle, i.e., probability that the system size n $(0\le n\le b-1)$ and $P_{n,1}(t)$ represents the probability that, at time t, there are n $(n\ge 0)$ customers are in the queue and server is busy in serving b customers, i.e., probability that the system size is $n+b$ $(n\ge 0)$.
Now relating the state of the system at time t and $t+dt$ the Kolmogorov equations of the model under consideration is given by

$$\begin{aligned}
\frac{d}{dt}P_{0,0}(t) &= -\lambda\beta_0 P_{0,0}(t)+\mu P_{0,1}(t), && (2.1)\\
\frac{d}{dt}P_{n,0}(t) &= -\lambda\beta_n P_{n,0}(t)+\mu P_{n,1}(t)+\lambda\beta_{n-1}P_{n-1,0}(t)\,,\ 1\le n\le b-1, && (2.2)\\
\frac{d}{dt}P_{0,1}(t) &= -(\lambda\beta_b+\mu)P_{0,1}(t)+\mu P_{b,1}(t)+\lambda\beta_{b-1}P_{b-1,0}(t)\,, && (2.3)\\
\frac{d}{dt}P_{n-b,1}(t) &= -(\lambda\beta_n+\mu)P_{n-b,1}(t)+\mu P_{n,1}(t)+\lambda\beta_{n-1}P_{n-b-1,1}(t)\,,\ n>b. && (2.4)
\end{aligned}$$

In steady-state, as $t\to\infty$, we define

$$\begin{aligned}
\lim_{t\to\infty} P_{n,0}(t) &= P_{n,0},\, 0\le n\le b-1,\\
\lim_{t\to\infty} P_{n,1}(t) &= P_{n,1},\, n\ge 0.
\end{aligned}$$

Therefore, the steady state governing equations of the queuing model under consideration are obtained as

$$\begin{aligned}
0 &= -\lambda\beta_0 P_{0,0}+\mu P_{0,1}, && (2.5)\\
0 &= -\lambda\beta_n P_{n,0}+\mu P_{n,1}+\lambda\beta_{n-1}P_{n-1,0}\,,\ 1\le n\le b-1, && (2.6)\\
0 &= -(\lambda\beta_b+\mu)P_{0,1}+\mu P_{b,1}+\lambda\beta_{b-1}P_{b-1,0}\,, && (2.7)
\end{aligned}$$

$$0 \;=\; -(\lambda\beta_n+\mu)P_{n-b,1}+\mu P_{n,1}+\lambda\beta_{n-1}P_{n-b-1,1}\,,\ n>b. \tag{2.8}$$

Now the steady state probabilities, i.e., $P_{n,0}$ and $P_{n,1}$, will be obtained by solving the equations (2.5) - (2.8) using probability generating function (pgf) method. Towards this end, we define the pgf as follows:

$$G(z)=\sum_{n=0}^{b-1}P_{n,0}z^n+\sum_{n=0}^{\infty}P_{n,1}z^{n+b}\ ,\ |\,z\,|\leq 1\ . \tag{2.9}$$

Now multiplying equations (2.5) - (2.8) by appropriate power of z and summing over n, after algebraic manipulations, we obtain

$$0 \;=\; \lambda(z-1)\left(\sum_{n=0}^{b-1}\beta_nP_{n,0}z^n+\sum_{n=0}^{\infty}\beta_{n+b}P_{n,1}z^{n+b}\right)+\mu(z^{-b}-1)\sum_{n=0}^{\infty}P_{n,1}z^{n+b}\ . \tag{2.10}$$

Now our main objective is to extract the unknown coefficients $P_{n,0}$ and $P_{n,1}$ from equation (2.10). However, right hand side (RHS) of equation (2.10) involves another unknown coefficients β_n in its first part, whereas, second part is independent of β_n. Without knowing the exact expression for β_n it is not possible to have an analytical expression for $P_{n,0}$ and $P_{n,1}$. Therefore, in the following sections we consider some special cases for β_n to obtain $P_{n,0}$ and $P_{n,1}$ analytically.

2.2.1.1 Special cases

Case 1:

In this case, we consider $\beta_n=\tilde{p}$, for $n\geq 0\,,n\neq b-1$ (where $\tilde{p}$ is a constant probability with $0<\tilde{p}\leq 1$) and $\beta_{b-1}=1$. Under this consideration, using (2.9), one can obtain $G(z)$ from (2.10) as follows:

$$G(z)=\frac{\lambda(1-\tilde{p})(1-z)z^{2b-1}P_{b-1,0}-\mu(z^b-1)\sum_{n=0}^{b-1}P_{n,0}z^n}{\lambda\tilde{p}z^{b+1}-(\mu+\lambda\tilde{p})z^b+\mu}\ ,\ |\,z\,|\leq 1. \tag{2.11}$$

