Chapter 4

Comparative Study of Three Numerical Schemes for Fractional Integro-Differential Equations

4.1 Introduction

Fractional calculus is a branch of mathematics that deals with generalization of the well-known operations of differentiations and integrations to arbitrary orders. Many papers on fractional calculus have been published during the second half of the 20th century. Recent developments of fractional calculus involve the real world applications in science and engineering such as viscoelasticity [137, 138, 176], bio-engineering [139], biology [136] and more can be found in [2, 5, 78]. Two main differences between fractional calculus and classical calculus are, 1) fractional integrals and derivatives are nonlocal, and 2) in the limit results obtained from fractional calculus coincide with those obtained from classical calculus. This makes

fractional calculus to be richer than classical calculus. Many real world problems are being modelled using fractional derivative and integral terms and such equations are known as the fractional integro-differential equations. Fractional integro-differential equations arise in the areas of signal processing [78], mechanics [177], econometrics [178], fluid dynamics [5], nuclear reactor dynamics, acoustic waves [1] and electromagnetics [179] etc. There have been much developments on the analytical and numerical methods for solving the fractional integro-differential equations in recent years. Some of them are described as follows: In [180], Saadatmandi and Dehghan applied the Legendre collocation method for solving the fractional integro-differential equations. Rawashdeh [181] presented the spline collocation scheme for solving fractional integro-differential equations. In [182, 183], the application of the collocation method has been extended for solving linear and nonlinear fractional integro-differential equations. Galerkin method, and wavelet Galerkin method based numerical methods are studied by the authors respectively in [184] and [185] for fractional integro-differential equations. In [186, 187], authors presented the second kind Chebyshev wavelet based approximation and cosine and sine wavelet method respectively for fractional integro-differential equations. Some other methods such as least squares method [188], Tau approximation methods [189, 190], Chebyshev pseudo spectral method [68], hybrid collocation method [191], Sinc-Legendre collocation method [192], Legendre spectral element method [193], meshless method [194], and operational matrix method [195] are discussed by the authors for solving fractional integro-differential equations and fractional differential equations in recent years. More recently in 2015, Tohidi and co-authors [196] presented the Euler function based operational matrix approach for solving the fractional integro-differential equations.

In this chapter, we present three numerical schemes such as Linear, Quadratic and

Quadratic-linear for the fractional integro-differential equations defined in terms of the Caputo derivatives and also discuss the comparative performances of these schemes. These schemes are based on the discretization of the domain into subdomains and then the unknown function is approximated between subdomain using linear and quadratic interpolating polynomials. These approximations schemes are also useful when the analytical integrations of the expression containing the fractional derivative and integral terms are rare or difficult to compute and thus the numerical schemes become important. The significant contributions in development of these schemes for fractional derivatives are discussed in [197], where the idea of discretized fractional calculus with linear multistep method is presented. Kumar and Agrawal [148] presented the numerical scheme for the fractional differential equations defined in terms of the Caputo derivatives using the quadratic approximations. In 2006, Odibat [149] discussed the algorithms named modified Trapezoidal rules for the approximations of the fractional integral and the Caputo fractional derivatives and also discussed the error estimates. Further, Agrawal [150] discussed the finite element approximation for the fractional variational problem. Later, Pandey and Agrawal [152] presented a comparative study of linear, quadratic and quadratic-linear approximations to the fractional variational problems defined in terms of the generalized derivatives. In [198], the authors presented the higher order approximations to the Caputo derivatives and then applied to solve the advection-diffusion equations. In 2015, Dehghan et al. [199] discussed the high order difference schemes together with Galerkin spectral technique for numerical solution of multi-term time fractional partial differential equations. These recent works motivated the authors to present and discuss the comparative study of Linear, Quadratic and Quadratic-linear schemes for fractional integro-differential equations. Here these schemes are discussed and compared only for linear fractional integro-differential equations. However, similar approach may also be applied for nonlinear case in future. The Quadratic-linear

scheme is proposed as combination of the Linear scheme and Quadratic scheme for fractional integro-differential equations. These schemes and their comparative study are analyzed on the illustrative test examples. Further, error estimates of the presented schemes are established. Numerical convergence order of all three schemes are calculated and presented through tables.

4.2 Statement of Problem

Now, we consider the general fractional integro-differential equation defined as,

$$D^\alpha \varphi(x) = f(x) + \int_0^x K(x,\tau)\varphi(\tau)\,d\tau, \qquad 0 \leq x,\tau \leq 1, 0 < \alpha < 1, \qquad (4.1)$$

with the following supplementary condition $\varphi(0) = \delta$, where $D^\alpha \varphi(x)$ indicates the α-th order Caputo derivatives of $\varphi(x)$, and $f(x), K(x,\tau)$ are known functions. The numerical schemes for this problem are presented in the upcoming section.

4.3 Numerical Schemes

In this section, three approximation schemes namely Linear, Quadratic and Quadratic-Linear schemes are presented and discussed for the fractional integro-differential equation defined by Eq. (4.1). These schemes are denoted as $S1$, $S2$ and $S3$ respectively and are described as follows:

4.3.1 The Linear Scheme ($S1$)

In the Linear scheme, the domain is divided into n subintervals $[x_j, x_{j+1}]$ and then the unknown function is approximated as the linear interpolation function into each subintervals with the uniform step size $h = \frac{1}{n}$, such that the node points are $x_k = kh, k = 0, 1, 2, \ldots, n$. We denote the values of functions $\varphi(x), f(x)$ at the node x_k by $\varphi_k = \varphi(x_k)$ and $f_k = f(x_k)$. The fractional integro-differential equation defined by Eq.(4.1) takes the form for $0 < \alpha < 1$,

$$\frac{1}{\Gamma(1-\alpha)} \int_0^{x_k} (x_k - \tau)^{-\alpha} \varphi'(\tau) \, d\tau = f(x_k) + \int_0^{x_k} K(x_k, \tau) \varphi(\tau) \, d\tau. \tag{4.2}$$

The left side of Eq. (4.2) can be approximated using the initial condition $D^\alpha \varphi(x_0) = 0$,

$$\frac{1}{\Gamma(1-\alpha)} \int_0^{x_k} (x_k - \tau)^{-\alpha} \varphi'(\tau) \, d\tau = \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{k-1} \int_{x_j}^{x_{j+1}} (x_k - \tau)^{-\alpha} \varphi'(\tau) \, d\tau$$

$$= \frac{1}{\Gamma(1-\alpha)} \sum_0^{k-1} \int_{x_j}^{x_{j+1}} (x_k - \tau)^{-\alpha} \left(\frac{\varphi_{j+1} - \varphi_j}{h} \right) d\tau,$$

or,

$$\frac{1}{\Gamma(1-\alpha)} \int_0^{x_k} (x_k - \tau)^{-\alpha} \varphi'(\tau) \, d\tau = \sum_{j=0}^{k} A(k,j) \varphi_j, \tag{4.3}$$

where,

$$A(k,j) = \frac{h^{-\alpha}}{\Gamma(1-\alpha)} \begin{cases} (k-1)^{1-\alpha} - (k)^{1-\alpha}, & j = 0, \\ (k-j-1)^{1-\alpha} - 2(k-j)^{1-\alpha} + (k-j+1)^{1-\alpha}, & 1 \le j \le k, \\ 1, & j = k. \end{cases} \tag{4.4}$$

Similarly the integral equation on right side of Eq. (4.2) is approximated as,

$$\int_0^{x_k} K(x_k, \tau)\varphi(\tau)\, d\tau = \sum_{j=0}^{k-1} \int_{x_j}^{x_{j+1}} K(x_j, \tau)\varphi(\tau)\, d\tau \tag{4.5}$$

