Chapter 5

Numerical Solutions of Generalized Fractional Integro-Differential Equations

5.1 Introduction

Fractional integro-differential equations (FIDEs) appears in the multidisciplinary area see [78, 177, 178, 179]. FIDEs are very difficult to solve analytically and to find the exact solution of this type of equations are very rare. A great deal of interest has been emphasized by several researchers to solve the FIDEs numerically. Some analytical methods like fractional differential transform method [200], variational iteration method [201], Adomian decomposition method [202, 203], and homotopy analysis method [204] have been used to solve FIDEs. Also, FIDEs have been solved numerically by many researchers. Galerkin method, and wavelet Galerkin method based numerical methods are studied by the authors respectively in [184] and [205] for FIDEs. In [180], authors studied the FIDEs using the collocation method. Further methods in [68, 189, 190, 191, 193, 206, 207, 208] and operational matrix method [196] are discussed by the authors for solving FIDEs in recent years. More recently in 2017, Kumar et al. [153] studied three numerical scheme for FIDEs. In this chapter, we defined a new GFIDEs with the help of generalized fractional derivatives given in [83]. GFIDEs are more attractive over the other FIDEs for three good properties. First, the weight $w(t)$ and scale function $z(t)$ appeared in the new generalized fractional derivatives. By selecting different weight and scale functions, we get Riemann-Liouville, Caputo, Erdelyi-Kober, Hadamard, Grunwald-Letnikov and Riesz derivatives. Thus, GFIDEs proposed in this paper is more global to FIDEs. Second, weight function appears in kernel of fractional derivatives play greater dimension of flexibility in modelling. Third, scale functions $z(t)$ shifted the domain $[0, 1]$ to $[z(0), z(1)]$ or $[z(1), z(0)]$ accordingly, it is monotonic increasing or decreasing respectively. As earlier discussion many of author studied the approximation of fractional derivative and using this solve fractional model see [148, 153, 209, 210]. This motivates me for the approximation of generalized fractional derivative defined in term of scale and weight function recently given by Agrawal. In [148], authors studied differential equations of fractional order using the quadratic polynomials. Kumar et al. [209, 210] presented approximation of fractional integrals and fractional derivatives with application in solving Abel's integral equation. In [172], Pandey et al. also studied the Abel's integral equation using collocation method. Further, Agrawal [150] studied fractional variational problem via finite element approximation. Later, in [152], Pandey and Agrawal studied the generalized fractional variational problems. Due to this motivation, in this chapter, first of all we present two approximations of GFD namely linear and quadratic approximation. Further, using this we discuss two numerical schemes for solving the GFIDEs. These schemes mainly based on discretization technique in which firstly, we divide the interval into subintervals and we approximate the unknown function on each subinterval using the linear and quadratic interpolating polynomials.

5.2 Statement of the Problem and Approximation of Fractional Derivatives

In this section, first we define the GFIDEs in term of Caputo-type GFD defined by Agrawal [83] recently and then we presented the linear and quadratic approximation of GFD. Now, we consider the GFIDEs defined as,

$$\mathbb{D}_*^{\alpha} u(t) = f(t) + \int_0^t K(t,s)u(s)ds, \ 0 \leq t, \ s \leq 1, \ 0 < \alpha < 1, \tag{5.1}$$

with the subsequent additional condition $u(0) = \delta$, where $\mathbb{D}_*^{\alpha} u(t)$ denotes Caputo type GFD of $u(t)$, of order α and $f(t)$, $K(t,s)$ are known functions. For the numerical scheme firstly, we present linear and quadratic approximation for the left side of Eq. (5.1).

5.2.1 Linear Approximation of Generalized Fractional Derivative

To find the linear approximation of GFD, first we divide the domain into n subdomains $[t_j, t_{j+1}]$, and using the piecewise linear interpolating polynomial for the approximation of unknown function into each subdomain with equal length $h = 1/n$ such that node point are $t_k = kh, k = 0, 1, 2, \ldots, n$. And denote the $u(t_k) = u_k, w(t_k) = w_k, z(t_k) = z_k$. Assuming $z(t)$ is strictly monotonic increasing on $[0, 1]$

and $w(t) > 0$, then $s = z^{-1}(v)$ by taking $v = z(s)$. Then the type 2 GFD of $u(t)$ at node point t_k can be discretized as

$$[\mathbb{D}_*^\alpha u(t)]_{t_k} = \frac{[w(t_k)]^{-1}}{\Gamma(1-\alpha)} \int_0^{t_k} \frac{\frac{d}{ds}[w(s)u(s)]}{[z(t_k) - z(s)]^\alpha} ds$$

$$= \frac{[w(t_k)]^{-1}}{\Gamma(1-\alpha)} \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} \frac{\frac{d}{ds}[w(s)u(s)]}{[z(t_k) - z(s)]^\alpha} ds, \tag{5.2}$$

where $\mathbb{D}_*^\alpha u(t),\ 0 < \alpha < 1$ denote the Caputo-type GFD of function $u(t)$.

$$= \frac{[w(t_k)]^{-1}}{\Gamma(1-\alpha)} \sum_{j=0}^{k-1} \int_{z_j}^{z_{j+1}} \frac{1}{[z(t_k) - v]^\alpha} \frac{d[w\left((z^{-1}(v)\right) u\left(z^{-1}(v)\right)]}{dz^{-1}(v)} dz^{-1}(v). \tag{5.3}$$