Now our main objective is to obtain the closed form expression for the steady state probabilities $P_{n,0}$ $(0\leq n\leq b-1)$ and $P_{n,1}$ $(n\geq 0)$, from (2.11). However, this is not straight forward as RHS of (2.11) contains b unknown terms $P_{n,0}$ $(0\leq n\leq b-1)$. To resolve this, let us denote the numerator of $G(z)$ by $f(z)$ and denominator by $g(z)$ as follows which will be used in sequel.

$$f(z) = \lambda(1-\tilde{p})(1-z)z^{2b-1}P_{b-1,0} - \mu(z^b-1)\sum_{n=0}^{b-1} P_{n,0}z^n, \quad (2.12)$$

$$g(z) = \lambda\tilde{p}z^{b+1} - (\mu+\lambda\tilde{p})z^b + \mu. \quad (2.13)$$

Case 2:

In this case, we consider $\beta_n = \tilde{p}_1$ for $0 \leq n \leq b-2$; $\beta_{b-1} = 1$ and $\beta_n = \tilde{p}$, for $n \geq b$ (where $\tilde{p}_1$ and $\tilde{p}$ are constant probabilities with $0 < \tilde{p}_1, \tilde{p} \leq 1$). Under this consideration, using (2.9), from (2.10) we obtain

$$G(z) = \frac{\lambda(1-z)z^b\left[(\tilde{p}_1-\tilde{p})\sum_{n=0}^{b-2} P_{n,0}z^n + (1-\tilde{p})P_{b-1,0}z^{b-1}\right] - \mu(z^b-1)\sum_{n=0}^{b-1} P_{n,0}z^n}{\lambda\tilde{p}z^{b+1} - (\mu+\lambda\tilde{p})z^b + \mu}, \; |z| \leq 1, \quad (2.14)$$

which again contain b unknown terms $P_{n,0}$ $(0 \leq n \leq b-1)$. Now by denoting the numerator of $G(z)$ by $f(z)$ and denominator by $g(z)$, we have

$$f(z) = \lambda(1-z)z^b\left[(\tilde{p}_1-\tilde{p})\sum_{n=0}^{b-2} P_{n,0}z^n + (1-\tilde{p})P_{b-1,0}z^{b-1}\right] - \mu(z^b-1)\sum_{n=0}^{b-1} P_{n,0}z^n, \quad (2.15)$$

$$g(z) = \lambda\tilde{p}z^{b+1} - (\mu+\lambda\tilde{p})z^b + \mu. \quad (2.16)$$

One may note here that the denominator of $G(z)$ is exactly same as that of *Case 1*, however, the numerator is of different form. But for both the cases numerator of $G(z)$ is a polynomial of degree $2b$ and denominator is of degree $(b+1)$.

Case 3:

In this case, we consider $\beta_n = \tilde{p}_n$ for $0 \leq n \leq b-2$, $\beta_{b-1} = 1$ and $\beta_n = \tilde{p}$ for $n \geq b$ (where $\tilde{p}_n$, and $\tilde{p}$ are constant probabilities with $0 < \tilde{p}_n, \tilde{p} \leq 1$). Under this consideration, using (2.9), from (2.10) we obtain $G(z)$, with b unknown terms $P_{n,0}$ $(0 \leq n \leq b-1)$, as follows

$$G(z) = \frac{\lambda(1-z)z^b\left[\sum_{n=0}^{b-2}(\tilde{p}_n - \tilde{p})P_{n,0}z^n + (1-\tilde{p})P_{b-1,0}z^{b-1}\right] - \mu(z^b-1)\sum_{n=0}^{b-1}P_{n,0}z^n}{\lambda\tilde{p}z^{b+1} - (\mu+\lambda\tilde{p})z^b + \mu}, \quad |z| \leq 1, \tag{2.17}$$

where the numerator $f(z)$ and denominator $g(z)$ of $G(z)$ is given by

$$f(z) = \lambda(1-z)z^b\left[\sum_{n=0}^{b-2}(\tilde{p}_n - \tilde{p})P_{n,0}z^n + (1-\tilde{p})P_{b-1,0}z^{b-1}\right] - \mu(z^b-1)\sum_{n=0}^{b-1}P_{n,0}z^n, \tag{2.18}$$

$$g(z) = \lambda\tilde{p}z^{b+1} - (\mu+\lambda\tilde{p})z^b + \mu. \tag{2.19}$$

Again in *Case 3* also one can observe that $g(z)$ is a polynomial of degree $(b+1)$ as obtained in *Case 1* and *Case 2*, and $f(z)$ is a polynomial of degree $2b$, however, of different form.