The function $\varphi(\tau)$ is approximated over the interval $[x_j, x_{j+1}]$ using formula $\varphi_{1,j}(\tau) \approx \frac{(x_{j+1}-\tau)}{h}\varphi_j + \frac{(\tau-x_j)}{h}\varphi_{j+1}$, then

$$\int_0^{x_k} K(x_k, \tau)\varphi(\tau)\, d\tau = \sum_{j=0}^{k} U(k, j)\varphi_j, \tag{4.6}$$

where,

$$U(k, j) = \begin{cases} S(k, 0), & j = 0, \\ S(k, j) + T(k, j-1), & 1 \le j \le k, \\ T(k, k-1), & j = k \end{cases} \tag{4.7}$$

$$S(k, j) = h \int_0^1 (1 - p)K(kh, hp + jh)\, dp \tag{4.8}$$

$$T(k, j) = h \int_0^1 pK(kh, hp + jh)\, dp. \tag{4.9}$$

Using Eq.(4.3) and Eq. (4.6), Eq. (4.2) is equivalent to the linear system of equations,

$$\sum_{j=0}^{k} \xi(k, j)\varphi_j = f_k, k = 1, 2, \ldots, n, \tag{4.10}$$

where,

$\xi(k, j) = A(k, j) - U(k, j).$

The approximated solution of the Eq. (4.1) can be obtained by solving the linear system given by Eq. (4.10). Now we discuss the Quadratic scheme ($S2$) as follows.

4.3.2 The Quadratic Scheme ($S2$)

In this scheme, the unknown function is approximated by linear interpolation function for $j = 0$ in the interval $[x_0, x_1]$ and for $j \geq 2$, the unknown function is approximated as quadratic interpolation function in the interval $[x_{j-1}, x_j]$. We follow the similar steps as described in the scheme $S1$ to derive the expression for the Quadratic scheme for Eq.(4.2). The derivations are obtained in parts and described below.

Part 1: Approximation of the left side of the Eq. (4.2) containing the Caputo derivative term: We assume the initial condition $D^\alpha \varphi(x_0) = 0$,

$$\frac{1}{\Gamma(1-\alpha)} \int_0^{x_k} (x_k - \tau)^{-\alpha} \varphi'(\tau) \, d\tau = \frac{1}{\Gamma(1-\alpha)} \sum_{j=1}^{k} \int_{x_{j-1}}^{x_j} (x_k - \tau)^{-\alpha} \varphi'(\tau) \, d\tau \quad (4.11)$$

$$= \frac{1}{\Gamma(1-\alpha)} \int_{x_0}^{x_1} (x_k - \tau)^{-\alpha} \varphi'(\tau) \, d\tau + \frac{1}{\Gamma(1-\alpha)} \sum_{j=2}^{k} \int_{x_{j-1}}^{x_j} (x_k - \tau)^{-\alpha} \varphi'(\tau) \, d\tau \quad (4.12)$$

On the first interval $[x_0, x_1]$, the linear interpolation is used for approximating the unknown function as like scheme $S1$ and for the other subintervals ($j \geq 2$), the quadratic interpolation function $(\varphi_{2,j-1}(\tau))'$ for the three points (x_{j-2}, φ_{j-2}), (x_{j-1}, φ_{j-1}) and (x_j, φ_j) is applied such that,

$$(\varphi_{2,j-1}(\tau))' = \left(\frac{(\tau - x_{j-1})(\tau - x_j)}{2h^2} \varphi_{j-2} - \frac{(\tau - x_{j-2})(\tau - x_j)}{h^2} \varphi_{j-1} \right.$$

$$\left. + \frac{(\tau - x_{j-2})(\tau - x_{j-1})}{2h^2} \varphi_j \right)'$$

$$(\varphi_{2,j-1}(\tau))' = \left(\frac{(2\tau - x_{j-1} - x_j)}{2h^2} \varphi_{j-2} - \frac{(2\tau - x_{j-1} - x_j)}{h^2} \varphi_{j-1} + \frac{(2\tau - x_{j-1} - x_j)}{2h^2} \varphi_j \right)$$

$$(4.13)$$

Thus from Eq. (4.12) and Eq. (4.13), we have,

$$\approx \frac{1}{\Gamma(1-\alpha)} \int_{x_0}^{x_1} (x_k - \tau)^{-\alpha} \left(\frac{\varphi_1 - \varphi_0}{h} \right) d\tau + \frac{1}{\Gamma(1-\alpha)} \sum_{j=2}^{k} \int_{x_{j-1}}^{x_j} (x_k - \tau)^{-\alpha} \left(\varphi_{2,j-1}(\tau) \right)' d\tau,$$

$$= \frac{1}{\Gamma(1-\alpha)} \int_{x_0}^{x_1} (x_k - \tau)^{-\alpha} \left(\frac{\varphi_1 - \varphi_0}{h} \right) d\tau$$

$$+ \frac{h^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=2}^{k} B(k-j)\varphi_{j-2} + C(k-j)\varphi_{j-1} + D(k-j)\varphi_j,$$

$$= \frac{h^{-\alpha} a_{k-1}}{\Gamma(2-\alpha)} (\varphi_1 - \varphi_0) + \frac{h^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=2}^{k} B(k-j)\varphi_{j-2} + C(k-j)\varphi_{j-1} + D(k-j)\varphi_j,$$

$$(4.14)$$

where,

$$a_{k-1} = k^{1-\alpha} - (k-1)^{1-\alpha},$$

$$B(k-j) = \frac{1}{2-\alpha} \left[(k-j+1)^{1-\alpha} \left(k-j+\frac{\alpha}{2} \right) - (k-j)^{1-\alpha} \left(k-j-\frac{\alpha}{2}+1 \right) \right],$$

$$C(k-j) = \frac{2}{(2-\alpha)} \left[(k-j)^{1-\alpha}(k-j-\alpha+2) - (k-j+1)^{2-\alpha} \right],$$

$$D(k-j) = \frac{1}{2-\alpha} \left[(k-j+1)^{1-\alpha} \left(k-j-\frac{\alpha}{2}+2 \right) - (k-j)^{1-\alpha} \left(k-j-\frac{3\alpha}{2}+3 \right) \right].$$

Eq. (4.14) can be written as,

$$\frac{h^{-\alpha} a_{k-1}}{\Gamma(2-\alpha)} (\varphi_1 - \varphi_0) + \frac{h^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=2}^{k} B(k-j)\varphi_{j-2} + C(k-j)\varphi_{j-1} + D(k-j)\varphi_j$$

$$= \frac{h^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^{k} s_j \varphi_{k-j}.$$

Or,

$$\frac{1}{\Gamma(1-\alpha)} \int_0^{x_k} (x_k - \tau)^{-\alpha} \varphi'(\tau) \, d\tau = \frac{h^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^k s_j \varphi_{k-j} \qquad (4.15)$$

Here the coefficients s_j in Eq. (4.15) for different k can be expressed as,

$$\text{For } k = 1, \begin{cases} s_0 = a_0 \\ s_1 = -a_0. \end{cases}$$

$$\text{For } k = 2, \begin{cases} s_0 = D(0) \\ s_1 = C(0) + a_1 \\ s_2 = B(0) - a_1. \end{cases}$$

$$\text{For } k = 3, \begin{cases} s_0 = D(0) \\ s_1 = C(0) + D(1) \\ s_2 = B(0) + C(1) + a_2 \\ s_3 = B(1) - a_2. \end{cases}$$

And for $k \geq 4$, the coefficients have the following relations:

$$\begin{cases} s_0 = D(0) \\ s_1 = C(0) + D(1) \\ s_j = B(j-2) + C(j-1) + D(j), \qquad (2 \leq j \leq k-2) \\ s_{k-1} = B(k-3) + C(k-2) + a_{k-1} \\ s_k = B(k-2) - a_{k-1}. \end{cases}$$

Part 2: Approximation of the integration term on the right side of Eq.(4.2):