On the each interval $[z_j, z_{j+1}]$, we approximate the unknown function with the help of linear interpolation polynomial $(p_u^1(v))'$

$$= \frac{[w(t_k)]^{-1}}{\Gamma(1-\alpha)} \sum_{j=0}^{k-1} \left[\frac{w_{j+1}u_{j+1} - w_j u_j}{z_{j+1} - z_j}\right] \int_{z_j}^{z_{j+1}} \frac{1}{[z(t_k) - v]^\alpha} dv \tag{5.4}$$

$$= \frac{[w(t_k)]^{-1}}{\Gamma(2-\alpha)} \sum_{j=0}^{k-1} \frac{w_{j+1}u_{j+1} - w_j u_j}{z_{j+1} - z_j} \left[(z_k - z_j)^{(1-\alpha)} - (z_k - z_{j+1})^{(1-\alpha)}\right] \tag{5.5}$$

$$= \frac{[w(t_k)]^{-1}}{\Gamma(2-\alpha)} \sum_{j=0}^{k-1} \left(\left[\frac{(z_k - z_j)^{(1-\alpha)} - (z_k - z_{j+1})^{(1-\alpha)}}{z_{j+1} - z_j}\right] w_{j+1}u_{j+1} - \left[\frac{(z_k - z_j)^{(1-\alpha)} - (z_k - z_{j+1})^{(1-\alpha)}}{z_{j+1} - z_j}\right] w_j u_j \right) \tag{5.6}$$

$$= \sum_{j=0}^{k} A(k,j) u_j. \tag{5.7}$$

Where

$$A(k,j) = \frac{[w(t_k)]^{-1}}{\Gamma(2-\alpha)} \begin{cases} \frac{g_1^k}{z_1 - z_0} w_0 & j = 0, \\ \left(\frac{g_j^k}{z_j - z_{j-1}} - \frac{g_{j+1}^k}{z_{j+1} - z_j}\right) w_j & 1 \leq j \leq k-1, \\ (z_k - z_j)^{-\alpha} w_k & j = k, \end{cases} \tag{5.8}$$

and

$$g_j^k = (z_k - z_{j-1})^{(1-\alpha)} - (z_k - z_j)^{(1-\alpha)}.$$

5.2.2 Quadratic Approximation of Generalized Fractional Derivative

Again, to find the quadratic approximation, first we divide the domain into n subintervals $[t_{j-1}, t_j]$ and using the piecewise liner interpolating polynomial for the approximation of unknown function into $[t_0, t_1]$ and for the other subintervals $[t_{j-1}, t_j]$ $(i.e. j \geq 2)$. We use the piecewise quadratic interpolation polynomial for the approximation of unknown function in the subinterval $[t_{j-1}, t_j]$ with equal step length $h = 1/n$ such that node points are $t_k = kh, k = 0, 1, 2, \ldots, n$ and denote $u(t_k) = u_k, w(t_k) = w_k,\ z(t_k) = z_k$. Assuming $z(t)$ is strictly monotonic increasing $[0, 1]$ and $w(t) > 0$, then $s = z^{-1}(v)$ by taking $v = z(s)$. Then the type 2 GFD of $u(t)$ at node point t_k can be discretized as

$$\begin{aligned} [\mathbb{D}_*^\alpha u(t)]_{t_k} &= \frac{[w(t_k)]^{-1}}{\Gamma(1-\alpha)} \int_0^{t_k} \frac{\frac{d}{ds}[w(s)u(s)]}{[z(t_k) - z(s)]^\alpha} ds \\ &= \frac{[w(t_k)]^{-1}}{\Gamma(1-\alpha)} \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} \frac{\frac{d}{ds}[w(s)u(s)]}{[z(t_k) - z(s)]^\alpha} ds \end{aligned}$$

$$= \frac{[w(t_k)]^{-1}}{\Gamma(1-\alpha)} \int_{t_0}^{t_1} \frac{\frac{d}{ds}[w(s)u(s)]}{[z(t_k)-z(s)]^{\alpha}} ds + \frac{[w(t_k)]^{-1}}{\Gamma(1-\alpha)} \sum_{j=2}^{k} \int_{t_{j-1}}^{t_j} \frac{\frac{d}{ds}[w(s)u(s)]}{[z(t_k)-z(s)]^{\alpha}} ds \tag{5.9}$$

$$= \frac{[w(t_k)]^{-1}}{\Gamma(1-\alpha)} \int_{z_0}^{z_1} \frac{1}{[z(t_k)-v]^{\alpha}} \frac{d[w\left((z^{-1}(v)\right) u\left(z^{-1}(v)\right)]}{dz^{-1}(v)} dz^{-1}(v)$$

$$+ \frac{[w(t_k)]^{-1}}{\Gamma(1-\alpha)} \sum_{j=2}^{k} \int_{z_{j-1}}^{z_j} \frac{1}{[z(t_k)-v]^{\alpha}} \frac{d[w\left((z^{-1}(v)\right) u\left(z^{-1}(v)\right)]}{dz^{-1}(v)} dz^{-1}(v). \tag{5.10}$$

On the first interval $[z_0, z_1]$, approximate the unknown function with the help of linear interpolation polynomial $(p_u^1(v))'$ and for the next subintervals $(k \geq 2)$, the quadratic interpolation function $(p_u^2(v))'$ for the three points $(z_{j-2}, u_{j-2}), (z_{j-1}, u_{j-1})$ and (z_j, u_j) is applied such that,

$$(p_u^1(v))' = \frac{w_1 u_1 - w_0 u_0}{z_1 - z_0}, \tag{5.11}$$

$$(p_u^2(v))' = \Bigg(\frac{(v-z_{j-1})(v-z_j)}{(z_{j-1}-z_{j-2})(z_j-z_{j-2})} w_{j-2}u_{j-2} - \frac{(v-z_{j-2})(v-z_j)}{(z_{j-1}-z_{j-2})(z_j-z_{j-1})} w_{j-1}u_{j-1}$$

$$- \frac{(v-z_{j-2})(v-z_{j-1})}{(z_j-z_{j-2})(z_j-z_{j-1})} w_j u_j \Bigg)'. \tag{5.12}$$

$$(p_u^2(v))' = \frac{(2v-z_{j-1}-z_j)}{(z_{j-1}-z_{j-2})(z_j-z_{j-2})} w_{j-2}u_{j-2} - \frac{(2v-z_{j-2}-z_j)}{(z_{j-1}-z_{j-2})(z_j-z_{j-1})} w_{j-1}u_{j-1}$$

$$+ \frac{(2v-z_{j-2}-z_{j-1})}{(z_j-z_{j-2})(z_j-z_{j-1})} w_j u_j. \tag{5.13}$$