Remark : Using the results given in Neuts (1967) one can conclude here that the states of the Markov chain of the considered model will be positive recurrent if and only if $\frac{\lambda\tilde{p}}{b\mu} < 1$, which ensures the existence of steady state solution, i.e., $P_{n,0}$ $(0 \leq n \leq b-1)$ and $P_{n,1}$ $(n \geq 0)$.

Analysis

It should be noted here that for all the three cases as discussed above $g(z)$ is the same polynomial of degree $(b+1)$ where as $f(z)$ is a polynomial of degree $2b$ and $z=1$ is an obvious zero for both $f(z)$ and $g(z)$. Therefore, analysis can be carried out with similar argument for all three cases.

Let C be a closed contour defined by $|z| = 1+\delta$, where δ is small positive real number. Now assuming $g_1(z) = -(\mu+\lambda\,\tilde{p})z^b$ and $g_2(z) = \left(\lambda\,\tilde{p}\,z^{b+1}+\mu\right)$, so that, $g_1+g_2 = g$, one can obtain

$$|g_1(z)| > |g_2(z)| \text{ on } C \text{ if and only if, } \frac{\lambda\tilde{p}}{b\mu} < 1. \tag{2.20}$$

Henceforth, by using Rouche's theorem, we conclude that $g(z)$ must have b zeros within the contour C. Now since $z=1$ is one zero of $g(z)$, the remaining $(b-1)$ zeros must lie within the unit disk $|z|=1$. Therefore, $g(z)$ has only one zero outside the unit disk $|z|=1$ and let us denote it by z_0.

Since $G(z)$ is analytic within and on C, $f(z)$ and $g(z)$ must have b common zeros, say z_i $(1 \leq i \leq b)$ with $z_b = 1$ within and on C. Hence,

$$f(z_i) = 0; \; \{z_i : |z_i| \leq 1, g(z_i) = 0, \; 1 \leq i \leq b\}. \tag{2.21}$$

Equation (2.21) yields b linear algebraic equations in b unknowns,out of which one will be $z_b = 1$. Therefore, ultimately we will get $(b-1)$ equations in b unknowns, solving which we will obtain the values of all $P_{n,0}$ $(1 \leq n \leq b-1)$, in terms of only one unknown, i.e., $P_{0,0}$.

It should be noted here that these common zeros (except $z_b = 1$) of $f(z)$ and $g(z)$ may be all distinct or some of them are repeated. Therefore, depending on the nature of the zeros we discuss following two cases.

Case A: The common zeros of $f(z)$ and $g(z)$ inside the closed contour C are all distinct

Let us consider that $z_i \neq z_j$ for all $i \neq j$ and $1 \leq i, j \leq b-1$ as appeared in (2.21). Then one can derive $(b-1)$ equations in b unknowns, $P_{n,0}$ $(0 \leq n \leq b-1)$, which will results in $P_{n,0} = \zeta_n P_{0,0}$, $1 \leq n \leq b-1$, where ζ_n's are constants.

Case B: Some of the common zeros of $f(z)$ and $g(z)$ inside the closed contour C are repeated

Let us suppose that some of z_i's as appeared in (2.21), are multiple roots. We denote the multiple roots by $x_1, x_2, \ldots, x_l$ with multiplicity $r_1, r_2, \ldots, r_l$, so that, $m = \sum_{i=1}^{l} r_i$. and the remaining distinct roots by $x_{m+1}, x_{m+2}, \ldots, x_b$ with $x_b = 1$. Using the property of analiticity of $G(z)$ in $|z| \leq 1$ we obtain

$$\begin{aligned} f^{(i-1)}(x_j) &= 0, \qquad j = 1,2,\ldots,l, \; i = 1,2,\ldots,r_j, \\ f(x_i) &= 0, \qquad i = m+1, m+2, \ldots, b-1, \end{aligned}$$

where $f^{(i)}(x)$ denotes the i^{th} derivative of $f(z)$ at $z = x$, which results in total $(b-1)$ system of linearly independent equations in b unknowns $P_{n,0}$ $(0 \leq n \leq b-1)$. Solving we obtain

$P_{n,0} = \zeta_n P_{0,0}$, $1 \le n \le b-1$ with ζ_n's as constants.