The integration term on the right side of Eq.(4.2) is expressed as,

$$\int_0^{x_k} K(x_k, \tau)\varphi(\tau)\, d\tau = \sum_{j=1}^{k} \int_{x_{j-1}}^{x_j} K(x_k, \tau)\varphi(\tau)\, d\tau, \tag{4.16}$$

$$= \int_{x_0}^{x_1} K(x_k, \tau)\varphi(\tau)\, d\tau + \sum_{j=2}^{k} \int_{x_{j-1}}^{x_j} K(x_k, \tau)\varphi(\tau)\, d\tau. \tag{4.17}$$

In Eq. (4.17), the function $\varphi(\tau)$ is approximated over the interval $[x_0, x_1]$ using the following formula:

$$\varphi_{1,0}(\tau) \approx \frac{(x_1 - \tau)}{h}\varphi_0 + \frac{(\tau - x_0)}{h}\varphi_1,$$

$$\int_{x_0}^{x_1} K(x_k, \tau)\varphi(\tau)\, d\tau = a_k\varphi_0 + b_k\varphi_1,$$

where,

$$a_k = h \int_0^1 K(kh, ph)(1 - p)\, dp \text{ and } b_k = h \int_0^1 K(kh, ph)p\, dp \tag{4.18}$$

And for $j \geq 2$, the function $\varphi(\tau)$ is approximated over the interval $[x_{j-1}, x_j]$ for three points $(x_{j-2}, \varphi_{j-2}), (x_{j-1}, \varphi_{j-1})$ and (x_j, φ_j), we use the quadratiic interpolation function $\varphi(\tau)$ as discussed in the discretization of the Part 1 such that,

$$\varphi_{2,j-1}(\tau) = \left(\frac{(\tau - x_{j-1})(\tau - x_j)}{2h^2}\varphi_{j-2} - \frac{(\tau - x_{j-2})(\tau - x_j)}{h^2}\varphi_{j-1} \right.$$

$$\left. + \frac{(\tau - x_{j-2})(\tau - x_{j-1})}{2h^2}\varphi_j \right).$$

Hence second integral of Eq. (4.17) is expressed as,

$$\sum_{j=2}^{k} \int_{x_{j-1}}^{x_j} K(x_k, \tau)\varphi(\tau)\, d\tau \approx \sum_{j=2}^{k} M(k,j)\varphi_{j-2} + N(k,j)\varphi_{j-1} + O(k,j)\varphi_j. \quad (4.19)$$

Now using the transformation $\frac{\tau - x_{j-1}}{h} = p$, Eq. (4.17) takes form,

$$\sum_{j=1}^{k} \int_{x_{j-1}}^{x_j} K(x_k, \tau)\varphi(\tau)\, d\tau = a_k\varphi_0 + b_k\varphi_1 + \sum_{j=2}^{k} M(k,j)\varphi_{j-2} + N(k,j)\varphi_{j-1} + O(k,j)\varphi_j$$

where,

$$M(k,j) = \frac{h}{2} \int_0^1 K(kh, hp + jh - h)p(p-1)\, dp,$$

$$N(k,j) = \frac{h}{2} \int_0^1 K(kh, hp + jh - h)p(1 - p^2)\, dp,$$

$$O(k,j) = \frac{h}{2} \int_0^1 K(kh, hp + jh - h)p(p+1)\, dp,$$

or,

$$\int_0^{x_k} K(x_k, \tau)\varphi(\tau)\, d\tau = \sum_{j=0}^{k} v_j\varphi_j. \quad (4.20)$$

The coefficients v_j for different k can be calculated as,

$$\text{For } k = 1, \begin{cases} v_0 = a_1 \\ v_1 = b_1. \end{cases}$$

$$\text{For } k = 2, \begin{cases} v_0 = M(2,2) + a_2 \\ v_1 = N(2,2) + b_2 \\ v_2 = O(2,2). \end{cases}$$

$$\text{For } k = 3, \begin{cases} v_0 = M(3,2) + a_k \\ v_1 = M(3,3) + N(3,2) + b_3 \\ v_2 = N(3,3) + O(3,2) \\ v_3 = O(3,3). \end{cases}$$

And for $k \geq 4$, the coefficients have the following relations:

$$\begin{cases} v_0 = M(k,2) + a_k \\ v_1 = M(k,3) + N(k,2) + b_k \\ v_j = M(k,j+2) + N(k,j+1) + O(k,j), \quad (2 \leq j \leq k-2) \\ v_{k-1} = N(k,k) + O(k,k-1) \\ v_k = O(k,k). \end{cases}$$

Using Eqs. (4.15) and (4.20), Eq. (4.2) can be expressed as,

$$A_k(\varphi_0, \varphi_1, \varphi_2, \ldots, \varphi_k) - C_k(\varphi_0, \varphi_1, \varphi_2, \ldots, \varphi_k) = f_k, \quad k = 1, 2, \ldots, n. \qquad (4.21)$$

where,

$$A_k(\varphi_0, \varphi_1, \varphi_2, \ldots, \varphi_k) = \frac{h^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^{k} s_j \varphi_{k-j},$$

$$C_k(\varphi_0, \varphi_1, \varphi_2, \ldots, \varphi_k) = \sum_{j=0}^{k} v_j \varphi_j.$$

Now, by solving the above system of linear equations given by Eq. (4.21), one obtains the desired approximate solution of the fractional integro-differential equation defined by Eq. (4.1).

4.3.3 The Quadratic Linear Scheme ($S3$)

Here, we discuss a hybrid scheme combining quadratic interpolation and linear interpolation approximation together to approximate the desired solution of Eq.(4.2). The left side of Eq. (4.2) (the term having Caputo derivative) is discretized using quadratic interpolation. However, the integration term on the right side of Eq.(4.2) is discretized using the linear interpolation approximation for the unknown function. We use the discretization of both the parts as discussed in schemes $S1$ and $S2$. Now, from Eq. (4.15) and Eq. (4.6), Eq. (4.2) takes the form,

$$\frac{h^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^{k} s_j \varphi_{k-j} - \sum_{j=0}^{k} U(k,j)\varphi_j = f_k, \quad \text{where } k = 1, 2, \ldots, n. \tag{4.22}$$

or,

$$A_k(\varphi_0, \varphi_1, \varphi_2, \ldots, \varphi_k) - B_k(\varphi_0, \varphi_1, \varphi_2, \ldots, \varphi_k) = f_k, \quad k = 1, 2, \ldots, n. \tag{4.23}$$

where,

$$A_k(\varphi_0, \varphi_1, \varphi_2, \ldots, \varphi_k) = \frac{h^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^{k} s_j \varphi_{k-j}, \tag{4.24}$$

$$B_k(\varphi_0, \varphi_1, \varphi_2, \ldots, \varphi_k) = \sum_{j=0}^{k} U(k,j)\varphi_j. \tag{4.25}$$

By solving the above system given by Eq. (4.23), the approximate solution is calculated.

4.4 Error Estimates of the Approximations

In this section, the error estimates of the schemes $S1$, $S2$ and $S3$ are discussed as follows.