Again consider first term of Eq. (5.10)

$$\frac{[w(t_k)]^{-1}}{\Gamma(1-\alpha)} \int_{z_0}^{z_1} \frac{1}{[z(t_k)-v]^{\alpha}} \frac{d[w\left(z^{-1}(v)\right) u\left(z^{-1}(v)\right)]}{dz^{-1}(v)} dz^{-1}(v)$$

$$= \frac{[w(t_k)]^{-1}}{\Gamma(2-\alpha)} \left(\frac{w_1 u_1 - w_0 u_0}{z_1 - z_0} \right) \left[(z_k - z_0)^{(1-\alpha)} - (z_k - z_1)^{(1-\alpha)} \right]$$

$$= \frac{[w(t_k)]^{-1}}{\Gamma(2-\alpha)} \left[\frac{(z_k - z_0)^{(1-\alpha)} - (z_k - z_1)^{(1-\alpha)}}{z_1 - z_0} \right] w_1 u_1$$

$$- \frac{[w(t_k)]^{-1}}{\Gamma(2-\alpha)} \left[\frac{(z_k - z_0)^{(1-\alpha)} - (z_k - z_1)^{(1-\alpha)}}{z_1 - z_0} \right] w_0 u_0. \tag{5.14}$$

And second term of Eq. (5.10)

$$\frac{[w(t_k)]^{-1}}{\Gamma(1-\alpha)} \sum_{j=2}^{k} \int_{z_{j-1}}^{z_j} \frac{1}{[z(t_k) - v]^{\alpha}} \frac{d[w\,(z^{-1}(v))\,u\,(z^{-1}(v))]}{dz^{-1}(v)} dz^{-1}(v)$$

$$= \sum_{j=2}^{k} a_j^k u_{j-2} - b_j^k u_{j-1} + c_j^k u_j. \tag{5.15}$$

Where,

$$a_j^k = \frac{w_{j-2}}{(z_{j-1} - z_{j-2})(z_j - z_{j-2})} \frac{[w(t_k)]^{-1}}{\Gamma(1-\alpha)} \left[\frac{(2z_k - z_{j-1} - z_j)}{(1-\alpha)} p_j^k - \frac{2}{(2-\alpha)} q_j^k \right], \tag{5.16}$$

$$b_j^k = \frac{w_{j-1}}{(z_{j-1} - z_{j-2})(z_j - z_{j-1})} \frac{[w(t_k)]^{-1}}{\Gamma(1-\alpha)} \left[\frac{(2z_k - z_{j-2} - z_j)}{(1-\alpha)} p_j^k - \frac{2}{(2-\alpha)} q_j^k \right], \tag{5.17}$$

$$c_j^k = \frac{w_j}{(z_j - z_{j-2})(z_j - z_{j-1})} \frac{[w(t_k)]^{-1}}{\Gamma(1-\alpha)} \left[\frac{(2z_k - z_{j-2} - z_{j-1})}{(1-\alpha)} p_j^k - \frac{2}{(2-\alpha)} q_j^k \right], \tag{5.18}$$

$$p_j^k = (z_k - z_{j-1})^{(1-\alpha)} - (z_k - z_j)^{(1-\alpha)}, \tag{5.19}$$

$$q_j^k = (z_k - z_{j-1})^{(2-\alpha)} - (z_k - z_j)^{(2-\alpha)}. \tag{5.20}$$

Finally we get approximation of first term of Eq. (5.1) as,

$$= \frac{[w(t_k)]^{-1}}{\Gamma(2-\alpha)} \left[\frac{p_1^k}{z_1 - z_0} \right] \left[\frac{w_1 u_1 - w_0 u_0}{z_1 - z_0} \right] + \sum_{j=2}^{k} (a_j^k u_{j-2} - b_j^k u_{j-1} + c_j^k u_j)$$

$$= p_k u_1 - q_k u_0 + \sum_{j=2}^{k}(a_j^k u_{j-2} - b_j^k u_{j-1} + c_j^k u_j), \tag{5.21}$$

where,

$$p_k = \frac{[w(t_k)]^{-1}}{\Gamma(2-\alpha)}\left[\frac{p_1^k w_1}{z_1 - z_0}\right], \tag{5.22}$$

$$q_k = \frac{[w(t_k)]^{-1}}{\Gamma(2-\alpha)}\left[\frac{p_1^k w_0}{z_1 - z_0}\right]. \tag{5.23}$$

5.3 Error Estimate of Approximation of Generalized Fractional Derivatives

In this section, we presents error estimate of approximation of GFD discussed in previous Section 5.2. For this, we present error estimate of linear and quadratic approximation of GFD. Let E_L^k and E_Q^k are the error estimate of liner and quadratic approximation of GFD respectively.

5.3.1 Error Estimate of the Linear Approximation

To analyze error of approximation for GFD, we use notation for simplicity, let $g(s) = w(s)u(s)$ and $s = z^{-1}(v)$ by taking $v = z(s)$. Hence $g(v) = w(z^{-1}(v))u(z^{-1}(v))$, $w(t_k) = w_k$ and $z(t_k) = z_k$. According to Newton interpolation method, we interpolate $g(v)$ on $[z_j, z_{j+1}]$ by $p_g^1(v)$ respectively such that,

$$g(v) - p_g^1(v) = \frac{g''(\xi_j)}{2!}(v - z_j)(v - z_{j+1}), \qquad \xi_j \in [z_j, z_{j+1}]. \tag{5.24}$$

Where, $p_g^1(v)$ is piecewise linear interpolation polynomial using the node (z_j, g_j), (z_{j+1}, g_{j+1}).

Let E_L^k be the truncation error given by

$$E_L^k = \frac{[w_k]^{-1}}{\Gamma(1-\alpha)} \sum_{j=0}^{k-1} \int_{z_j}^{z_{j+1}} \frac{1}{[z_k - v]^{\alpha}} [g(v) - p_g^1(v)]' dv \tag{5.25}$$

$$= \frac{[w_k]^{-1}}{\Gamma(1-\alpha)} \sum_{j=0}^{k-1} \int_{z_j}^{z_{j+1}} \frac{g''(\xi_j)}{2!} \frac{1}{[z_k - v]^{\alpha}} [(v - z_j)(v - z_{j+1})]' dv \tag{5.26}$$

using integration by part we get,

$$= -\frac{\alpha[w_k]^{-1}}{\Gamma(1-\alpha)} \sum_{j=0}^{k-1} \int_{z_j}^{z_{j+1}} \frac{g''(\xi_j)}{2!} [(v - z_j)(v - z_{j+1})] \, d[z_k - v]^{-\alpha} \tag{5.27}$$

