Extended Abstract

This thesis describes the application of discretization methods that can be used to quickly evaluate solutions of various real-world problems. These methods can also be understood by someone who has an understanding of the topic from the engineering or science perspective.

The thesis consists of six chapters. Chapter 1 includes definitions, notations and literature review on Abel's integral equations, integro-differential equations, numerical methods for partial differential equations and generalized fractional calculus which is used in subsequence chapters.

In chapter 2, we presented two approximate methods such as Quadratic and Cubic approximations for the Riemann-Liouville fractional integral and Caputo fractional derivatives. The approximations error estimates are also obtained. Numerical simulations for these approximation schemes are performed with the test examples from literature and obtained numerical results are also compared. To establish the application of the presented schemes, the problem of Abel's inversion is considered. Numerical inversion of Abel's equation is obtained using Quadratic and Cubic approximations of the Caputo derivative. Test examples from literature are considered to validate the effectiveness of the presented schemes. It is observed that the Quadratic and Cubic approximations schemes produce the convergence of orders h^3 and h^4 respectively. The entire chapter in form of a paper has been published in Journal of King Saud University-Science.

"K. Kumar, R. K. Pandey, S. Sharma, Approximations of Fractional Integrals and Caputo Derivatives with Application in Solving Abel's Integral Equations, ***Journal of King Saud University-Science*****, 2018 (Elsevier).** "

In chapter 3, we consider the integral equation known as generalized Abel's integral equations (GAIEs) [1-2] and present two different numerical schemes to solve it. The GAIEs [1] is defined as,

$$a(t)\int_0^t \frac{\varphi(\tau)}{(t-\tau)^\alpha}d\tau + b(t)\int_t^1 \frac{\varphi(\tau)}{(\tau-t)^\alpha}d\tau = g(t), \qquad (1)$$

where, $0<\alpha<1$, the function φ unknown function and $a(t), b(t), g(t)$is the known function. Further, we studied two numerical approximations for the generalized Abel's integral equations (GAIEs). Two numerical schemes such as linear scheme and quadratic scheme are proposed to solve GAIEs numerically. The error convergence of the presented schemes is also established where it is observed that the quadratic scheme achieves the convergence order up to 3. Some examples of GAIEs from literature are considered to perform the numerical investigations and the obtained numerical results are shown in tabular form. We analyze that the presented schemes work well and provide good numerical results. It is also observed that the accuracy in the numerical solutions can be achieved with smaller value of the step size. As the GAIEs reduces to the Abel's integral equation of the first kind in special case, therefore a similar scheme could be developed to solve such equations. The entire chapter in form of a paper has been published in International Journal of Applied and Computational Mathematics.

"K. Kumar, R. K. Pandey, S. Sharma, Numerical Schemes for the Generalized Abel's Integral Equations, ***International Journal of Applied and Computational Mathematics,*** **4 (2018) pp. 68. (Springer).** "

In chapter 4, we consider the Fractional Integro-Differential Equations (FIDEs) defined in terms of the Caputo fractional derivatives as:

$$D^{\alpha}\varphi(x) = f(x) + \int_0^x K(x,\tau)\varphi(\tau)d\tau\ ,\ \ 0 \le x,\tau \le 1,\ \ 0 < \alpha < 1, \tag{2}$$

with the following supplementary condition $\varphi(0) = \delta$, where $D^{\alpha}\varphi(x)$ indicates the $\alpha -$ th order Caputo derivatives of $\varphi(x)$, and $f(x)$, $K(x,\tau)$ are known functions. We presented a comparative study three numerical schemes such as Linear, Quadratic and Quadratic-Linear scheme for the fractional integro-differential equations defined in terms of the Caputo fractional derivatives. The error estimates of the respective approximations are also established. Numerical tests of the discussed schemes show that all schemes work well, and when the number of terms approximating the solution are increased, the desired solution is achieved. The accuracy of the numerical schemes with respect to the step size h is analyzed and illustrated through various tables. Finally, comparative performances of the schemes are discussed. The entire chapter in form of a paper has been published in Journal of Computational and Applied Mathematics.