Now corresponding to each zero z_i, both $f(z)$ and $g(z)$ have a common factor of the form $(z-z_i)$. On canceling the common factors from $f(z)$ and $g(z)$, and using *Case A* or *Case B*, $G(z)$ can be rewritten as

$$G(z) = \frac{\eta P_{0,0} A(z)}{(z_0 - z)}, \qquad |z| \le 1, \tag{2.22}$$

where $P_{0,0}$ is the only unknown term in $G(z)$, η is a constant and $A(z)$ is a monic polynomial of degree b, and can be written as

$$A(z) = \prod_{i=1}^{b} (z - \alpha_i), \qquad |z| \le 1, \tag{2.23}$$

where α_i's, are those zeros of $f(z)$ which are not a zero of $g(z)$. As $A(z)$ is a monic polynomial, (2.23) can be rewritten as

$$A(z) = \sum_{r=0}^{b} (-1)^{b-r} S_{b-r} z^r, \qquad |z| \le 1, \tag{2.24}$$

where $S_0 = 1$ and $S_r = \sum_{\substack{i_1,i_2,i_3,..,i_k,.,i_r=1 \\ i_1<i_2<i_3<...<i_k<...<i_r}}^{b} \left(\prod_{k=1}^{r} \alpha_{i_k} \right)$, $1 \le r \le b$; and α_{i_k} is obtained by using $f(\alpha_{i_k}) = 0$ *and* $g(\alpha_{i_k}) \neq 0$.

Lemma 2.1. The constant η as appeared in (2.22) is given by

$$\eta = \frac{z_0}{(-1)^b S_b}. \tag{2.25}$$

Proof: Using Binomial expansion, (2.22) can be rewritten as

$$G(z) = \eta P_{0,0} A(z) \sum_{n=0}^{\infty} \frac{z^n}{z_0^{n+1}}, \qquad |z| \le 1. \tag{2.26}$$

Using the result $G(0) = P_{0,0}$ in (2.26) and assuming that $P_{0,0} > 0$ after little algebraic manipulation we obtain the desired result (2.25) by using (2.24).

Theorem 2.1. The Steady state probabilities $P_{n,0}\,(0 \le n \le b-1)$ and $P_{n,1}\,(n \ge 0)$ are given as follows

$$P_{n,0} = \left(\frac{z_0-1}{\gamma}\right)\sum_{i=0}^{n} x_{n-i}y_i, \quad 0 \le n \le b-1, \tag{2.27}$$

$$P_{n,1} = \left(\frac{z_0-1}{\gamma}\right)\sum_{i=0}^{b} x_{n+b-i}y_i, \quad n \ge 0, \tag{2.28}$$

where $x_n = \frac{1}{z_0^{n+1}}$; $n \ge 0$, $y_n = (-1)^{b-n}S_{b-n}; 0 \le n \le b$ and $\gamma = \sum_{n=0}^{b} y_n$.

Proof: Using the result $G(1) = 1$ and Lemma 2.1 in expression (2.26), we obtain

$$P_{0,0} = \left(\frac{(-1)^b S_b}{\gamma}\right)\left(\frac{z_0-1}{z_0}\right). \tag{2.29}$$

Now using Lemma 2.1 and (2.29), in equation (2.26) after little algebraic manipulation we obtain

$$G(z) = \left(\frac{z_0-1}{\gamma}\right)\left[\sum_{n=0}^{b-1}\sum_{i=0}^{n} x_{n-i}y_i z^n + \sum_{n=0}^{\infty}\sum_{i=0}^{b} x_{n+b-i}y_i z^{n+b}\right]. \tag{2.30}$$

Expression (2.30) will generate the all the steady state probabilities. Comparing the coefficients of the corresponding powers of z^n of RHS and LHS of the expression (2.30), we get the steady state probabilities in our desired form (2.27) and (2.28).

2.3 Performance measures

In this section, we present some important performance measures of the model under consideration. The expected system length and queue length are given by $L = \sum_{n=0}^{b-1} nP_{n,0} + \sum_{n=0}^{\infty}(n+b)P_{n,1}$ and $L_q = \sum_{n=0}^{b-1} nP_{n,0} + \sum_{n=0}^{\infty} nP_{n,1}$, respectively. Now using Little's law the expected waiting time of a customer in the system as well as in the queue are obtained as $W = \frac{L}{\Lambda}$ and $W_q = \frac{L_q}{\Lambda}$, respectively. Here, Λ is the effective arrival rate of the system and is given by $\Lambda = \sum_{n=0}^{b-1} \lambda\beta_n P_{n,0} + \sum_{n=0}^{\infty} \lambda\beta_{n+b}P_{n,1}$ (see, Ancker and Gafarian (1963a,b)). Therefore,

- for *Case 1*, $\Lambda = \lambda(1-\tilde{p})P_{b-1,0} + \lambda\tilde{p}$;

- for *Case 2*, $\Lambda = \lambda(\tilde{p}_1 - \tilde{p})\sum_{n=0}^{b-2} P_{n,0} + \lambda(1-\tilde{p})P_{b-1,0} + \lambda\tilde{p}$;

- for *Case 3*, $\Lambda = \lambda\sum_{n=0}^{b-2}(\tilde{p}_n - \tilde{p})P_{n,0} + \lambda(1-\tilde{p})P_{b-1,0} + \lambda\tilde{p}$.