$$= \frac{\alpha[w_k]^{-1}}{\Gamma(1-\alpha)} \sum_{j=0}^{k-1} \int_{z_j}^{z_{j+1}} \frac{g''(\xi_j)}{2!} [(v - z_j)(z_{j+1} - v)] \, d[z_k - v]^{-\alpha} \tag{5.28}$$

$$= \frac{\alpha[w_k]^{-1}}{\Gamma(1-\alpha)} \sum_{j=0}^{k-2} \int_{z_j}^{z_{j+1}} \frac{g''(\xi_j)}{2!} [(v - z_j)(z_{j+1} - v)] \, d[z_k - v]^{-\alpha}$$

$$+ \frac{\alpha[w_k]^{-1}}{\Gamma(1-\alpha)} \int_{z_{k-1}}^{z_k} \frac{g''(\xi_j)}{2!} [(v - z_{k-1})(z_k - v)] \, d[z_k - v]^{-\alpha} \tag{5.29}$$

$$\leq \frac{\alpha \max_{t_0 \leq \xi \leq t_k} |g''(\xi_j)| \max_{0 \leq j \leq k} (z_{j+1} - z_j)^2}{8 w_k \Gamma(1-\alpha)} \sum_{j=0}^{k-2} \int_{z_j}^{z_{j+1}} [z_k - v]^{-\alpha-1} dv$$

$$+ \frac{\alpha \max_{t_0 \leq \xi \leq t_k} |g''(\xi_j)|}{2 w_k \Gamma(1-\alpha)} \int_{z_{k-1}}^{z_k} [(v - z_{k-1})(z_k - v)][z_k - v]^{-\alpha-1} dv, \tag{5.30}$$

since

$$\sum_{j=0}^{k-2} \int_{z_j}^{z_{j+1}} [z_k - v]^{-\alpha-1} dv = \frac{1}{\alpha}[(z_k - z_{k-1})^{-\alpha} - (z_k - z_0)^{-\alpha}] \leq \frac{1}{\alpha}(z_k - z_{k-1})^{-\alpha} \tag{5.31}$$

and

$$\int_{z_{k-1}}^{z_k} [(v - z_{k-1})(z_k - v)][z_k - v]^{-\alpha-1} dv = \frac{1}{(1-\alpha)} \int_{z_{k-1}}^{z_k} [v - z_k]^{1-\alpha} dv$$

$$= \frac{1}{(1-\alpha)(2-\alpha)} (z_k - z_{k-1})^{2-\alpha}. \qquad (5.32)$$

Thus, from Eq. (5.30) and (5.31)-(5.32), we get

$$E_L^k \leq \left[\frac{[w_k]^{-1}}{8\Gamma(1-\alpha)} + \frac{\alpha[w_k]^{-1}}{2\Gamma(3-\alpha)} \right] \max_{t_0 \leq v \leq t_k} |g''(v)| \max_{0 \leq j \leq k} (z_{j+1} - z_j)^{2-\alpha}. \qquad (5.33)$$

Denoting $[t_{j^*}, t_{j^*+1}]$ as the interval satisfying $\max_{0 \leq j \leq k}(z_{j+1} - z_j) = [z_{j^*+1} - z_{j^*}]$, then the error of approximation has the form

$$E_L^k \leq \left[\frac{[w_k]^{-1}}{8\Gamma(1-\alpha)} + \frac{\alpha[w_k]^{-1}}{2\Gamma(3-\alpha)} \right] \max_{t_0 \leq v \leq t_k} |g''(v)|(Lh)^{2-\alpha} \qquad (5.34)$$

when $z(t)$ satisfying the Lipschitz condition on $[t_{j^*}, t_{j^*+1}]$ and L is the Lipschitz constant.

5.3.2 Error Estimate of the Quadratic Approximation

To analyze error of approximation for generalized time fractional derivative, we use notation for simplicity, let $g(s) = w(s)u(s)$ and $s = z^{-1}(v)$ by taking $v = z(s)$. Hence $g(v) = w(z^{-1}(v))u(z^{-1}(v))$, $w(t_k) = w_k$ and $z(t_k) = z_k$. According to Newton interpolation method, we interpolate $g(v)$ on $[z_0, z_1]$ and $[z_{j-1}, z_j]$ by $p_g^1(v)$ and $p_g^2(v)$ respectively such that

$$g(v) - p_g^1(v) = \frac{g''(\xi_1)}{2!}(v - z_0)(v - z_1), \qquad \xi_1 \in [z_0, z_1]. \qquad (5.35)$$

$$g(v) - p_g^2(v) = \frac{g'''(\xi_1)}{3!}(v - z_{j-2})(v - z_{j-1})(v - z_j), \qquad \xi_2 \in [z_{j-2}, z_j]. \tag{5.36}$$

Where, $p_g^1(v)$ and $p_g^2(v)$ are piecewise linear and quadratic interpolation polynomials using the node $(z_0, g_0), (z_1, g_1)$ and $(z_{j-2}, g_{j-2}), (z_{j-1}, g_{j-1}), (z_j, g_j)$ respectively.

Let E_Q^k be the truncation error of quadratic approximation then

$$E_Q^k = \frac{[w_k]^{-1}}{\Gamma(1-\alpha)} \int_{z_0}^{z_1} \frac{1}{[z_k - v]^\alpha}[g(v) - p_g^1(v)]' dv$$
$$+ \frac{[w_k]^{-1}}{\Gamma(1-\alpha)} \sum_{j=2}^{k} \int_{z_{j-1}}^{z_j} \frac{1}{[z_k - v]^\alpha}[g(v) - p_g^2(v)]' dv. \tag{5.37}$$

Consider first term of Eq. (5.37)

$$\frac{[w_k]^{-1}}{\Gamma(1-\alpha)} \int_{z_0}^{z_1} \frac{1}{[z_k - v]^\alpha}[g(v) - p_g^1(v)]' dv$$
$$= \frac{[w_k]^{-1}}{2\Gamma(1-\alpha)} \int_{z_0}^{z_1} \frac{g''(\xi_1)}{(z_k - v)^\alpha}[(v - z_0)(v - z_1)]' dv \tag{5.38}$$

