Chapter 3

Numerical Schemes for the Generalized Abel's Integral Equations

3.1 Introduction

Integral Equations arise naturally in many applications of mathematics, science and technology and have been studied extensively both at the theoretical and practical levels. Integral equations build the base for modelling of various phenomena in basic and engineering sciences. For example, Abel's equation is one of the integral equation which is directly derived from problem of physics [43, 160, 161, 162, 163]. In this chapter, we consider the integral equation known as generalized Abel's integral equation (GAIE) [164, 165] and present two numerical schemes to solve it. The

GAIE [166] is defined as,

$$a(t)\int_0^t \frac{\varphi(t)}{(t-\tau)^\alpha}d\tau + b(t)\int_t^1 \frac{\varphi(t)}{(\tau-t)^\alpha}d\tau = g(t) \tag{3.1}$$

where $0 < \alpha < 1, \varphi$ be an unknown function and $a(t), b(t), g(t)$ are the known functions. It is interesting to note that Eq.(3.1) reduces to two standard forms of the Abel's integral equation of first kind such as,

$$\int_0^t \frac{\varphi(t)}{\sqrt{t-\tau}}d\tau = g(t) \tag{3.2}$$

and

$$2\int_t^1 \frac{\tau\varphi(t)}{\sqrt{\tau^2-t^2}}d\tau = g(t) \tag{3.3}$$

and these integral equations occur in many branches of science such as the 3-D image reconstruction from cone-beam projections in computerized tomography [167], reconstructions of the radially distributed emissivity from the line of sight projected intensity [53, 60, 168, 169, 170], as these lead naturally in the case of radial symmetry to the study of Abel's type integral equations. Several methods (analytical and numerical) have been purposed to solve GAIEs defined by Eq.(3.1). Some methods use the idea of the fractional integral operator and these are investigated first by Chakrabarti [164, 165].

Further, Dixit et al. [166, 171] provided an operational matrix based approach for solving GAIEs. Recently in [172], Pandey et al. discussed the collocation approach for solving GAIEs and also established the error convergence. Saadatmandi and Dehghan [158] applied collocation method to solve Abel's integral equations of first and second kind using shifted Legendre's polynomials. In [174], authors presented meshless based method for solving integral equations. Yousefi [175] proposed the Legendre wavelets based method for solving Abel's integral equations of first and second kind.

In this chapter, we will focus on obtaining the approximate solution of the GAIEs. First, the numerical schemes are developed and then these schemes are applied to solve the GAIEs. The numerical schemes presented here are based on the idea of dividing the whole interval into a set of small subintervals and between two successive subintervals the unknown function is approximated in terms of the linear and quadratic polynomials. Thus, these numerical schemes are named as linear and quadratic schemes. The error estimates for these approximations are also obtained where, we observe that the quadratic approximation achieves the convergence order up to 3. Further, illustrative examples are considered to validate the proposed schemes. The derivation of the schemes is described in the upcoming section.

3.2 Description of the Numerical Schemes

Let us denote the left part of Eq. (3.1) in form of an operator,

$$T^{\alpha}\varphi(t) = a(t)\int_0^t \frac{\varphi(t)}{(t-\tau)^{\alpha}}d\tau + b(t)\int_t^1 \frac{\varphi(t)}{(\tau-t)^{\alpha}}d\tau \tag{3.4}$$

such that the Eq.(3.1) takes the form $T^{\alpha}\varphi(t) = g(t)$.

Now, we present the discretization schemes for the operator defined by Eq. (3.4). First, we divide the domain into several subdomains and then approximate the unknown function into each subdomain. The considered schemes are linear and quadratic polynomial approximations of the unknown function into each subdomain. Similar schemes are earlier presented for fractional variational problems by Pandey and Agrawal [152]. The numerical schemes for T-operator are presented in sequence below.