Another important performance measure of the model is the average balking rate of the system which is given as follows.

Balking Rate

The instantaneous balking rate of the system is given by $\lambda(1-\beta_n)$, when an arriving customer, upon his arrival, finds n $(n \geq 0)$ customers in the system ahead of him. Hence, the average balking rate (BR) of the system is given by $BR = \sum_{n=0}^{b-1}\lambda(1-\beta_n)P_{n,0} + \sum_{n=0}^{\infty}\lambda(1-\beta_{n+b})P_{n,1}$ (see, Ancker and Gafarian (1963a,b)). Therefore,

- for *Case 1*, $BR = \lambda(1-\tilde{p})(1-P_{b-1,0})$,

- for *Case 2*, $BR = \lambda(\tilde{p} - \tilde{p}_1)\sum_{n=0}^{b-2} P_{n,0} + \lambda(1-\tilde{p})(1-P_{b-1,0})$,

- for *Case 3*, $BR = \lambda\sum_{n=0}^{b-2}(\tilde{p} - \tilde{p}_n)P_{n,0} + \lambda(1-\tilde{p})(1-P_{b-1,0})$.

Lemma 2.2. The closed form expressions for L, L_q, W and W_q are given by

$$L = \left[(z_0-1)^{-1} + \gamma^{-1}\sum_{r=0}^{b} r y_r\right], \tag{2.31}$$

$$L_q = L - b + \frac{b(z_0-1)}{\gamma}\sum_{r=0}^{b-1}\sum_{n=0}^{b-r-1}\frac{y_r}{{z_0}^{n+1}}, \tag{2.32}$$

$$W = \frac{1}{\Lambda}\left[(z_0-1)^{-1} + \gamma^{-1}\sum_{r=0}^{b} r y_r\right], \tag{2.33}$$

$$W_q = \frac{1}{\Lambda}\left[L - b + \frac{b(z_0-1)}{\gamma}\sum_{r=0}^{b-1}\sum_{n=0}^{b-r-1}\frac{y_r}{{z_0}^{n+1}}\right]. \tag{2.34}$$

Proof: Differentiating (2.26) with respect to (w.r.t.) z and then letting $z \to 1$ and using the result $G'(1) = L$, we obtain

$$L = \eta P_{0,0}A(1)\sum_{n=0}^{\infty}\frac{n}{z_0^{n+1}} + \eta P_{0,0}A'(1)\sum_{n=0}^{\infty}\frac{1}{z_0^{n+1}}. \tag{2.35}$$

The RHS of equation (2.35) contains two convergent infinite series $\sum_{n=0}^{\infty}\frac{n}{z_0^{n+1}}$ and $\sum_{n=0}^{\infty}\frac{1}{z_0^{n+1}}$, as $|z_0| > 1$. Hence, using Lemma 2.1 and (2.29) in equation (2.35) we obtain

$$L = \frac{1}{(z_0 - 1)} + \left[\frac{A'(z)}{A(z)}\right]_{z=1}. \tag{2.36}$$

Using expression (2.24) in (2.36) we obtain the desired result (2.31). Now L_q can be rewritten as

$$L_q = L - b\sum_{n=0}^{\infty} P_{n,1}, \tag{2.37}$$

Using Theorem 2.1 in equation (2.37), after algebraic manipulation, we obtain the result (2.32). Now using Little's Law, the expected waiting time of a customer in the system (W) and in the queue (W_q) is easily obtained in the desired form (2.33) and (2.34), respectively.

2.4 Numerical results

In this section, we present some numerical results in the form of table and graphs to illustrate the effect of different system parameters on key performance measures of the model under consideration. Table 2.1 presents the steady state probabilities $P_{n,0}$ $(0 \le n \le b-1)$ and $P_{n,1}$ $(n \ge 0)$ for *Case 1*, *Case 2* and *Case 3*, as discussed in section 2.2.1.1. (It should be noted here that the 1st column of Table 2.1 represents the number of customers present in the queue excluding the customers with the server. The 2nd column of Table 2.1 represents the probability that the system is in state $(n,0)$, i.e., system size is n, $(0 \le n \le a-1)$. The 3rd column of Table 2.1 represents the probability that the system is in state $(n,1)$, i.e., the system size is $n+b$, $(n \ge 0)$. Similar notation are carried out in column 4 to 7 of Table 2.1.) The input parameters are taken as $\lambda = 13.5$, $\mu = 0.8$, $b = 15$ for all three case, whereas, the joining probabilities are taken as $\tilde{p} = 0.5$ for *Case 1*, $\tilde{p}_1 = 0.7$ and $\tilde{p} = 0.5$ for *Case 2*, and $\tilde{p}_n = (1-(b-n)0.05)$ $(0 \le n \le b-2)$ for *Case 3*. The performance measures for respective cases are presented at the last row of the table.