Using integration by part we get

$$\frac{[w_k]^{-1}}{2\Gamma(1-\alpha)} \int_{z_0}^{z_1} g''(\xi_1)(z_k - v)^{-\alpha-1}(v - z_0)(v - z_1) dv$$
$$\leq \frac{[w_k]^{-1}}{2\Gamma(1-\alpha)} \max_{t_0 \leq \xi_1 \leq t_k} |g''(\xi_1)|(z_k - z_1)^{-\alpha-1} \int_{z_0}^{z_1} (v - z_0)(z_1 - v) dv$$
$$\leq \frac{-\alpha \max_{t_0 \leq \xi_1 \leq t_k} |g''(\xi_1)|(z_k - z_1)^{-\alpha-1}(z_1 - z_0)^3}{12 w_k \Gamma(1-\alpha)}. \tag{5.39}$$

Consider second term Eq. (5.37)

$$\frac{[w_k]^{-1}}{\Gamma(1-\alpha)} \sum_{j=2}^{k} \int_{z_{j-1}}^{z_j} \frac{1}{[z_k - v]^\alpha}[g(v) - p_g^2(v)]' dv$$

Since $g(v) - p_g^2(v) = \frac{g'''(\xi_2)}{3!}(v - z_{j-2})(v - z_{j-1})(v - z_j)$ and applying integration by part we get

$$\frac{\alpha[w_k]^{-1}}{\Gamma(1-\alpha)} \sum_{j=2}^{k} \int_{z_{j-1}}^{z_j} \frac{g'''(\xi_2)}{3!}(v - z_{j-2})(v - z_{j-1})(v - z_j)[z(t_k) - v]^{-\alpha-1} dv$$

$$= \frac{\alpha[w_k]^{-1}}{\Gamma(1-\alpha)} \sum_{j=2}^{k-1} \int_{z_{j-1}}^{z_j} \frac{g'''(\xi_2)}{3!}(v - z_{j-2})(v - z_{j-1})(v - z_j)[z(t_k) - v]^{-\alpha-1} dv$$

$$+ \frac{\alpha[w_k]^{-1}}{\Gamma(1-\alpha)} \int_{z_{k-1}}^{z_k} \frac{g'''(\xi_2)}{3!}(v - z_{k-2})(v - z_{k-1})(v - z_k)[z(t_k) - v]^{-\alpha-1} dv \tag{5.40}$$

Consider the first part of Eq. (5.40)

$$\frac{\alpha[w_k]^{-1}}{\Gamma(1-\alpha)} \sum_{j=2}^{k-1} \int_{z_{j-1}}^{z_j} \frac{g'''(\xi_2)}{3!}(v - z_{j-2})(v - z_{j-1})(v - z_j)[z(t_k) - v]^{-\alpha-1} dv$$

$$\leq \frac{\alpha[w_k]^{-1}}{\Gamma(1-\alpha)} \varphi(z_{j-2}, z_{j-1}, z_j) \sum_{j=2}^{k-1} \int_{z_{j-1}}^{z_j} \frac{g'''(\xi_2)}{3!}[z(t_k) - v]^{-\alpha-1} dv \tag{5.41}$$

$$= \frac{\alpha[w_k]^{-1} \max_{t_0 \leq \xi_2 \leq t_k} |g'''(\xi_2)|}{6\Gamma(1-\alpha)} \varphi(z_{j-2}, z_{j-1}, z_j) \int_{z_1}^{z_{k-1}} [z(t_k) - v]^{-\alpha-1} dv. \tag{5.42}$$

Where,

$$\varphi(z_{j-2}, z_{j-1}, z_j) = \frac{1}{27} \varphi_1(z_{j-2}, z_{j-1}, z_j) \varphi_2(z_{j-2}, z_{j-1}, z_j) \varphi_3(z_{j-2}, z_{j-1}, z_j), \tag{5.43}$$

$$\varphi_1(z_{j-2}, z_{j-1}, z_j) = (z_{j-1} + z_j - 2z_{j-2}) - \sigma(z_{j-2}, z_{j-1}, z_j), \tag{5.44}$$

$$\varphi_2(z_{j-2}, z_{j-1}, z_j) = (z_j + z_{j-2} - 2z_{j-1}) - \sigma(z_{j-2}, z_{j-1}, z_j), \tag{5.45}$$

$$\varphi_3(z_{j-2}, z_{j-1}, z_j) = (z_{j-2} + z_{j-1} - 2z_j) - \sigma(z_{j-2}, z_{j-1}, z_j), \tag{5.46}$$

$$\sigma(z_{j-2}, z_{j-1}, z_j) = \sqrt{z_{j-2}(z_{j-2} - z_{j-1}) + z_{j-1}(z_{j-1} - z_j) + z_j(z_j - z_{j-2})}. \tag{5.47}$$

Again since,

$$\int_{z_1}^{z_{k-1}} [z(t_k) - v]^{-\alpha-1} dv = \frac{1}{\alpha}[(z_k - z_{k-1})^{-\alpha} - (z_k - z_1)^{-\alpha}] \leq (z_k - z_{k-1})^{-\alpha}. \quad (5.48)$$

Hence, from Eq. (5.42) and Eq. (5.48), we have first term of Eq. (5.40) is less than or equal to

$$\frac{\alpha[w_k]^{-1} \max_{t_0 \leq \xi_1 \leq t_k} |g'''(\xi_2)|}{6\Gamma(1-\alpha)} \varphi(z_{j-2}, z_{j-1}, z_j)(z_k - z_{k-1})^{-\alpha}. \quad (5.49)$$

Consider the second term of Eq. (5.40)

$$\frac{\alpha[w_k]^{-1}}{\Gamma(1-\alpha)} \int_{z_{k-1}}^{z_k} \frac{g'''(\xi_2)}{3!}(v - z_{k-2})(v - z_{k-1})(v - z_k)[z(t_k) - v]^{-\alpha-1} dv$$

$$= -\frac{\alpha[w_k]^{-1}}{6\Gamma(1-\alpha)} \int_{z_{k-1}}^{z_k} g'''\xi_2(v - z_{k-2})(v - z_{k-1})[z(t_k) - v]^{-\alpha} dv \quad (5.50)$$

$$= -\frac{\alpha[w_k]^{-1} \max_{t_0 \leq \xi_2 \leq t_k} |g'''(\xi_2)|}{6\Gamma(1-\alpha)}(z_k - z_{k-1})^{2-\alpha}$$

$$\times \left[\frac{(z_{k-1} - z_{k-2})}{(1-\alpha)(2-\alpha)} + \frac{2(z_k - z_{k-1})}{(1-\alpha)(2-\alpha)(3-\alpha)}\right] \quad (5.51)$$

$$= -\frac{\alpha[w_k]^{-1} \max_{t_0 \leq \xi_2 \leq t_k} |g'''(\xi_2)|}{3\Gamma(1-\alpha)} \frac{(z_k - z_{k-1})^{2-\alpha}}{(1-\alpha)(2-\alpha)} \left[\frac{(z_{k-1} - z_{k-2})}{2} + \frac{(z_k - z_{k-1})}{(3-\alpha)}\right]. \quad (5.52)$$