3.2.1 Linear Scheme ($N1$)

In this scheme, the interval $[0, 1]$ is divided into k subintervals $[t_n, t_{n+1}]$ with uniform step size (or time interval)h , where $h = 1/k$ such that the node points are $t_n = nh, n = 0, 1, 2, 3 \dots k$. We label the unknown $f(\tau)$ at the node point t_n as f_n and $a(t), b(t)$ at the node point t_n as a_n, b_n respectively.

$$T^{\alpha} f(t)|_{t=t_n} = a(t_n) \int_0^{t_n} \frac{f(\tau)}{(t_n - \tau)^{\alpha}} d\tau + b(t_n) \int_{t_n}^{1} \frac{f(\tau)}{(\tau - t_n)^{\alpha}} d\tau \approx TL(f, n, h, \alpha) \quad (3.5)$$

where, $TL(f, n, h, \alpha)$ represents the linear approximation at the grid point t_n of T-operator and $E_{TL}(f, n, h, \alpha)$ represents the error term of the linear approximation such that,

$$E_{TL}(f, n, h, \alpha) = T^{\alpha} f(t) - TL(f, n, h, \alpha). \quad (3.6)$$

Theorem 3.2.1. *Let $f \in C^2[0, 1]$ and suppose that the interval [0,1] is subdivided into k subintervals $[t_n, t_{n+1}]$ of equal width $h = 1/k$ by using the nodes $t_n = nh$ for $n = 0, 1, 2, \dots, k$, then the linear approximation at the grid point t_n of $T^{\alpha} f(t)$ represented by $TL(f, n, h, \alpha)$ can be expressed as,*

$$TL(f, n, h, \alpha) = \frac{h^{1-\alpha}}{(1-\alpha)(2-\alpha)} \sum_{j=0}^{k} V(n, j) f_j \quad ,$$

where,

$$V(n,j)=\begin{cases} a_n\left((n-1)^{2-\alpha}-n^{1-\alpha}(n+\alpha-2)\right), & j=0,\\ a_n\left((n-j-1)^{2-\alpha}-2(n-j)^{2-\alpha}+(n-j+1)^{2-\alpha}\right), & 1\le j\le n-1,\\ a_n+b_n, & j=n,\\ b_n\left((j-n-1)^{2-\alpha}-2(j-n)^{2-\alpha}+(j-n+1)^{2-\alpha}\right), & n+1\le j\le k-1,\\ b_n\left((k-n-1)^{2-\alpha}-(k-n)^{1-\alpha}(k-n+\alpha-2)\right), & j=k. \end{cases}$$

Proof. The proof of this theorem is given as follows:

From Eq. (3.5), we have,

$$\begin{aligned} T^{\alpha}f(t)|_{t=t_n} &= a(t_n)\int_0^{t_n}\frac{f(\tau)}{(t_n-\tau)^{\alpha}}d\tau+b(t_n)\int_{t_n}^{1}\frac{f(\tau)}{(\tau-t_n)^{\alpha}}d\tau\\ &= a(t_n)\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}\frac{f(\tau)}{(t_n-\tau)^{\alpha}}d\tau+b(t_n)\sum_{j=n}^{k-1}\int_{t_j}^{t_{j+1}}\frac{f(\tau)}{(\tau-t_n)^{\alpha}}d\tau. \end{aligned} \tag{3.7}$$

The function $f(\tau)$ is approximated over the interval $[t_j, t_{j+1}]$ using the following formula,

$$f_{1,j}(\tau)=\frac{(t_{j+1}-\tau)}{h}f_j+\frac{(\tau-t_j)}{h}f_{j+1}. \tag{3.8}$$

The remaining few steps are straight forward and can be obtained by using Eqs. (3.7) and (3.8), so we omit the proof.

Theorem 3.2.2. *Suppose $a(t)$ and $b(t)$ be continuous on $[0,1]$ then the error estimate of the approximation given by Theorem 3.2.1, denoted as $E_{TL}(f,n,h,\alpha)$ satisfies,*

$$|E_{TL}(f,n,h,\alpha)| \leq C_\alpha M_1 \|f''\|_\infty (t_n)^{1-\alpha} h^2 + D_\alpha M_2 \|f''\|_\infty (t_k - t_n)^{1-\alpha} h^2 \approx O(h^2),$$

where C_α, D_α are constant depending on α and $\|\cdot\|_\infty$ denotes the infinite norm on the interval $[0,1]$.