Theorem 4.4.1 ($S1$). *Let $\varphi(\tau) \in C^2[0, x_k]$, $\alpha(0 < \alpha < 1)$ and suppose $K(x,\tau), 0 \leq x, \tau \leq 1$ is continuous and $|K(x,\tau)| \leq M$, where $M > 0$. Let $E^k(\varphi, h, \alpha)$ be the error of the approximation to Eq.(4.1) using Linear scheme (S1) then,*

$$|E^k(\varphi, h, \alpha)| \leq \frac{1}{\Gamma(1-\alpha)} \left[\frac{1}{8} + \frac{\alpha}{(1-\alpha)(2-\alpha)} \right] \max_{x_0 \leq \tau \leq x_k} |\varphi''(\tau)| h^{2-\alpha}$$

$$+ \frac{M}{8} \max_{x_0 \leq \tau \leq x_k} |\varphi''(\tau)| x_k h^2.$$

Proof. From Eqs. (4.1) and (4.10), we have,

$$|E^k(\varphi, h, \alpha)| \leq \frac{1}{\Gamma(1-\alpha)} \left| \sum_{j=0}^{k-1} \int_{x_j}^{x_{j+1}} (x_k - \tau)^{-\alpha} \left(\varphi'(\tau) - \frac{\varphi_{j+1} - \varphi_j}{h} \right) d\tau \right|$$

$$+ \left| \sum_{j=0}^{k-1} \int_{x_j}^{x_{j+1}} K(x_k, \tau)(\varphi(\tau) - \varphi_{1,j}(\tau)) \, d\tau \right|. \tag{4.26}$$

For simplicity, we use notation,

$$|E_1^k(\varphi, h, \alpha)| = \frac{1}{\Gamma(1-\alpha)} \left| \sum_{j=0}^{k-1} \int_{x_j}^{x_{j+1}} (x_k - \tau)^{-\alpha} \left(\varphi'(\tau) - \frac{\varphi_{j+1} - \varphi_j}{h} \right) d\tau \right|, \quad (4.27)$$

$$|E_2^k(\varphi, h, \alpha)| = \left| \sum_{j=0}^{k-1} \int_{x_j}^{x_{j+1}} K(x_k, \tau)(\varphi(\tau) - \varphi_{1,j}(\tau)) \, d\tau \right|. \quad (4.28)$$

Now, we consider the first part given by Eq. (4.27),

$$|E_1^k(\varphi, h, \alpha)| = \frac{1}{\Gamma(1-\alpha)} \left| \sum_{j=0}^{k-1} \int_{x_j}^{x_{j+1}} (x_k - \tau)^{-\alpha} \left(\varphi'(\tau) - \frac{\varphi_{j+1} - \varphi_j}{h} \right) d\tau \right|$$

$$= \frac{1}{\Gamma(1-\alpha)} \left| \sum_{j=0}^{k-1} \int_{x_j}^{x_{j+1}} (x_k - \tau)^{-\alpha} \left(\varphi(\tau) - \varphi_{1,j}(\tau) \right)' d\tau \right|$$

$$= \frac{1}{\Gamma(1-\alpha)} \left| \sum_{j=0}^{k-1} \int_{x_j}^{x_{j+1}} (x_k - \tau)^{-\alpha} \, d \left(\varphi(\tau) - \varphi_{1,j}(\tau) \right) \right|$$

$$= \frac{1}{\Gamma(1-\alpha)} \left| \sum_{j=0}^{k-1} \left(\varphi(\tau) - \varphi_{1,j}(\tau) \right) (x_k - \tau)^{-\alpha} \big|_{\tau=x_j}^{x_{j+1}} \right.$$

$$\left. + \alpha \sum_{j=0}^{k-1} \int_{x_j}^{x_{j+1}} (x_k - \tau)^{-\alpha-1} \left(\varphi(\tau) - \varphi_{1,j}(\tau) \right) d\tau \right|$$

$$= \frac{\alpha}{\Gamma(1-\alpha)} \left| \sum_{j=0}^{k-1} \int_{x_j}^{x_{j+1}} (x_k - \tau)^{-\alpha-1} \left(\varphi(\tau) - \varphi_{1,j}(\tau) \right) d\tau \right|$$

$$\leq \frac{\alpha}{\Gamma(1-\alpha)} \left| \sum_{j=0}^{k-2} \int_{x_j}^{x_{j+1}} (x_k - \tau)^{-\alpha-1} \left(\varphi(\tau) - \varphi_{1,j}(\tau) \right) d\tau \right|$$

$$+ \frac{\alpha}{\Gamma(1-\alpha)} \left| \int_{x_{k-1}}^{x_k} (x_k - \tau)^{-\alpha-1} \left(\varphi(\tau) - \varphi_{1,k-1}(\tau) \right) d\tau \right|$$

$$\leq \frac{1}{2}|\varphi''(\eta)|\frac{\alpha}{\Gamma(1-\alpha)}\left|\sum_{j=0}^{k-2}\int_{x_j}^{x_{j+1}}(x_k-\tau)^{-\alpha-1}(\tau-x_j)(\tau-x_{j+1})\,d\tau\right|$$

$$+\frac{\alpha}{\Gamma(1-\alpha)}\frac{1}{2}|\varphi''(\xi)|\left|\int_{x_{k-1}}^{x_k}(x_k-\tau)^{-\alpha}(\tau-x_{k-1})\,d\tau\right|$$

$$\leq \frac{1}{8}|\varphi''(\eta)|\frac{1}{\Gamma(1-\alpha)}h^{2-\alpha}+\frac{\alpha}{\Gamma(1-\alpha)}\frac{1}{2}|\varphi''(\xi)|\frac{h^{2-\alpha}}{(1-\alpha)(2-\alpha)}, \tag{4.29}$$

where, $\eta \in (x_0, x_{k-1}), \xi \in (x_{k-1}, x_k)$.

Now, we take second part given by Eq. (4.28),

$$|E_2^k(\varphi, h, \alpha)| = \left|\sum_{j=0}^{k-1}\int_{x_j}^{x_{j+1}}K(x_k, \tau)(\varphi(\tau)-\varphi_{1,j}(\tau))\,d\tau\right|$$

$$\leq \frac{|\varphi''(\xi_1)|}{2}\left|\sum_{j=0}^{k-1}\int_{x_j}^{x_{j+1}}K(x_k, \tau)(\tau-x_j)(\tau-x_{j+1})\,d\tau\right|$$

$$\leq \frac{M}{8}\max_{x_0\leq\tau\leq x_k}|\varphi''(\tau)|x_k h^2, \tag{4.30}$$

where, $\xi_1 \in (x_0, x_k)$. From Eq. (4.29)-(4.30) and Eq. (4.26),

$$|E^k(\varphi, h, \alpha)| \leq \frac{1}{\Gamma(1-\alpha)}\left[\frac{1}{8}+\frac{\alpha}{(1-\alpha)(2-\alpha)}\right]\max_{x_0\leq\tau\leq x_k}|\varphi''(\tau)|h^{2-\alpha}$$

$$+\frac{M}{8}\max_{x_0\leq\tau\leq x_k}|\varphi''(\tau)|x_k h^2.$$

This completes the proof.

Theorem 4.4.2 *(S2). Let $\varphi(\tau) \in C^3[0, x_k], \alpha(0 < \alpha < 1)$ and suppose $K(x, \tau)$, $0 \leq x, \tau \leq 1$ is continuous and $|K(x, \tau)| \leq M$, where $M > 0$. Let $G^k(\varphi, h, \alpha)$ be the*

error of the approximation to Eq. (4.1) using Quadratic scheme (S2) then,

(i) *For,* $k = 1, |G^1(\varphi, h, \alpha)| \leq \dfrac{\alpha}{2\Gamma(3 - \alpha)} \max_{x_0 \leq \tau \leq x_1} |\varphi''(\tau)| h^{2-\alpha}$

$$+ \frac{M}{2} \max_{x_0 \leq \tau \leq x_1} |\varphi''(\tau)| x_1 h^2 \ and$$

(ii) *For,* $k \geq 2, |G^k(\varphi, h, \alpha)| \leq \dfrac{1}{\Gamma(1 - \alpha)} \Bigg(\dfrac{\alpha}{12} \max_{x_0 \leq \tau \leq x_k} |\varphi''(\tau)| (x_k - x_1)^{-\alpha-1} h^3$

$$+ \left[\frac{1}{12} + \frac{\alpha}{3(1 - \alpha)(2 - \alpha)} \left(\frac{1}{2} + \frac{1}{3 - \alpha} \right) \right]$$

$$\times \max_{x_0 \leq \tau \leq x_k} |\varphi'''(\tau)| h^{3-\alpha} \Bigg)$$

$$+ \frac{M}{2} \max_{x_0 \leq \tau \leq x_1} |\varphi''(\tau)| x_1 h^2 + \frac{M}{12} \max_{x_0 \leq \tau \leq x_1} |\varphi'''(\tau)| (x_k - x_1) h^3.$$