From Eqs. (5.39), (5.49) and (5.52) we have

$$E_Q^k \leq \frac{-\alpha \max_{t_0 \leq \xi_1 \leq t_k} |g''(\xi_1)|(z_k - z_1)^{-\alpha-1}(z_1 - z_0)^3}{12 w_k \Gamma(1-\alpha)}$$
$$+ \frac{\alpha [w_k]^{-1} \max_{t_0 \leq \xi_1 \leq t_k} |g'''(\xi_2)|}{6\Gamma(1-\alpha)} \varphi(z_{j-2}, z_{j-1}, z_j)(z_k - z_{k-1})^{-\alpha}$$
$$- \frac{\alpha [w_k]^{-1} \max_{t_0 \leq \xi_2 \leq t_k} |g'''(\xi_2)|}{3\Gamma(1-\alpha)} \frac{(z_k - z_{k-1})^{2-\alpha}}{(1-\alpha)(2-\alpha)} \left[\frac{(z_{k-1} - z_{k-2})}{2} + \frac{(z_k - z_{k-1})}{(3-\alpha)} \right]. \tag{5.53}$$

Lemma 5.3.1. *If scale function $z(t)$ is satisfying the Lipschitz condition with constant L, that is $|z_j - z_{j-1}| \leq Lh$ then we have following relation*
(i) $\sigma(z_{j-2}, z_{j-1}, z_j) \leq \sqrt{3}Lh.$
(ii) $\varphi_1(z_{j-2}, z_{j-1}, z_j) \leq (3 - \sqrt{3})Lh.$
(iii) $\varphi_2(z_{j-2}, z_{j-1}, z_j) \leq -(\sqrt{3})Lh.$
(iv) $\varphi_3(z_{j-2}, z_{j-1}, z_j) \leq (-3 - \sqrt{3})Lh.$
(v) $\varphi(z_{j-2}, z_{j-1}, z_j) \leq \left(\frac{2\sqrt{3}}{9}\right) L^3h^3 \leq \frac{1}{2}L^3h^3.$

It is clear from above Lemma 5.3.1 and Eq. (5.53) that error estimate E_Q^k is the order of convergence $h^{3-\alpha}$, if the scale function satisfying the Lipschitz condition with constant L.

$$|E_Q^k| \leq \frac{-\alpha \max_{t_0 \leq v \leq t_k} |g''(v)|(t_k - t_1)^{-\alpha-1} L^{2-\alpha}(h)^3}{12 w_k \Gamma(1-\alpha)}$$
$$+ \frac{\alpha [w_k]^{-1} \max_{t_0 \leq \xi_2 \leq t_k} |g'''(\xi_2)|}{\Gamma(1-\alpha)} \frac{1}{12} L^{3-\alpha}(h)^{3-\alpha}$$
$$- \frac{\alpha [w_k]^{-1} \max_{t_0 \leq \xi_2 \leq t_k} |g'''(\xi_2)|}{3\Gamma(1-\alpha)} \frac{1}{(1-\alpha)(2-\alpha)} \left[\frac{1}{2} + \frac{1}{(3-\alpha)} \right] L^{3-\alpha} h^{3-\alpha}.$$

5.4 Numerical Schemes

In this section, we present two numerical schemes namely linear and quadratic for the GFIDEs. Since, the GFIDEs has mainly two term one is fractional term and second one is integral term which are present on the left and right side of Eq.(5.1) respectively. We get linear system after approximation of left side GFD term and right side integral term of Eq.(5.1) and after solving it, we get the numerical solution of GFIDEs. The linear and quadratic schemes are symbolized as $P1$ and $P2$ respectively. Now, we are presenting numerical schemes as follows:

5.4.1 The Linear Scheme ($P1$)

For studying the linear scheme firstly, we divide the domain into n subdomains $[t_j, t_{j+1}]$ and then the linear interpolation polynomial is used for the approximation of unknown function into each subdomain with the uniform step size $h = 1/n$ such that the node points are $t_k = kh, k = 0, 1, \dots, n$. The function $u(t), f(t)$ denoted as $u_k = u(t_k)$, and $f_k = f(t_k)$ at the node point t_k. The GFIDEs defined by Eq. (5.1) takes the form for $0 < \alpha < 1$ at the node point t_k ,

$$[\mathbb{D}_*^\alpha u(t)]_{t_k} = f(t_k) + \int_0^{t_k} K(t_k, s)u(s)ds, \ 0 \le t, \ s \le 1. \tag{5.54}$$

The left side of Eq.(5.54) can be approximated using linear approximation and we get from Eq. (5.7), $[\mathbb{D}_*^\alpha u(t)]_{t_k} = \sum_{j=0}^{k} A(k,j)u_j$, where $A(k,j)$ defined in Eq. (5.8). Now we approximate integral part of Eq. (5.54) at node point t_k like Linear Scheme ($S1$) (see [153]). Hence we get,

$$\int_0^{t_k} K(t_k, s)u(s)ds = \sum_{j=0}^{k} B(k,j)u_j, \tag{5.55}$$

where

$$B(k,j) = \begin{cases} S(k,0), & j=0 \\ S(k,j)+T(k,j-1), & 1\le j\le k \\ T(k,k-1) & j=k, \end{cases} \tag{5.56}$$

$$S(k,j) = h\int_0^1 (1-p)K(kh,hp+jh)dp, \tag{5.57}$$

$$T(k,j) = h\int_0^1 pK(kh,hp+jh)dp. \tag{5.58}$$

Using Eq. (5.54) and Eq. (5.55), Eq. (5.7) reduces to linear system ,

$$\sum_{j=0}^{k} \xi(k,j)u_j = f_k, \qquad k=1,2,\ldots,n, \tag{5.59}$$

where

$$\xi(k,j) = A(k,j) - B(k,j).$$

Clearly, by solving linear system given by Eq.(5.59) we get the approximate solution of Eq.(5.54).