Proof Using Eq. (3.6), we have

$$\begin{aligned}
|E_{TL}(f,n,h,\alpha)| &= |T^\alpha f(t) - TL(f,n,h,\alpha)| \\
&\leq \left| a(t_n) \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \frac{(f(\tau) - f_{1,j}(\tau))}{(t_n - \tau)^\alpha} d\tau \right| + \left| b(t_n) \sum_{j=n}^{k-1} \int_{t_j}^{t_{j+1}} \frac{(f(\tau) - f_{1,j}(\tau))}{(\tau - t_n)^\alpha} d\tau \right| \\
&\leq \frac{1}{2} \|f''\|_\infty \left| a(t_n) \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \frac{(\tau - t_j)(\tau - t_{j+1})}{(t_n - \tau)^\alpha} d\tau \right| \\
&+ \frac{1}{2} \|f''\|_\infty \left| b(t_n) \sum_{j=n}^{k-1} \int_{t_j}^{t_{j+1}} \frac{(\tau - t_j)(\tau - t_{j+1})}{(\tau - t_n)^\alpha} d\tau \right|,
\end{aligned}$$

since $a(t), b(t)$ be continuous on $[0,1]$ therefore $|a(t)| \leq M_1$ and $|b(t)| \leq M_2$ and thus we have,

$$\begin{aligned}
&\leq \frac{h^2}{8} \|f''\|_\infty \left\{ M_1 \left| \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} (t_n - \tau)^{-\alpha} d\tau \right| + M_2 \left| \sum_{j=n}^{k-1} \int_{t_j}^{t_{j+1}} (\tau - t_n)^{-\alpha} d\tau \right| \right\} \\
&\leq C_\alpha M_1 \|f''\|_\infty (t_n)^{1-\alpha} h^2 + D_\alpha M_2 \|f''\|_\infty (t_k - t_n)^{1-\alpha} h^2 \quad \approx O(h^2).
\end{aligned}$$

This completes the proof.

3.2.2 Quadratic Scheme ($N2$)

Here, the interval $[0, 1]$ is divided into k subintervals $[t_n, t_{n+1}]$ with uniform step size (or time interval) h, where $h = 1/k$ such that the node points are $t_n = nh, n = 0, 1, 2, 3, \ldots, k$. We label the unknown $f(\tau)$ at the node point t_n as f_n and $a(t), b(t)$ at the node point t_n as a_n, b_n respectively.

$$T^\alpha f(t)|_{t=t_n} = a(t_n)\int_0^{t_n} \frac{f(\tau)}{(t_n-\tau)^\alpha}d\tau + b(t_n)\int_{t_n}^1 \frac{f(\tau)}{(\tau-t_n)^\alpha}d\tau \approx TQ(f, n, h, \alpha), \tag{3.9}$$

where, $TQ(f, n, h, \alpha)$ represents the Quadratic approximation at the grid point t_n of T-operator and $E_{TQ}(f, n, h, \alpha)$ represents the error term of the Quadratic approximation such that,

$$E_{TQ}(f, n, h, \alpha) = T^\alpha f(t) - TQ(f, n, h, \alpha). \tag{3.10}$$

Theorem 3.2.3. *Let $f \in C^3[0, 1]$ and suppose that the interval [0,1] is subdivided into k subintervals $[t_{j-1}, t_j]$ of equal width $h = 1/k$ by using the nodes $t_j = jh$ for $j = 0, 1, 2, \ldots, k$, then the Quadratic approximation at the grid point t_n of $T^\alpha f(t)$, represented by $TQ(f, n, h, \alpha)$ can be expressed as,*

$$\begin{aligned} TQ(f, n, h, \alpha) = {} & \frac{h^{1-\alpha}}{(1-\alpha)(2-\alpha)}a_n(u_n f_0 + v_n f_1) \\ & + h^{1-\alpha}a_n \sum_{j=2}^{n} A_{nj}f_{j-2} + B_{nj}f_{j-1} + C_{nj}f_j \\ & + h^{1-\alpha}b_n \sum_{j=n+1}^{k} D_{nj}f_{j-2} + E_{nj}f_{j-1} + F_{nj}f_j, \end{aligned}$$

where, coefficients of f_j for $j = 0, 1, 2, \ldots, k$ are given below in the proof.