Proof. Part (i): From Eqs. (4.1) and (4.21), we have the error of the approximation for $k = 1$,

$$|G^1(\varphi, h, \alpha)| \leq \frac{1}{\Gamma(1 - \alpha)} \left| \int_{x_0}^{x_1} (x_1 - \tau)^{-\alpha} \left(\varphi'(\tau) - \frac{\varphi_1 - \varphi_0}{h} \right) d\tau \right|$$

$$+ \left| \int_{x_0}^{x_1} K(x_1, \tau) (\varphi(\tau) - \varphi_{1,j}(\tau)) \, d\tau \right|. \tag{4.31}$$

For simplicity, we introduce the notations for each right side part of Eq. (4.31) as,

$$|G_1^1(\varphi, h, \alpha)| = \frac{1}{\Gamma(1 - \alpha)} \left| \int_{x_0}^{x_1} (x_1 - \tau)^{-\alpha} \left(\varphi'(\tau) - \frac{\varphi_1 - \varphi_0}{h} \right) d\tau \right|, \tag{4.32}$$

$$|G_2^1(\varphi, h, \alpha)| = \left| \int_{x_0}^{x_1} K(x_1, \tau) (\varphi(\tau) - \varphi_{1,j}(\tau)) \, d\tau \right|. \tag{4.33}$$

Now we consider the first part given by Eq. (4.32),

$$|G_1^1(\varphi, h, \alpha)| = \frac{1}{\Gamma(1-\alpha)} \left| \int_{x_0}^{x_1} (x_1 - \tau)^{-\alpha} \left(\varphi'(\tau) - \frac{\varphi_1 - \varphi_0}{h} \right) d\tau \right|$$

$$\leq \frac{\alpha}{2\Gamma(3-\alpha)} \max_{x_0 \leq \tau \leq x_1} |\varphi''(\tau)| h^{2-\alpha}, \tag{4.34}$$

$$|G_2^1(\varphi, h, \alpha)| = \left| \int_{x_0}^{x_1} K(x_1, \tau) \left(\varphi(\tau) - \varphi_{1,j}(\tau) \right) d\tau \right|$$

$$\leq \frac{M}{2} \max_{x_0 \leq \tau \leq x_1} |\varphi''(\tau)| x_1 h^2. \tag{4.35}$$

From Eqs. (4.34)-(4.35) and Eq. (4.31),

$$|G^1(\varphi, h, \alpha)| \leq \frac{\alpha}{2\Gamma(3-\alpha)} \max_{x_0 \leq \tau \leq x_1} |\varphi''(\tau)| h^{2-\alpha} + \frac{M}{2} \max_{x_0 \leq \tau \leq x_1} |\varphi''(\tau)| x_1 h^2.$$

This completes the proof for Part (i). The proof of Part (ii) is given below.

Part (ii): From Eqs. (4.1) and (4.21), we have, for $k \geq 2$

$$|G^k(\varphi, h, \alpha)| \leq \frac{1}{\Gamma(1-\alpha)} \left(\left| \int_{x_0}^{x_1} (x_k - \tau)^{-\alpha} (\varphi(\tau) - \varphi_{1,0}(\tau))' \, d\tau \right| \right.$$

$$+ \left| \sum_{j=2}^{k} \int_{x_{j-1}}^{x_j} (x_k - \tau)^{-\alpha} (\varphi(\tau) - \varphi_{2,j-1}(\tau))' \, d\tau \right| \right)$$

$$+ \left(\left| \int_{x_0}^{x_1} K(x_k, \tau)(\varphi(\tau) - \varphi_{1,0}(\tau)) \, d\tau \right| \right.$$

$$+ \left| \sum_{j=2}^{k} \int_{x_{j-1}}^{x_j} K(x_k, \tau)(\varphi(\tau) - \varphi_{2,j-1}(\tau)) \, d\tau \right| \right). \tag{4.36}$$

We denote Eq. (4.36) as,

$$|G^k(\varphi, h, \alpha)| \leq |G_1^k(\varphi, h, \alpha)| + |G_2^k(\varphi, h, \alpha)| \tag{4.37}$$

where,

$$|G_1^k(\varphi, h, \alpha)| = \frac{1}{\Gamma(1-\alpha)} \left(\left| \int_{x_0}^{x_1} (x_k - \tau)^{-\alpha} (\varphi(\tau) - \varphi_{1,0}(\tau))' \, d\tau \right| \right.$$

$$\left. + \left| \sum_{j=2}^{k} \int_{x_{j-1}}^{x_j} \frac{(\varphi(\tau) - \varphi_{2,j-1}(\tau))'}{(x_k - \tau)^\alpha} \, d\tau \right| \right) \tag{4.38}$$

$$|G_2^k(\varphi, h, \alpha)| = \left| \int_{x_0}^{x_1} K(x_k, \tau)(\varphi(\tau) - \varphi_{1,0}(\tau)) \, d\tau \right|$$

$$+ \left| \sum_{j=2}^{k} \int_{x_{j-1}}^{x_j} K(x_k, \tau)(\varphi(\tau) - \varphi_{2,j-1}(\tau)) \, d\tau \right| \tag{4.39}$$

Now we consider Eq. (4.38),

$$|G_1^k(\varphi, h, \alpha)| = \frac{1}{\Gamma(1-\alpha)} \left(\left| \int_{x_0}^{x_1} (x_k - \tau)^{-\alpha} (\varphi(\tau) - \varphi_{1,0}(\tau))' \, d\tau \right| \right.$$

$$\left. + \left| \sum_{j=2}^{k} \int_{x_{j-1}}^{x_j} \frac{(\varphi(\tau) - \varphi_{2,j-1}(\tau))'}{(x_k - \tau)^\alpha} \, d\tau \right| \right),$$

$$\leq \frac{\alpha}{\Gamma(1-\alpha)} \left[\left| \int_{x_0}^{x_1} (\varphi(\tau) - \varphi_{1,0}(\tau))(x_k - \tau)^{-\alpha-1} \, d\tau \right| \right.$$

$$\left. + \left| \sum_{j=2}^{k} \int_{x_{j-1}}^{x_j} (\varphi(\tau) - \varphi_{2,j-1}(\tau))(x_k - \tau)^{-\alpha-1} \, d\tau \right| \right]. \tag{4.40}$$