"K. Kumar, R. K. Pandey, S. Sharma, Comparative study of three numerical schemes for fractional integro-differential equations, ***Journal of Computational and Applied Mathematics***, **315 (2017) 287-302. (Elsevier).** "

In chapter 5, first we consider the Generalized Fractional Integro-Differential Equations (GFIDEs) in term of Caputo-type Generalized Fractional Derivative (GFD) defined by Agrawal [3] recently. We consider the GFIDEs given as,

$$\mathbb{D}_*^{\alpha}u(t) = f(t) + \int_0^t K(t,s)u(s)ds\ ,\ \ 0 \le t,s \le 1,\ \ 0 < \alpha < 1, \tag{3}$$

with the subsequent additional condition $u(0) = \delta$, where $\mathbb{D}_*^{\alpha}u(t)$ denotes Caputo type GFD of $u(t)$, of order α and $f(t)$, $K(t,s)$ are known functions. Caputo type GFD say $\mathbb{D}_*^{\alpha}u(t)$ is defined using the scale function $z(t)$ and weight function $w(t)$ for $0 < \alpha < 1$,

$$\mathbb{D}_{*}^{\alpha}u(t) = \left(D_{0+;[z;w;2]}^{\alpha}u\right)(t) = \frac{[w(t)]^{-1}}{\Gamma(1-\alpha)}\int_0^t \frac{\left(w(\tau)u(\tau)\right)'}{\left(z(t)-z(\tau)\right)^{\alpha}}\,d\tau. \tag{4}$$

Further, we have discussed two numerical schemes namely *P1* and *P2* for Generalized Fractional Integro-Differential Equations (GFIDEs) using approximations of Generalized Fractional Derivative (GFD). Two approximation schemes namely linear approximation and quadratic approximation are presented for GFD which are later used in solving GFIDEs defined using GFD. For the special choice of scale and weight functions, GFD reduces to Caputo and Riemann–Liouville derivatives. Error estimate and convergence order (CO) of linear and quadratic approximations are also investigated. Further, an illustrative example is considered to validate the effectiveness of proposed numerical schemes.

In chapter 6, we consider Generalized Time-Fractional Telegraph-Type Equation (GTFTTE) with Generalized Time Fractional Partial Derivative (GTFPD) which reduces in Caputo derivative in special choice of scale and weight functions. For this, GTFTTE, we consider GTFPD of type 2 with order α $(0 < \alpha < 1)$, in the Caputo sense as,

$$\frac{{}_{*}\partial^{\alpha}u(x,t)}{{}_{*}\partial t^{\alpha}} = \frac{[w(t)]^{-1}}{\Gamma(1-\alpha)}\int_0^t \frac{1}{[z(t)-z(\tau)]^{\alpha}}\frac{\partial}{\partial\tau}[w(\tau)u(x,\tau)]\,d\tau. \tag{5}$$

Now, we consider the GTFTTE in term of the GTFPD as given in Eq. (5), we get GTFTTE with force term $f(x,t)$ as

$$\frac{{}_{*}\partial^{\alpha}u(x,t)}{{}_{*}\partial t^{\alpha}} + \frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial^2 x} + f(x,t), \tag{6}$$

with initial and boundary conditions,

$$u(x,0) = \varphi(x) \quad x \in [a,b], \tag{7}$$

$$u(a,t) = u(b,t) = 0, \qquad t > 0,\ 0 < \alpha < 1. \tag{8}$$

Further, a numerical scheme is formulated and analysed for solving a Generalized Time-Fractional Telegraph-Type Equation (GTFTTE) defined using Generalized Time Fractional Derivative (GTFD) proposed recently. The GTFD involves the scale and weight functions and it reduces to the traditional Caputo derivative for a particular choice of weight and scale functions. Weight and scale functions play an important role in describing the behaviour of real-life physical systems and thus we study the solution behaviour of the GTFTTE by varying weight and scale functions in GTFD. Some cases of the solution profiles of the GTFTTE under these choices are investigated and obtained results are discussed. The convergence and stability of the presented numerical scheme for GTFTTE are also given. We consider three examples to perform numerical simulations. Furthermore, we describe the effects of scale function and weight functions on the numerical solution of GTFTTE. The entire chapter has been communicated in Numerical Methods for Partial Differential Equations.

"K. Kumar, R. K. Pandey, S. Sharma, Y. Xu, Numerical Scheme with Convergence for Generalized Time–Fractional Telegraph-Type Equation, ***Numerical Methods for Partial Differential Equations***, *(Under Revision).* "

References

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[3]. O.P. Agrawal, Some generalized fractional operators and their applications in integral equations, Fractional Calculus & Applied Analysis **4** (2012) 700-711.