5.4.2 The Quadratic Scheme ($P2$)

For the quadratic scheme, first we divide the interval into n subintervals $[t_{j-1}, t_j]$ and we approximate the unknown function using the linear interpolating polynomial on subinterval $[t_0, t_1]$ and for the other subintervals $[t_{j-1}, t_j]$($i.e.$ $j \ge 2$), we use piecewise quadratic interpolation polynomial for the approximation of unknown function with equal lenth $h = 1/n$ such that the node points are $t_k = kh, k = 0, 1, 2, \dots, n$. Again the left side of Eq. (5.54) can be approximated using quadratic approximation and we get from Eq. (5.21)

$$[\mathbb{D}_*^\alpha u(t)]_{t_k} = p_k u_1 - q_k u_0 + \sum_{j=2}^{k} (a_j^k u_{j-2} - b_j^k u_{j-1} + c_j^k u_j), \tag{5.60}$$

where coefficients are defined in Eq.(5.16-5.20). Now we approximate integral part of Eq. (5.54) at node point t_k like Quadratic Scheme (S2) (see [153]). Hence we get,

$$\int_0^{t_k} K(t_k, s)u(s)ds = \sum_{j=0}^{k} s_j u_j. \tag{5.61}$$

The coefficients v_j for different k can be designed as,

$$\text{For } k = 1, \qquad \begin{cases} s_0 = a_1 \\ s_1 = b_1 \end{cases}$$

$$\text{For } k = 2, \qquad \begin{cases} s_0 = S_1(2,2) + a_2 \\ s_1 = S_2(2,2) + b_2 \\ s_2 = S_3(2,2) \end{cases}$$

$$\text{For } k = 3, \qquad \begin{cases} s_0 = S_1(3,2) + a_3 \\ s_1 = S_1(3,3) + S_2(3,2) + b_3 \\ s_2 = S_2(3,3) + S_3(3,2) \\ s_3 = S_3(3,3) \end{cases}$$

And for $k \geq 4$, the coefficients are calculated as:

$$\begin{cases} s_0 = S_1(k,2) + a_k \\ s_1 = S_1(k,3) + S_2(k,2) + b_k \\ s_j = S_1(k,j+2) + S_2(k,j+1) + S_3(k,j)(2 \leq j \leq k-2) \\ s_{k-1} = S_2(k,k) + S_3(k,k-1) \\ s_k = S_3(k,k) \end{cases}$$

Where,

$$\begin{aligned} a_k &= h\int_0^1 K(kh,ph)(1-p)dp, \\ b_k &= h\int_0^1 K(kh,ph)pdp, \\ S_1(k,j) &= \frac{h}{2}\int_0^1 K(kh,hp+jh-h)p(p-1)dp, \\ S_2(k,j) &= h\int_0^1 K(kh,hp+jh-h)(1-p^2)dp, \\ S_3(k,j) &= \frac{h}{2}\int_0^1 K(kh,hp+jh-h)p(p+1)dp. \end{aligned}$$

Using Eq. (5.60) and Eq. (5.61), the Eq. (5.54) can be expressed as

$$A_k(u_0,u_1,u_2,\ldots,u_k) - C_k(u_0,u_1,u_2,\ldots,u_k) = f_k, \qquad k = 1,2,\ldots,n \tag{5.62}$$

where,

$$A_k(u_0,u_1,u_2,\ldots,u_k) = [\mathbb{D}_*^{\alpha}u(t)]_{t_k},$$

$$C_k(u_0, u_1, u_2, \ldots, u_k) = \sum_{j=0}^{k} s_j u_j.$$

Clearly, by solving linear system given by Eq.(5.62) we get the approximate solution of Eq.(5.1).

5.5 Numerical Results

Numerical outcomes for the linear and quadratic schemes will be presented in this section. We also present numerical outcomes for the linear and quadratic approximation of generalized fractional Caputo-type derivatives for different scale function $z(t)$. For this, we consider example from literature and investigate the performance of the presented scheme. The examples are considered from [153] such that it has same exact solution. The numerical results obtained using the presented numerical schemes $P1$ and $P2$ are displayed through the tables. For the numerical simulation we take weight function $w(t) = 1$. We also calculate the convergence order (CO) of linear and quadratic approximation of GFD for different scale functions. The maximum absolute error (MAE) and convergence order (CO) are calculated. In the Tables 5.1-5.5 and Tables 5.6-5.7 shows the numerical results for linear and quadratic approximation respectively.

TABLE 5.1: MAE and CO for $z(t) = t^2, \alpha = 0.5$, at $t = 0.3$ for $u(t) = t^2 - t$ using linear approximation.

h	MAE	CO
1/10	3.48007×10^{-2}	
1/20	1.17389×10^{-2}	1.56782
1/40	4.06577×10^{-3}	1.5297
1/80	1.42373×10^{-3}	1.51385

TABLE 5.2: MAE and CO for $z(t) = t^2, \alpha = 0.4$, at $t = 0.3$ for $u(t) = t^2 - t$ using linear approximation.

h	MAE	CO
1/10	2.02214×10^{-2}	
1/20	6.50385×10^{-3}	1.63652
1/40	2.13389×10^{-3}	1.60781
1/80	7.04818×10^{-4}	1.59816
1/160	2.33267×10^{-24}	1.59527

TABLE 5.3: MAE and CO for $z(t) = t^3, \alpha = 0.8$, at $t = 0.6$ for $u(t) = t^2 - t$ using linear approximation.

h	MAE	CO
1/10	3.69401×10^{-2}	
1/20	1.46143×10^{-2}	1.33781
1/40	6.04967×10^{-3}	1.27245
1/80	2.56371×10^{-3}	1.23862

TABLE 5.4: MAE and CO for $z(t) = t^{0.4}, \alpha = 0.6$, at $t = 0.6$ for $u(t) = t^2 - t$ using linear approximation.

h	MAE	CO
1/10	2.67273×10^{-2}	
1/20	1.06875×10^{-2}	1.32239
1/40	4.19023×10^{-3}	1.35082
1/80	1.62402×10^{-3}	1.36746

TABLE 5.5: MAE and CO for $z(t) = t^4, \alpha = 0.6$, at $t = 0.6$ for $u(t) = t^2 - t$ using linear approximation.