Figures 2.1 to 2.4 present the sensitivity of some system parameters on important performance measures of the model under consideration. The effect of serving batch size 'b' on the performance measures, L, L_q, W, W_q and BR, for *Case 1* are presented in Figure 2.1. In this figure we consider $\lambda = 9.25$, $\mu = 2.5$ and $\tilde{p} = 0.5$ and it is evident from the figure that as 'b' increases, all the performance measures are also increases.

The effect of BR on L, for *Case 1*, is displayed in Figure 2.2 for fixed values of $\lambda = 11.2$, $\mu = 0.56$ and $b = 20$. In this figure the values of BR varies by varying the values of $\tilde{p}$ in such a way that as $\tilde{p}$ decreases BR increases linearly. Therefore, it is clearly evident from Figure 2.2 that, L decreases with the increase in the value of BR, eventually, with the decrease in the value of $\tilde{p}$. This behavior is on the expected direction as decrease in the value of joining probability ($\tilde{p}$) will obviously decreases the expected system length. The effect of service rate (μ) on expected system length (L) for *Case 1* (Figure 2.3(a)), *Case 2* (Figure 2.3(b)) and *Case 3* (Figure 2.3(c)) are presented in Figure 2.3, for three values of fixed serving batch size, i.e., $b = 10, 15, 20$, and for fixed values of the parameters λ and β_n. We have considered $\lambda = 4.0$, $\tilde{p} = 0.45$ for Figure 2.3(a); $\lambda = 4.0$, $\tilde{p}_1 = 0.7$, $\tilde{p} = 0.45$ for Figure 2.3(b) and $\tilde{p}_n$ $(0 \le n \le b-2)$, $\tilde{p} = 0.45$ for Figure 2.3(c). Now $\tilde{p}_n$ for Figure 2.3(c) are chosen in three different ways depending on the values of b, i.e., when $b = 10$ then $\tilde{p}_n = (1.1-(b-n)0.1)$; when $b = 15$ then $\tilde{p}_n = (1.0-(b-n)0.05)$ and when $b = 20$, then $\tilde{p}_n = (1.05-(b-n)0.05)$. It is clearly observed from Figure 2.3 that, L decreases with the increases in the values of μ, for all three cases and all three values of b, and this behavior is also on expected direction as increase in service rate will obviously decrease the expected system length.

It is also observed from Figure 2.3 that for fixed values of μ (when $\mu > 0.4$ for *Case 1*; $\mu > 0.5$ for *Case 2* and $\mu > 0.8$ for *Case 3*), as b increases L also increases and the similar behavior is also observed in Figure 2.1 with $\mu = 2.5$ for *Case 1*. However, the reversed behavior is observed in Figure 2.3 when $\mu < 0.4$ for *Case 1*; $\mu < 0.5$ for *Case 2* and $\mu < 0.8$ for *Case 3*. In Figure 2.4 we present the effect of ρ, where ρ is a parameter defined as $\rho = \frac{\lambda \tilde{p}}{\mu b}$, on average system length (L) and average waiting time of a customer in the system (W) for *Case 1*, and for the fixed values of $\mu = 4.5$, $b = 15$ and $\tilde{p} = 0.7$. In this figure, ρ varies from 0.3 to 0.77 by varying the values of λ from 30 to 75 at a constant increment 2. It is clearly evident from the figure that as ρ increases L increases slowly when $\rho < 0.55$, however, it is increasing rapidly when $\rho > 0.55$. Now increase in ρ by increasing λ will obviously increase the average system length. From the figure it is also evident that, in comparison to L, the value of W remain almost constant as λ increases. However, a close view of the Figure 2.4 reveals that with the increase in the value of ρ, (by increasing λ) W decreases initially, i.e., upto $\rho = 0.55$, and then starts increasing with the increase in the value of ρ. This behavior of W is observed because Λ is strictly increasing function, which is calculated using the formulae $(= \lambda(1-\tilde{p})P_{b-1,0} + \lambda\tilde{p}$.

In Figure 2.4 we have considered the values of λ (ρ) is increasing, in such a way from 30 to 75 (corresponding ρ from 0.311 to 0.77) with a difference of 2. As a result Λ $(= \lambda(1-\tilde{p})P_{b-1,0} + \lambda\tilde{p}$; for *Case 1*) is strictly increasing function. Also, L is increasing slowly with the increase in the value of λ when $\rho < 0.55$, However it is increasing very rapidly when $\rho > 0.55$. As a result W decreases initially when $\rho < 0.55$ however it starts increasing slowly when $\rho > 0.55$, which is shown in the box of the figure.