Since,

$$\left| \int_{x_0}^{x_1} (\varphi(\tau) - \varphi_{1,0}(\tau))(x_k - \tau)^{-\alpha-1} \, d\tau \right|$$

$$= \left| \int_{x_0}^{x_1} \frac{\varphi''(\eta_1)}{2} (\tau - x_0)(\tau - x_1)(x_k - \tau)^{-\alpha-1} \, d\tau \right|$$

$$= \left| \frac{\varphi''(\eta_1)}{2} \int_{x_0}^{x_1} (\tau - x_0)(\tau - x_1)(x_k - \tau)^{-\alpha-1} \, d\tau \right|$$

$$\leq \frac{1}{12} |\varphi''(\eta_1)|(x_k - x_1)^{-\alpha-1} h^3. \tag{4.41}$$

where $\eta_1 \in (x_0, x_1)$, We also know that

$$\left| \sum_{j=2}^{k-1} \int_{x_{j-1}}^{x_j} (\varphi(\tau) - \varphi_{2,j-1}(\tau))(x_k - \tau)^{-\alpha-1} \, d\tau \right|$$

$$= \left| \sum_{j=2}^{k-1} \int_{x_{j-1}}^{x_j} \frac{\varphi'''(\vartheta_j)}{6} (\tau - x_{j-2})(\tau - x_{j-1})(\tau - x_j)(x_k - \tau)^{-\alpha-1} \, d\tau \right|$$

$$= \frac{1}{6} \left| \varphi'''(\eta_j) \sum_{j=2}^{k-1} \int_{x_{j-1}}^{x_j} (\tau - x_{j-2})(\tau - x_{j-1})(\tau - x_j)(x_k - \tau)^{-\alpha-1} \, d\tau \right|$$

$$\leq \frac{1}{6} |\varphi'''(\eta)| \sum_{j=2}^{k-1} \int_{x_{j-1}}^{x_j} (\tau - x_{j-2})(\tau - x_{j-1})(\tau - x_j)(x_k - \tau)^{-\alpha-1} \, d\tau$$

$$\leq \frac{1}{12} |\varphi'''(\eta)| h^3 \int_{x_1}^{x_{k-1}} (x_k - \tau)^{-\alpha-1} \, d\tau$$

$$\leq \frac{1}{12} |\varphi'''(\eta)| h^{3-\alpha}, \tag{4.42}$$

where $\eta_j \in (x_{j-2}, x_j)$, $2 \leq j \leq k-1$, $\eta \in (x_0, x_{k-1})$ and

$$\int_{x_{k-1}}^{x_k} (\varphi(\tau) - \varphi_{2,k-1}(\tau))(x_k - \tau)^{-\alpha-1} \, d\tau$$

$$= \int_{x_{k-1}}^{x_k} \frac{\varphi'''(\vartheta_k)}{6}(\tau - x_{k-2})(\tau - x_{k-1})(\tau - x_k)(x_k - \tau)^{-\alpha-1}\,d\tau$$

$$= -\frac{\varphi'''(\eta_k)}{6}\int_{x_{k-1}}^{x_k}(\tau - x_{k-2})(\tau - x_{k-1})(x_k - \tau)^{-\alpha}\,d\tau$$

$$= -\frac{1}{3}\frac{1}{(2-\alpha)(1-\alpha)}\left(\frac{1}{2}+\frac{1}{3-\alpha}\right)\varphi'''(\eta_k)h^{3-\alpha}. \tag{4.43}$$

From Eqs. (4.39)-(4.43) and (4.37) we get,

$$|G_1^k(\varphi,h,\alpha)| \leq \left(\frac{\alpha}{12}\max_{x_0\leq\tau\leq x_1}|\varphi''(\tau)|(x_k-x_1)^{-\alpha-1}h^3\right.$$

$$+\left[\frac{1}{12}+\frac{\alpha}{3(1-\alpha)(2-\alpha)}\left(\frac{1}{2}+\frac{1}{3-\alpha}\right)\right]$$

$$\left.\times\max_{x_0\leq\tau\leq x_k}\frac{1}{\Gamma(1-\alpha)}|\varphi'''(\tau)|h^{3-\alpha}\right). \tag{4.44}$$

Now consider Eq. (4.39),

$$|G_2^k(\varphi,h,\alpha)| = \left|\int_{x_0}^{x_1}K(x_k,\tau)(\varphi(\tau)-\varphi_{1,0}(\tau))\,d\tau\right|$$

$$+\left|\sum_{j=2}^{k}\int_{x_{j-1}}^{x_j}K(x_k,\tau)(\varphi(\tau)-\varphi_{2,j-1}(\tau))\,d\tau\right|$$

$$= \frac{M}{2}\max_{x_0\leq\tau\leq x_1}|\varphi''(\tau)|x_1 h^2$$

$$+\frac{1}{6}\max_{x_0\leq\tau\leq x_k}|\varphi'''(\tau)|\left|\sum_{j=2}^{k}\int_{x_{j-1}}^{x_j}K(x_k,\tau)(\tau-x_{j-2})(\tau-x_{j-1})(\tau-x_j)\,d\tau\right|$$

$$\leq \frac{M}{2}\max_{x_0\leq\tau\leq x_1}|\varphi''(\tau)|x_1 h^2 + \frac{M}{12}\max_{x_0\leq\tau\leq x_k}|\varphi'''(\tau)|(x_k-x_1)h^3. \tag{4.45}$$

From Eqs. (4.44),(4.45) and (4.37),

$$|G^k(\varphi,h,\alpha)| \le \frac{1}{\Gamma(1-\alpha)} \left(\frac{\alpha}{12} \max_{x_0 \le \tau \le x_k} |\varphi''(\tau)|(x_k - x_1)^{-\alpha-1}h^3 \right.$$

$$+ \left[\frac{1}{12} + \frac{\alpha}{3(1-\alpha)(2-\alpha)} \left(\frac{1}{2} + \frac{1}{3-\alpha} \right) \right] \max_{x_0 \le \tau \le x_k} |\varphi'''(\tau)|h^{3-\alpha} \right)$$

$$+ \frac{M}{2} \max_{x_0 \le \tau \le x_1} |\varphi''(\tau)|x_1 h^2 + \frac{M}{12} \max_{x_0 \le \tau \le x_1} |\varphi'''(\tau)|(x_k - x_1)h^3.$$

This completes the proof.

Theorem 4.4.3 (S3). *Let $\varphi(\tau) \in C^3[0, x_k], \alpha(0 < \alpha < 1)$ and suppose $K(x, \tau)$,*
$0 \le x, \tau \le 1$ is continuous and $|K(x, \tau)| \le M$, where $M > 0$. Let $F^k(\varphi, h, \alpha)$ be the
error of the approximation to Eq. (4.1) using Quadratic-linear scheme (S3) then,

$$(i) \; For, k = 1, |F^1(\varphi, h, \alpha)| \le \frac{\alpha}{2\Gamma(3-\alpha)} \max_{x_0 \le \tau \le x_1} |\varphi''(\tau)|h^{2-\alpha}$$

$$+ \frac{M}{2} \max_{x_0 \le \tau \le x_1} |\varphi''(\tau)|x_1 h^2.$$

$$(ii) \; For, k \ge 2, |F^k(\varphi, h, \alpha)| \le \left(\frac{\alpha}{12} \max_{x_0 \le \tau \le x_k} |\varphi''(\tau)|(x_k - x_1)^{-\alpha-1}h^3 \right.$$

$$+ \left[\frac{1}{12} + \frac{\alpha}{3(1-\alpha)(2-\alpha)} \left(\frac{1}{2} + \frac{1}{3-\alpha} \right) \right]$$

$$\times \left. \max_{x_0 \le \tau \le x_k} |\varphi'''(\tau)|h^{3-\alpha} \right) + \frac{M}{2} \max_{x_0 \le \tau \le x_k} |\varphi''(\tau)|x_k h^2.$$

Proof (i) From Eqs. (4.1) and (4.22), we have,

$$|F^1(\varphi, h, \alpha)| \le \frac{1}{\Gamma(1-\alpha)} \left| \int_{x_0}^{x_1} (x_1 - \tau)^{-\alpha} \left(\varphi'(\tau) - \frac{\varphi_1 - \varphi_0}{h} \right) d\tau \right|$$

$$+ \left| \int_{x_0}^{x_1} K(x_1, \tau)(\varphi(\tau) - \varphi_{1,0}(\tau)) \, d\tau \right|. \qquad (4.46)$$

For simplicity, we use notation,

$$|F_1^1(\varphi, h, \alpha)| = \frac{1}{\Gamma(1-\alpha)} \left| \int_{x_0}^{x_1} (x_1 - \tau)^{-\alpha} \left(\varphi'(\tau) - \frac{\varphi_1 - \varphi_0}{h} \right) d\tau \right|, \qquad (4.47)$$

$$|F_2^1(\varphi, h, \alpha)| = \left| \int_{x_0}^{x_1} K(x_1, \tau) \left(\varphi(\tau) - \varphi_{1,0}(\tau) \right) d\tau \right|, \qquad (4.48)$$

such that,

$$|F^1(\varphi, h, \alpha)| \le |F_1^1(\varphi, h, \alpha)| + |F_2^1(\varphi, h, \alpha)|. \qquad (4.49)$$

Proof of this theorem could be obtained using Theorems 4.4.1 and 4.4.2.