h	MAE	CO
1/10	1.69605×10^{-2}	
1/20	5.60052×10^{-3}	1.59855
1/40	1.9611×10^{-3}	1.5139
1/80	7.09303×10^{-4}	1.46719
1/160	2.61351×10^{-4}	1.44041

TABLE 5.6: MAE and CO for $z(t) = t^2, \alpha = 0.5$, at $t = 0.6$ for $u(t) = t - t^3$ using linear approximation.

h	MAE	CO
1/10	6.84003×10^{-3}	
1/20	1.04274×10^{-3}	2.71362
1/40	1.70108×10^{-4}	2.61586
1/80	2.85787×10^{-5}	2.57344
1/160	4.87585×10^{-6}	2.55121

TABLE 5.7: MAE and CO for $z(t) = t^2, \alpha = 0.2$, at $t = 0.6$ for $u(t) = t - t^3$ using linear approximation.

h	MAE	CO
1/10	1.48829×10^{-3}	
1/20	2.07611×10^{-4}	2.8417
1/40	2.99599×10^{-5}	2.79278
1/80	4.35344×10^{-6}	2.78281
1/160	6.31831×10^{-7}	2.78455

Example 5.5.1. *Consider the following GFIDEs such as [188],*

$$\mathbb{D}_*^{\alpha}u(t) = (t^{-m})^{-2/m}(t^{-\alpha})\left[-\frac{(t^{-m})^{1/m}\Gamma(1+\frac{1}{m})}{\Gamma(1-\alpha+\frac{1}{m})} + \frac{\Gamma(\frac{2+m}{m})}{\Gamma(1-\alpha+\frac{2}{m})}\right]$$

$$-\left(\frac{3t^5-4t^4}{12}\right) + \int_0^t tsu(s)ds, \qquad 0 \le t, s \le 1$$

subject to $u(0) = 0$*, having exact solution* $u(t) = t^2 - t$ *with* $z(t) = t^m$*.*

TABLE 5.8: Numerical solutions obtained using scheme $P1$ for Example 5.5.1 for $z(t) = t^2,\ \alpha = 0.5$.

t_j	Exact solution	$n = 10$	$n = 20$
0.0	0.00	0.00	0.00
0.2	-0.16	-0.138111	-0.150582
0.4	-0.24	-0.220973	-0.232561
0.6	-0.24	-0.222632	-0.233403
0.8	-0.16	-0.142442	-0.153422
1.0	0.00	0.0203025	0.007532

TABLE 5.9: Numerical solutions obtained using scheme $P1$ for Example 5.5.1 for $z(t) = t^3$, $\alpha = 0.6$.

t_j	Exact solution	$n = 10$	$n = 20$
0.0	0.00	0.00	0.00
0.2	-0.16	-0.120531	-0.140827
0.4	-0.24	-0.205677	-0.225512
0.6	-0.24	-0.211454	-0.228492
0.8	-0.16	-0.134236	-0.149798
1.0	0.00	0.02808	0.0110503

TABLE 5.10: Numerical solutions obtained using scheme $P1$ for Example 5.5.1 for $z(t) = t^3$, $\alpha = 0.5$.

t_j	Exact solution	$n = 10$	$n = 20$
0.0	0.00	0.00	0.00
0.2	-0.16	-0.130655	-0.147345
0.4	-0.24	-0.217511	-0.23155
0.6	-0.24	-0.222635	-0.233723
0.8	-0.16	-0.144898	-0.154617
1.0	0.00	0.0163511	0.00579

TABLE 5.11: Numerical solutions obtained using scheme $P1$ for Example 5.5.1 for $z(t) = t^2$, $\alpha = 0.4$.

t_j	Exact solution	$n = 10$	$n = 20$
0.0	0.00	0.00	0.00
0.2	-0.16	-0.167992	-0.166132
0.4	-0.24	-0.252355	-0.242535
0.6	-0.24	-0.247608	-0.241731
0.8	-0.16	-0.166332	-0.16152
1.0	0.00	0.006744	0.001666

TABLE 5.12: Numerical solutions obtained using scheme $P2$ for Example 5.5.1 for $z(t) = t^2$, $\alpha = 0.2$.

t_j	Exact solution	$n = 10$	$n = 20$
0.0	0.00	0.00	0.00
0.2	-0.16	-0.167595	-0.161961
0.4	-0.24	-0.244027	-0.240622
0.6	-0.24	-0.242174	-0.240395
0.8	-0.16	-0.161867	-0.160353
1.0	0.00	0.002078	0.000404

TABLE 5.13: Numerical solutions obtained using scheme $P2$ for Example 5.5.1 for $z(t) = t^3, \ \alpha = 0.2$.

t_j	Exact solution	$n = 10$	$n = 20$
0.0	0.00	0.00	0.00
0.2	-0.16	-0.194829	-0.167586
0.4	-0.24	-0.254488	-0.241494
0.6	-0.24	-0.245509	-0.240813
0.8	-0.16	-0.164602	-0.16073
1.0	0.00	0.005415	0.000913

TABLE 5.14: Numerical solutions obtained using scheme $P2$ for Example 5.5.1 for $z(t) = t^{0.4}, \ \alpha = 0.2$.

t_j	Exact solution	$n = 10$	$n = 20$
0.0	0.00	0.00	0.00
0.2	-0.16	-0.166931	-0.162657
0.4	-0.24	-0.244439	-0.241968
0.6	-0.24	-0.24392	-0.241773
0.8	-0.16	-0.163941	-0.161787
1.0	0.00	0.004389	0.00198

The numerical solutions of Example 5.5.1 are using schemes $P1$ and $P2$ are presented through Tables 5.8-5.14 for different step size and scale function respectively.

5.6 Conclusions

In this chapter, we studied two approximations namely linear and quadratic of GFD which is defined using the scale and weight function. Using this approximation we presented two numerical schemes for GFIDEs. The order of linear and quadratic approximation is $h^{2-\alpha}$ and $h^{3-\alpha}$ respectively. For the numerical simulation we consider example and solved it using the numerical schemes $P1$ and $P2$ for different scale functions $z(t)$. It is clear from tables that linear and quadratic approximation validate the theoretical results.

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