Table 2.1: Steady state queue length distribution of $M/M^{15}/1$ queue with balking for *Case 1*, *Case 2* and *Case 3*

	Case 1		*Case 2*		*Case 3*	
n	$P_{n,0}$	$P_{n,1}$	$P_{n,0}$	$P_{n,1}$	$P_{n,0}$	$P_{n,1}$
0	0.0051929990	0.0219089100	0.0041885570	0.0247393000	0.0111536800	0.0235277900
1	0.0099909680	0.0202417900	0.0080583440	0.0228568100	0.0178821000	0.0217374900
2	0.0144239600	0.0187015300	0.0116336800	0.0211175600	0.0221282800	0.0200834100
3	0.0185196100	0.0172784800	0.0149370200	0.0195106600	0.0248600000	0.0185552100
4	0.0223039200	0.0159637000	0.0179890200	0.0180260400	0.0266125200	0.0171432800
5	0.0257998400	0.0147489800	0.0208089700	0.0166543800	0.0277057200	0.0158388000

6	0.0290300000	0.0136266800	0.0234142400	0.0153871000	0.0283402400	0.0146335800
7	0.0320141900	0.0125897900	0.0258212800	0.0142162500	0.0286495700	0.0135200600
8	0.0347713400	0.0116317900	0.0280450800	0.0131344900	0.0287236500	0.0124912800
9	0.0373187500	0.0107466900	0.0301000500	0.0121350500	0.0286259600	0.0115407800
10	0.0396718400	0.0099289450	0.0319983700	0.0112116500	0.0284026100	0.0106626100
11	0.0418463600	0.0091734210	0.0337519600	0.0103585300	0.0280869600	0.0098512580
12	0.0438555800	0.0084753880	0.0353725700	0.0095703140	0.0277038400	0.0091016460
13	0.0457113000	0.0078304700	0.0368693100	0.0088420800	0.0272718200	0.0084090740
14	0.0237133300	0.0072346260	0.0267768300	0.0081692590	0.0254655400	0.0077692020
15		0.0066841210		0.0075476350		0.0071780200
16		0.0061755060		0.0069733130		0.0066318220
17		0.0057055930		0.0064426920		0.0061271870
18		0.0052714370		0.0059524480		0.0056609500
19		0.0048703170		0.0054995080		0.0052301920
20		0.0044997200		0.0050810340		0.0048322100
21		0.0041573230		0.0046944030		0.0044645130
22		0.0038409790		0.0043371910		0.0041247940
23		0.0035487070		0.0040071610		0.0038109260
24		0.0032786760		0.0037022440		0.0035209410
⋮		⋮		⋮		⋮
41		0.0008538439		0.0009641511		0.0009169356
42		0.0007888723		0.0008907859		0.0008471632
43		0.0007288446		0.0008230033		0.0007826999
44		0.0006733846		0.0007603785		0.0007231419
45		0.0006221447		0.0007025189		0.0006681158
46		0.0005748038		0.0006490621		0.0006172769
47		0.0005310653		0.0005996730		0.0005703064
48		0.0004906549		0.0005540421		0.0005269101
49		0.0004533195		0.0005118833		0.0004868159

50		0.0004188250		0.0004729326		0.0004497726
⋮		⋮		⋮		⋮
70		0.0000860196		0.0000971324		0.0000923757
71		0.0000794741		0.0000897413		0.0000853466
72		0.0000734267		0.0000829126		0.0000788523
73		0.0000678394		0.0000766035		0.0000728522
74		0.0000626773		0.0000707745		0.0000673086
75		0.0000579080		0.0000653891		0.0000621869
76		0.0000535016		0.0000604134		0.0000574549
77		0.0000494305		0.0000558164		0.0000530830
78		0.0000456692		0.0000515691		0.0000490437
79		0.0000421941		0.0000476451		0.0000453119
⋮		⋮		⋮		⋮
90		0.0000176670		0.0000199494		0.0000189724
91		0.0000163226		0.0000184313		0.0000175287
92		0.0000150806		0.0000170289		0.0000161949
93		0.0000139331		0.0000157331		0.0000149626
94		0.0000128729		0.0000145359		0.0000138241
95		0.0000118933		0.0000134298		0.0000127721
96		0.0000109883		0.0000124079		0.0000118003
97		0.0000101522		0.0000114637		0.0000109024
98		0.0000093797		0.0000105914		0.0000100728
99		0.0000086660		0.0000097855		0.0000093063
100		0.0000080065		0.0000090409		0.0000085981
⋮		⋮		⋮		⋮
	L=11.437, L_q=7.11940 W=1.65510, W_q=1.03030 BR=6.58990		L=11.849, L_q=6.9736 W=1.5185, W_q=0.89374 BR=5.6972		L=11.255, L_q=6.6186 W=1.5167, W_q=0.89191 BR=6.0793	