Now we consider Eq. (4.47),

$$|F_1^1(\varphi, h, \alpha)| = \frac{1}{\Gamma(1-\alpha)} \left| \int_{x_0}^{x_1} (x_1 - \tau)^{-\alpha} \left(\varphi'(\tau) - \frac{\varphi_1 - \varphi_0}{h} \right) d\tau \right|$$

$$\le \frac{\alpha}{2\Gamma(3-\alpha)} \max_{x_0 \le \tau \le x_1} |\varphi''(\tau)| h^{2-\alpha}. \qquad (4.50)$$

$$|F_2^1(\varphi, h, \alpha)| = \left| \int_{x_0}^{x_1} K(x_1, \tau) \left(\varphi(\tau) - \varphi_{1,0}(\tau) \right) d\tau \right|$$

$$\le \frac{M}{2} \max_{x_0 \le \tau \le x_1} |\varphi''(\tau)| x_1 h^2. \qquad (4.51)$$

From Eqs. (4.49)-(4.51),

$$|F^1(\varphi, h, \alpha)| = \frac{\alpha}{2\Gamma(3-\alpha)} \max_{x_0 \le \tau \le x_1} |\varphi''(\tau)| h^{2-\alpha} + \frac{M}{2} \max_{x_0 \le \tau \le x_1} |\varphi''(\tau)| x_1 h^2.$$

(ii) From Eqs. (4.1) and (4.22), we have, for $k \ge 2$

$$|F^k(\varphi, h, \alpha)| \leq \frac{1}{\Gamma(1-\alpha)} \left(\left| \int_{x_0}^{x_1} (x_k - \tau)^{-\alpha} (\varphi(\tau) - \varphi_{1,0}(\tau))' \, d\tau \right| \right.$$

$$+ \left| \sum_{j=2}^{k} \int_{x_{j-1}}^{x_j} (x_k - \tau)^{-\alpha} (\varphi(\tau) - \varphi_{2,j-1}(\tau))' \, d\tau \right| \right)$$

$$+ \left| \sum_{j=0}^{k-1} \int_{x_j}^{x_{j+1}} K(x_k, \tau)(\varphi(\tau) - \varphi_{1,j}(\tau)) \, d\tau \right| \tag{4.52}$$

Eq. (4.52) can be written directly using Eqs. (4.30) and (4.38),

$$|F^k(\varphi, h, \alpha)| \leq \left(\frac{\alpha}{12} \max_{x_0 \leq \tau \leq x_k} |\varphi''(\tau)|(x_k - x_1)^{-\alpha-1} h^3 \right.$$

$$+ \left[\frac{1}{12} + \frac{\alpha}{3(1-\alpha)(2-\alpha)} \left(\frac{1}{2} + \frac{1}{3-\alpha} \right) \right] \times \max_{x_0 \leq \tau \leq x_k} |\varphi'''(\tau)| h^{3-\alpha} \right)$$

$$+ \frac{M}{2} \max_{x_0 \leq \tau \leq x_1} |\varphi''(\tau)| x_k h^2.$$

This completes the proof.

4.5 Numerical Results

Here, we take the examples from the literature and investigate the performance of the presented schemes from Section 4.3. The examples are solved in [188]. The numerical solution obtained using the discussed schemes $S1$, $S2$ and $S3$ are presented through the tables. Further, the convergence order of the presented schemes is calculated varying the step size h. For calculating the convergence order (CO) in each case, the maximum absolute error (MAE) is calculated using the formula,

$$E(x) = \varphi_{exact}(x) - \varphi_{Numerical}(x),$$

$MAE(h) := \max\{|E(x)|, x \in \{x_0, x_1, \ldots, x_k\}\},$

and then the convergence order(CO) is obtained using the formula

$CO=lg[MAE(h)/MAE(h/2)]/lg(2).$

Example 4.5.1. *Consider the following linear fractional integro-differential equation [188],*

$$D^{1/2}\varphi(x) = \frac{(8/3)x^{3/2} - 2x^{1/2}}{\sqrt{\pi}} - \frac{3x^5 - 4x^4}{12} + \int_0^x x\tau\varphi(\tau)\,d\tau, \quad 0 \le x, \tau \le 1$$

subject to $\varphi(0) = 0$, having exact solution $\varphi(x) = x^2 - x$.

The numerical solutions of Example 4.5.1 using schemes $S1$, $S2$ and $S3$ are presented through Tables 4.1-4.2 for varying the step size $h = 1/5$ and $h = 1/10$ respectively. Further, maximum absolute errors and the convergence orders for Example 4.5.1 are provided in Tables 4.3-4.5 for the schemes $S1$, $S2$ and $S3$. From the Tables 4.3-4.5, it is observed that the scheme $S3$ which is the combination of the Quadratic and Linear schemes provide comparatively better convergence order than the schemes $S1$ and $S2$.

TABLE 4.1: Numerical solutions obtained using schemes $S1$, $S2$ and $S3$ for Example 4.5.1 for $n = 5$.

x_j	Exact Solution	$S1$	$S2$	$S3$
0.0	0.00	0.00	0.00	0.00
0.2	-0.16	-0.146642	-0.146720	-0.146642
0.4	-0.24	-0.217607	-0.228706	-0.228487
0.6	-0.24	-0.209663	-0.231148	-0.230553
0.8	-0.16	-0.120963	-0.152223	-0.150925
1.0	0.00	0.0505046	0.0078693	0.0104518

TABLE 4.2: Numerical solutions obtained using schemes $S1$, $S2$ and $S3$ for Example 4.5.1 for $n = 10$.

x_j	Exact Solution	$S1$	$S2$	$S3$
0.0	0.00	0.00	0.00	0.00
0.2	-0.16	-0.154475	-0.157177	-0.157166
0.4	-0.24	-0.231231	-0.238252	-0.238209
0.6	-0.24	-0.228457	-0.238611	-0.23849
0.8	-0.16	-0.145454	-0.158719	-0.154838
1.0	0.00	0.0184683	0.00132716	0.00190422

TABLE 4.3: MAE and CO for Example 4.5.1 using scheme $S1$.

h	MAE	CO
1/5	5.05046×10^{-2}	—
1/10	1.84683×10^{-2}	1.45136
1/20	6.70253×10^{-3}	1.46227
1/40	2.41481×10^{-3}	1.4728
1/80	8.65157×10^{-4}	1.48087

TABLE 4.4: MAE and CO for Example 4.5.1 using scheme $S2$.

h	MAE	CO
1/5	1.32804×10^{-2}	–
1/10	3.32983×10^{-3}	1.9958
1/20	8.33153×10^{-4}	1.9988
1/40	2.08325×10^{-4}	1.9998
1/80	5.20829×10^{-5}	1.9995

TABLE 4.5: MAE and CO for Example 4.5.1 using scheme $S3$.

h	MAE	CO
1/5	1.3358×10^{-2}	–
1/10	3.33388×10^{-3}	2.00243
1/20	8.33345×10^{-4}	2.00022
1/40	2.08334×10^{-4}	2.00002
1/80	5.20833×10^{-5}	2.00001

Example 4.5.2. *Here, we consider the fractional integro-differential equation [188],*

$$D^{5/6}\varphi(x) = f(x) + \int_0^x xe^\tau \varphi(\tau)\, d\tau, \quad 0 \le x, \tau \le 1,$$

subject to $\varphi(0) = 0$, where $f(x) = -\frac{3}{91}\frac{\Gamma(5/6)x^{(1/6)}\left(-91+216x^2\right)}{\pi} + 5x - xe^x(5 - 5x + 3x^2 - x^3)$. This has exact solution $\varphi(x) = x - x^3$.