Proof The proof of this theorem is obtained by discretizing Eq. (3.4) as,

$$
\begin{aligned}
T^{\alpha}f(t)|_{t=t_n} &= a(t_n)\int_0^{t_n}\frac{f(\tau)}{(t_n-\tau)^{\alpha}}d\tau + b(t_n)\int_{t_n}^{1}\frac{f(\tau)}{(\tau-t_n)^{\alpha}}d\tau,\\
&= a(t_n)\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\frac{f(\tau)}{(t_n-\tau)^{\alpha}}d\tau + \sum_{j=n+1}^{k}\int_{t_{j-1}}^{t_j}\frac{f(\tau)}{(\tau-t_n)^{\alpha}}d\tau,\\
&= a(t_n)\int_0^{t_1}\frac{f(\tau)}{(t_n-\tau)^{\alpha}}d\tau + a(t_n)\sum_{j=2}^{n}\int_{t_{j-1}}^{t_j}\frac{f(\tau)}{(t_n-\tau)^{\alpha}}d\tau\\
&\qquad + b(t_n)\sum_{j=n+1}^{k}\int_{t_{j-1}}^{t_j}\frac{f(\tau)}{(\tau-t_n)^{\alpha}}d\tau.
\end{aligned}
$$

On the first interval $[t_0, t_1]$, we use the linear interpolation to approximate $f(\tau)$ and for the other subdomains ($j \geq 2$), we use the quadratic interpolation to approximate $f(\tau)$ by using three points $(t_{j-2}, f_{j-2}), (t_{j-1}, f_{j-1})$ and (t_j, f_j) such that,

$$
\begin{aligned}
f_{2j}(\tau) = {}& \frac{(\tau-t_{j-1})(\tau-t_j)}{2h^2}f_{j-2} - \frac{(\tau-t_{j-2})(\tau-t_j)}{h^2}f_{j-1}\\
&+ \frac{(\tau-t_{j-2})(\tau-t_{j-1})}{2h^2}f_j.
\end{aligned} \tag{3.11}
$$

Thus we have,

$$
\approx a_n\int_{t_0}^{t_1}\frac{f_{11}(\tau)}{(t_n-\tau)^{\alpha}}d\tau + a_n\sum_{j=2}^{n}\int_{t_{j-1}}^{t_j}\frac{f_{2j}(\tau)}{(t_n-\tau)^{\alpha}}d\tau + b_n\sum_{j=n+1}^{k}\int_{t_{j-1}}^{t_j}\frac{f_{2j}(\tau)}{(\tau-t_n)^{\alpha}}d\tau
$$

Therefore,

$$TQ(f,n,h,\alpha) = \frac{h^{1-\alpha}}{(1-\alpha)(2-\alpha)} a_n(u_n f_0 + v_n f_1)$$

$$+ h^{1-\alpha} a_n \sum_{j=2}^{n} (A_{nj} f_{j-2} + B_{nj} f_{j-1} + C_{nj} f_j)$$

$$+ h^{1-\alpha} b_n \sum_{j=n+1}^{k} (D_{nj} f_{j-2} + E_{nj} f_{j-1} + F_{nj} f_j),$$

where,

$$u_n = (n-1)^{2-\alpha} + (2-\alpha-n)n^{1-\alpha},$$

$$v_n = (n)^{2-\alpha} - (n+1-\alpha)(n-1)^{1-\alpha},$$

$$A_{nj} = \frac{1}{2}\int_0^1 x(x-1)(n-j-x+1)^{-\alpha}dx,$$

$$B_{nj} = \int_0^1 (x+1)(x-1)(n-j-x+1)^{-\alpha}dx,$$

$$C_{nj} = \frac{1}{2}\int_0^1 x(x+1)(n-j-x+1)^{-\alpha}dx,$$

$$D_{nj} = \frac{1}{2}\int_0^1 x(x-1)(j-n+x-1)^{-\alpha}dx,$$

$$E_{nj} = \int_0^1 (x+1)(x-1)(j-n+x-1)^{-\alpha}dx,$$

$$F_{nj} = \frac{1}{2}\int_0^1 x(x+1)(j-n+x-1)^{-\alpha}dx.$$

This completes the proof.

