Appendix A

Lemma A1. $\int_0^\infty \hat{P}(n,t)Ddt = \tilde{D}^{n+1}$**,** $n \geq 0$**.**

Proof. The matrices $\hat{P}(n,t)$ $(n \geq 0, t \geq 0)$ associated with the counting process $\left\{\hat{N}(t), J(t); t \geq 0\right\}$ satisfy the following system of difference-differential equations.

$$\hat{P}'(0,t) = \hat{P}(0,t)C, \quad \text{(A.1)}$$

$$\hat{P}'(n,t) = \hat{P}(n,t)C + \hat{P}(n-1,t)D; \, n \geq 1, \quad \text{(A.2)}$$

with $\hat{P}(0,0) = I_m$ and $\hat{P}(n,0) = \mathbf{0}_m \, (n \geq 1)$.

Let us define a matrix generating function

$$\hat{P}^*(z,t) = \sum_{n=0}^{\infty} \hat{P}(n,t)z^n, \qquad |\, z \,| \leq 1. \quad \text{(A.3)}$$

It satisfies

$$\frac{d}{dt}\hat{P}^*(z,t) = \hat{P}^*(z,t)\,(C + zD)\,,$$
$$\hat{P}^*(z,0) = I_m. \quad \text{(A.4)}$$

Solving the above matrix differential equation we obtain

$$\hat{P}^*(z,t) = e^{(C+zD)}, \qquad |\, z \,| \leq 1, \, t \geq 0. \quad \text{(A.5)}$$

Taking Laplace transform of (A.3) we have

$$\hat{P}^*(z,\theta) = \int_0^\infty e^{-\theta t}\hat{P}^*(z,t)dt. \tag{A.6}$$

Using (A.5) in (A.6) we have

$$\begin{aligned}\int_0^\infty e^{-\theta t}\hat{P}^*(z,t)dt &= \int_0^\infty e^{-\theta t}e^{(C+zD)t}dt,\\ &= \int_0^\infty e^{-[\theta I-(C+zD)]t}dt,\end{aligned}$$

$$or, \quad \int_0^\infty e^{-\theta t}\sum_{n=0}^{\infty}\hat{P}(n,t)z^n dt = \left[\int_0^\infty e^{-[\theta I-(C+zD)]t}[\theta I-(C+zD)]^{-1}dt\right]_0^\infty,$$

$$\begin{aligned} or, \quad \sum_{n=0}^{\infty} z^n \int_0^\infty e^{-\theta t}\hat{P}(n,t)dt &= [\theta I-(C+zD)]^{-1},\\ &= \left[(\theta I-C)\left(I-z(\theta I-C)^{-1}D\right)^{-1}\right],\\ &= \left[\left(I-z(\theta I-C)^{-1}D\right)^{-1}\right]^{-1}(\theta I-C)^{-1},\\ or, \quad \sum_{n=0}^{\infty} z^n \int_0^\infty e^{-\theta t}\hat{P}(n,t)dt &= \sum_{n=0}^{\infty}\left((\theta I-C)^{-1}D\right)^n z^n(\theta I-C)^{-1}. \end{aligned} \tag{A.7}$$

Now equating the coefficients of z^n both sides of (A.6) we get

$$\int_0^\infty e^{-\theta t}\hat{P}(n,t)dt = -\left((\theta I-C)^{-1}D\right)^n(-\theta I+C)^{-1}. \tag{A.8}$$

Now setting $\theta = 0$ in (A.8) we get

$$\int_0^\infty \hat{P}(n,t)dt = -\left((-C)^{-1}D\right)^n(C)^{-1}. \tag{A.9}$$

Post-multiplying (A.9) by D we get

$$\int_0^\infty \hat{P}(n,t)Ddt = -\left((-C)^{-1}D\right)^n (C)^{-1}D,$$
$$= \left((-C)^{-1}D\right)^{n+1}. \qquad \text{(A.10)}$$

Which implies

$$\int_0^\infty \hat{P}(n,t)Ddt = \tilde{D}^{n+1}, \qquad n \geq 0.$$