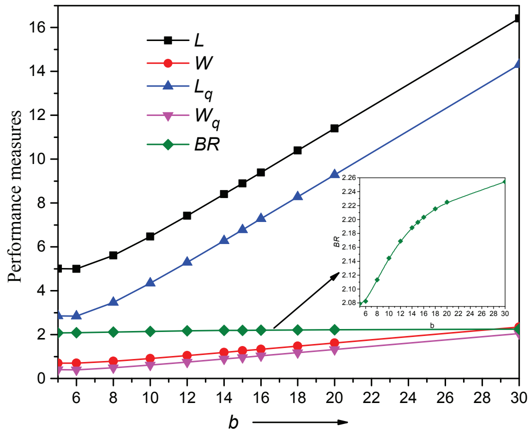


Figure 2.1: Effect of b on Performance Measures

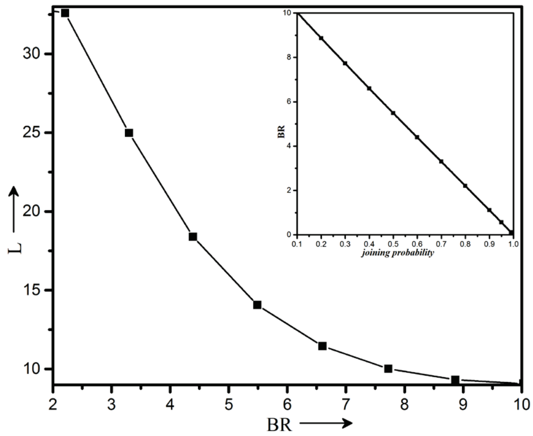
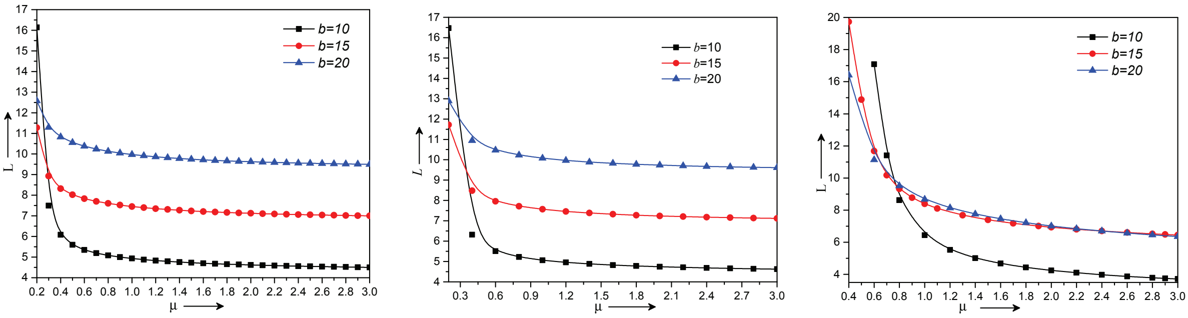


Figure 2.2: Effect of BR on L



(a) *Case 1*

(b) *Case 2*

(c) *Case 3*

Figure 2.3: Effect of μ on L

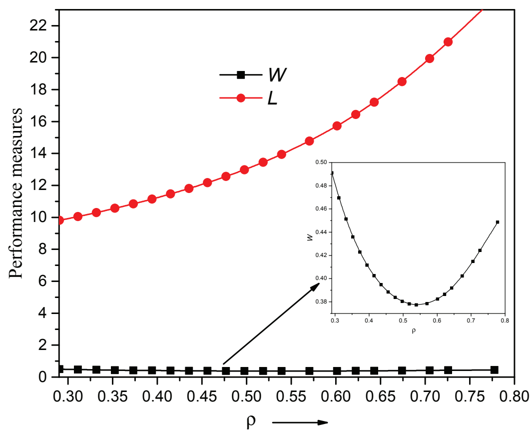


Figure 2.4: Effect of ρ on L and W for Case I

2.5 Concluding remarks

In this chapter, we studied the bulk service (with fixed batch size) queueing system with system size based balking behavior of the arriving customers in $M/M/1$ queue. We employed probability generating function method to obtain the steady state probability distribution of the queue length. The inclusion of balking in $M/M^b/1$ queueing model make it more complex to analyze. Also, to the best of the author's knowledge, the study of bulk service queue with balking is not available in the literature. The study of this chapter may encourage the researchers to study balking behavior of the customers in bulk service queueing model by considering more general bulk service rules available in the literature. In the next chapter we will discuss the analysis of system size based balking in bulk service queue with 'general bulk service' rule. In future, it would be interesting and more complex to study the bulk service queue with reneging behavior along with balking behavior of the customers.