Theorem 3.2.4. *Suppose $a(t)$ and $b(t)$ be continuous on $[0,1]$ then the error estimate of the approximation given by Theorem 3.2.3 denoted by $E_{TQ}(f,n,h,\alpha)$, satisfies,*

$$\begin{aligned}|E_{TQ}(f,n,h,\alpha)| \leq E_\alpha \|f''\|_\infty M_1 &\left((1-\alpha)(t_n)^{-\alpha}h^3 + \frac{\alpha(1-\alpha)}{2}(t_n)^{(-\alpha-1)}h^4 + \cdots\right) \\ &+ F_\alpha \|f'''\|_\infty M_1 (t_n - t_1)^{1-\alpha}h^3 + G_\alpha \|f'''\|_\infty M_2 (t_k - t_n)^{1-\alpha}h^3 \\ &\approx O(h^3), \end{aligned} \tag{3.12}$$

where $E_\alpha, F_\alpha, G_\alpha$ are constants depending upon α.

Proof. From Eq. (3.10) we get,

$$\begin{aligned}|E_{TQ}(f,n,h,\alpha)| &= |T^\alpha f(t) - TQ(f,n,h,\alpha)| \\ &\leq \left| a(t_n)\int_{t_0}^{t_1} \frac{(f(\tau) - f_{1,0}(\tau))}{(t_n-\tau)^\alpha} d\tau \right| + \left| a(t_n)\sum_{j=2}^{n}\int_{t_{j-1}}^{t_j} \frac{(f(\tau) - f_{2,j}(\tau))}{(t_n-\tau)^\alpha} d\tau \right| \\ &\quad + \left| b(t_n)\sum_{j=n+1}^{k}\int_{t_{j-1}}^{t_j} \frac{(f(\tau) - f_{2,j}(\tau))}{(\tau - t_n)^\alpha} d\tau \right|, \end{aligned}$$

since $a(t)$ and $b(t)$ be continuous on $[0,1]$ then $|a(t)| \leq M_1$ and $|b(t)| \leq M_2$ for positive M_1 and M_2, then we have,

$$\leq \frac{h^2}{8}\|f''\|_\infty M_1 \left|\int_{t_0}^{t_1} \frac{f_{11}(\tau)}{(t_n-\tau)^\alpha}d\tau\right| + \frac{1}{6}\|f'''\|_\infty M_1 \left|\sum_{j=2}^{n}\int_{t_{j-1}}^{t_j} \frac{(\tau-t_{j-2})(\tau-t_{j-1})(\tau-t_j)}{(t_n-\tau)^\alpha}d\tau\right|$$

$$+ \frac{1}{6}\|f'''\|_\infty M_2 \left|\sum_{j=n+1}^{k}\int_{t_{j-1}}^{t_j} \frac{(\tau-t_{j-2})(\tau-t_{j-1})(\tau-t_j)}{(\tau-t_n)^\alpha}d\tau\right|$$

$$\leq \frac{h^2}{8}\|f''\|_\infty M_1 \left|\int_{t_0}^{t_1} \frac{1}{(t_n-\tau)^\alpha}d\tau\right| + \frac{h^3}{12}\|f'''\|_\infty M_1 \left|\sum_{j=2}^{n}\int_{t_{j-1}}^{t_j} \frac{1}{(t_n-\tau)^\alpha}d\tau\right|$$

$$+ \frac{h^3}{12}\|f'''\|_\infty M_2 \left|\sum_{j=n+1}^{k}\int_{t_{j-1}}^{t_j} \frac{1}{(\tau-t_n)^\alpha}d\tau\right|$$

$$\leq E_\alpha\|f''\|_\infty M_1 \left((t_n)^{1-\alpha} - (t_n-t_1)^{1-\alpha}\right)h^2 + F_\alpha\|f'''\|_\infty M_1 (t_n-t_1)^{1-\alpha}h^3$$