The Example 4.5.2 is solved using the numerical schemes $S1$, $S2$ and $S3$ and the obtained approximate solutions are presented through Tables 4.6-4.7 varying the

step size $h = 1/5$ and $h = 1/10$ respectively. However, Tables 4.8-4.10 present the maximum absolute errors and convergence orders for each schemes. One can observe that the scheme $S1$ achieves convergence order more than 1 and schemes $S2$ and $S3$ obtain the convergence order greater than 2. It is to mention here that the quadratic scheme $S2$ in this case achieves better convergence order (Table 4.9) than the schemes $S3$ (Table 4.10). This phenomena may be due to the behaviour of the kernel, $k(x, \tau) = xe^\tau$ in the right side part of the fractional integro-differential equation.

TABLE 4.6: Numerical solutions obtained using schemes $S1$, $S2$ and $S3$ for Example 4.5.2 for $n = 5$.

x_j	Exact Solution	$S1$	$S2$	$S3$
0.0	0.00	0.00	0.00	0.00
0.2	0.192	0.180928	0.180928	0.180928
0.4	0.336	0.297013	0.314326	0.314249
0.6	0.384	0.298545	0.351121	0.350619
0.8	0.288	0.130392	0.241245	0.239352
1.0	0.000	-0.273753	-0.0670569	-0.0726013

TABLE 4.7: Numerical solutions obtained using schemes $S1$, $S2$ and $S3$ for Example 4.5.2 for $n = 10$.

x_j	Exact Solution	$S1$	$S2$	$S3$
0.0	0.00	0.00	0.00	0.00
0.2	0.192	0.187244	0.189354	0.189353
0.4	0.336	0.318846	0.330992	0.330968
0.6	0.384	0.346531	0.376553	0.376414
0.8	0.288	0.219913	0.277566	0.277066
1.0	0.000	-0.115363	-0.0147491	-0.0161695

TABLE 4.8: MAE and CO for Example 4.5.2 using scheme $S1$.

h	MAE	CO
1/5	2.73753×10^{-1}	–
1/10	1.15363×10^{-1}	1.24669
1/20	5.05187×10^{-2}	1.19129
1/40	2.24081×10^{-2}	1.1728
1/80	9.97983×10^{-3}	1.16693

TABLE 4.9: MAE and CO for Example 4.5.2 using scheme $S2$.

h	MAE	CO
1/5	6.70569×10^{-2}	–
1/10	1.47491×10^{-2}	2.18476
1/20	3.28518×10^{-3}	2.16658
1/40	7.33189×10^{-4}	2.16373
1/80	1.63575×10^{-4}	2.16423

TABLE 4.10: MAE and CO for Example 4.5.2 using scheme $S3$.

h	MAE	CO
1/5	7.26013×10^{-2}	–
1/10	1.61695×10^{-2}	2.16672
1/20	3.65192×10^{-3}	2.14655
1/40	8.26938×10^{-4}	2.1428
1/80	1.87313×10^{-4}	2.14233

Example 4.5.3. *In this example, we consider the fractional integro-differential equation similar to the one considered in [182] such that,*

$$D^{1/3}\varphi(x) = \frac{3\sqrt{\pi}x^{7/6}}{4\Gamma(13/6)} - \frac{2}{63}x^{9/2}(9 + 7x^2) + \int_0^x (x\tau + x^2\tau^2)\varphi(\tau)\,d\tau, \quad 0 \leq x, \tau \leq 1,$$

subject to $\varphi(0) = 0$ with the exact solution $\varphi(x) = x^{3/2}$.

The numerical schemes $S1$, $S2$ and $S3$ are performed on this example and the obtained numerical results varying the step size h are shown through Tables. Tables

4.11-4.12 present the obtained numerical solution using schemes $S1$, $S2$ and $S3$ for step sizes $h = 1/5$ and $h = 1/10$ respectively. And Tables 4.13-4.15 represent the maximum absolute errors and convergence orders for each schemes. From Tables 4.13-4.15, we observe that the scheme $S1$ performs comparatively better than the schemes $S2$ and $S3$. The occurrence of such behaviour may be due to the appearance of the fractional powers terms in the right side of the integro-differential equations and approximation of such fractional terms becomes difficult.

TABLE 4.11: Numerical solutions obtained using schemes $S1$, $S2$ and $S3$ for Example 4.5.3 for $n = 5$.

x_j	Exact Solution	$S1$	$S2$	$S3$
0.0	0.00	0.00	0.00	0.00
0.2	0.089443	0.099203	0.099203	0.099203
0.4	0.252982	0.263841	0.257785	0.257849
0.6	0.464758	0.476607	0.467637	0.467942
0.8	0.715542	0.72966	0.718179	0.719065
1.0	1.00	1.01967	1.00327	1.00547

TABLE 4.12: Numerical solutions obtained using schemes $S1$, $S2$ and $S3$ for Example 4.5.3 for $n = 10$.

x_j	Exact Solution	$S1$	$S2$	$S3$
0.0	0.00	0.00	0.00	0.00
0.2	0.089443	0.093188	0.091088	0.09109
0.4	0.252982	0.256826	0.253615	0.253637
0.6	0.464758	0.46881	0.465257	0.465341
0.8	0.715542	0.72024	0.716039	0.716264
1.0	1.00	1.00634	1.00061	1.00115

TABLE 4.13: MAE and CO for Example 4.5.3 using scheme $S1$.

h	MAE	CO
1/5	1.96715×10^{-2}	–
1/10	6.34357×10^{-3}	1.63274
1/20	2.05203×10^{-3}	1.62824
1/40	6.61465×10^{-4}	1.63332
1/80	2.12232×10^{-4}	1.64002

TABLE 4.14: MAE and CO for Example 4.5.3 using scheme $S2$.

h	MAE	CO
1/5	9.76046×10^{-3}	—
1/10	3.44045×10^{-3}	1.50435
1/20	1.21603×10^{-3}	1.50042
1/40	4.29918×10^{-4}	1.50005
1/80	1.51999×10^{-4}	1.5

TABLE 4.15: MAE and CO for Example 4.5.3 using scheme $S3$.

h	MAE	CO
1/5	9.76046×10^{-3}	—
1/10	3.44045×10^{-3}	1.50435
1/20	1.21603×10^{-3}	1.50042
1/40	4.29918×10^{-4}	1.50005
1/80	1.51999×10^{-4}	1.5

4.6 Conclusions

We studied comparative study of the different approximations schemes such as Linear, Quadratic and Quadratic-linear schemes for the fractional integro-differential equations. The convergences of the presented numerical schemes are also established. The discussed schemes successfully validate the numerical results. The convergence of the numerical schemes is also discussed and validated through numerical results. It is observed that the Quadratic-linear scheme $S3$ performs comparatively better to

schemes $S1$ and $S2$ for the Example 4.5.1. However, it appears that the scheme $S2$ performs comparatively good than schemes $S1$ and $S3$ for Example 4.5.2. Schemes $S2$ and $S3$ achieve second order convergence in both the examples. In the third Example 4.5.3, the fractional integro-differential equation having exact solution as fractional power of x is considered and numerical schemes are performed. And, in this case, it is observed that the scheme $S1$ achieves better convergence order than the schemes $S2$ and $S3$. Thus from these examples, it is concluded that all the schemes perform well and provides accurate numerical results. It is also observed that the performance of the schemes depends on the nature of the problem. Here, the developed schemes are discussed only for the linear fractional integro-differential equations.

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