$$+ G_\alpha\|f'''\|_\infty M_2 (t_k-t_n)^{1-\alpha}h^3$$

$$= E_\alpha\|f''\|_\infty M_1 \left((1-\alpha)(t_n)^{-\alpha}h^3 - \frac{\alpha(1-\alpha)}{2}(t_n)^{-\alpha-1}h^4 + \cdots\right)$$

$$+ F_\alpha\|f'''\|_\infty M_1 (t_n-t_1)^{1-\alpha}h^3 + G_\alpha\|f'''\|_\infty M_2 (t_k-t_n)^{1-\alpha}h^3 \approx O(h^3).$$

3.3 Numerical Illustration

In this section, the application of the numerical schemes discussed in Section 3.2 is established. The schemes are used to obtain the numerical solution of the GAIEs given by Eq.(3.1).Theorems 3.2.3 and 3.2.4 are used to find an approximation solution of Eq.(3.1) with $a(t) = b(t) = 1$. Consider Eq.(3.1) with $a(t) = b(t) = 1$,

$$\int_0^t \frac{f(t)}{(t-\tau)^\alpha}d\tau + \int_t^1 \frac{f(t)}{(\tau-t)^\alpha}d\tau = g(t),\ \ 0 < \alpha < 1,\ \ 0 \leq t \leq 1, \tag{3.13}$$

and $f(t)$ is the unknown function. Using the definition of T-operator, we can write Eq.(3.13) in the equivalent form,

$$T^{\alpha} f(t) = g(t), \tag{3.14}$$

and hence using the schemes $N1$ and $N2$ for approximation of the T-operator, the numerical solution of Eq. (3.14) is obtained. We consider some examples from the literature to validate the presented theoretical results. To establish the numerical convergence, let us define, $E(t)$ be error of the exact and numerical solution and $e(t)$ is the maximum absolute error(MAE) such as, $E(t) = f_{exact}(t) - f_{numerical}(t)$ and $MAE(h) := max\{|E(t)|, t \in \{t_0, t_1, \ldots, t_k\}\}$ respectively. Hence the numerical convergence order(CO) is obtained using the formula $CO = lg[MAE(h)/MAE(h/2)]/lg(2)$.

Example 3.3.1. *Consider the Generalized Abel integral equation [171, 172] given by Eq.(3.14) with* $\alpha = 1/2$ *and* $g(t) = \frac{4}{105} t^{3/2}(35 - 24t^2) + \frac{8}{105}(1-t)^{1/2}(5 + 13t - 6t^2 - 12t^3)$. *It has exact solution,* $f(t) = t - t^3$.

The numerical solutions of Example 3.3.1 using schemes $N1$ and $N2$ are presented through Tables 3.1 and 3.2 for varying the step size $h = 1/10$ and $h = 1/20$ respectively. Further, maximum absolute errors and the convergence orders for Example 3.3.1 are provided in Tables 3.3 and 3.4. From Tables 3.3 and 3.4, it is observed that the scheme $N1$ achieves the convergence order up to 2 and $N2$ achieves the convergence order very close to 3.

Table 3.1: Numerical solution using schemes $N1$ and $N2$ for Example 3.3.1 for $k = 10$.

t_j	Exact solution	$N1$	$N2$
0.0	0.000	0.000733	0.000253
0.1	0.099	0.099575	0.099213
0.2	0.192	0.193026	0.192226
0.3	0.273	0.274438	0.273226
0.4	0.336	0.337858	0.336227
0.5	0.375	0.377276	0.375228
0.6	0.384	0.386687	0.384227
0.7	0.357	0.360084	0.357226
0.8	0.288	0.291459	0.288224
0.9	0.171	0.174607	0.171209
1.0	0.000	0.004778	0.000306

Example 3.3.2. *Consider the Generalized Abel integral equation [166] given by Eq.(3.14) with* $\alpha = 1/2$, *and* $g(t) = \frac{4}{3}t^{3/2}{}_1F_2(1; \frac{5}{4}; \frac{7}{4}; -\frac{t^2}{4}) + \sqrt{2\pi}\left[S\left(\frac{\sqrt{2-2t}}{\sqrt{\pi}}\right)\cos t + C\left(\frac{\sqrt{2-2t}}{\sqrt{\pi}}\right)\sin t\right]$ *having the exact solution* $f(t) = \sin t$, *where* $C(z)$ *and* $S(z)$ *are called Fresnel integrals, defined by*

$$C(z) = \int_0^z \cos\left(\frac{\pi t^2}{2}\right)dt, \quad S(z) = \int_0^z \sin\left(\frac{\pi t^2}{2}\right)dt,$$

where ${}_pF_q(a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q; z)$ *denotes the generalized hypergeometric function.*

TABLE 3.2: Numerical solution using schemes $N1$ and $N2$ for Example 3.3.1 for $k = 20$.

t_j	Exact solution	$N1$	$N2$
0.0	0.000	0.000152	0.000033
0.1	0.099	0.099149	0.099028
0.2	0.192	0.192256	0.192029
0.3	0.273	0.273365	0.273029
0.4	0.336	0.336476	0.336029
0.5	0.375	0.375587	0.375029
0.6	0.384	0.384696	0.384029
0.7	0.357	0.357803	0.357029
0.8	0.288	0.288906	0.288029
0.9	0.171	0.171994	0.171029
1.0	0.000	0.001287	0.000039

TABLE 3.3: MAE and CO for Example 3.3.1 using scheme $N1$.

h	MAE	CO
1/5	1.72519×10^{-2}	
1/10	4.77831×10^{-3}	1.85218
1/20	1.28742×10^{-3}	1.89202
1/40	3.40749×10^{-4}	1.9177

TABLE 3.4: MAE and CO for Example 3.3.1 using scheme $N2$.

h	MAE	CO
1/5	2.3205×10^{-3}	
1/10	3.0626×10^{-4}	2.92161
1/20	3.99102×10^{-5}	2.93993
1/40	5.15496×10^{-6}	2.95272

TABLE 3.5: Numerical solution using schemes $N1$ and $N2$ for Example 3.3.2 for $k = 10$.

t_j	Exact solution	$N1$	$N2$
0.0	0.000	0.000116	0.000041
0.1	0.099833	0.099927	0.099868
0.2	0.198669	0.198837	0.198706
0.3	0.29552	0.295754	0.295556
0.4	0.389418	0.389717	0.389453
0.5	0.479426	0.479787	0.479459
0.6	0.564642	0.565062	0.564675
0.7	0.644218	0.64469	0.644248
0.8	0.717356	0.717873	0.717384
0.9	0.783356	0.783852	0.783351
1.0	0.841471	0.842159	0.841505

TABLE 3.6: Numerical solution using schemes $N1$ and $N2$ for Example 3.3.2 for $k = 20$.

t_j	Exact solution	$N1$	$N2$
0.0	0.000	0.000024	0.000005
0.1	0.099833	0.0998579	0.099838
0.2	0.198669	0.198711	0.198674
0.3	0.29552	0.29558	0.295525
0.4	0.389418	0.389495	0.389423
0.5	0.479426	0.479519	0.47943
0.6	0.564642	0.564751	0.564647
0.7	0.644218	0.644341	0.644221
0.8	0.717356	0.717491	0.71736
0.9	0.783356	0.783472	0.78333
1.0	0.841471	0.841655	0.841475

TABLE 3.7: MAE and CO for Example 3.3.2 using scheme $N1$.

h	MAE	CO
1/5	2.51102×10^{-3}	
1/10	6.8795×10^{-4}	1.8679
1/20	1.84114×10^{-4}	1.9017
1/40	4.85133×10^{-5}	1.92415

TABLE 3.8: MAE and CO for Example 3.3.2 using scheme $N2$.

h	MAE	CO
1/5	3.0766×10^{-4}	
1/10	4.10266×10^{-5}	2.90671
1/20	5.39757×10^{-6}	2.92618
1/40	7.02946×10^{-7}	2.94082

Example 3.3.2 is solved using schemes $N1$ and $N2$ and the numerical results are presented through Tabels 3.5 and 3.6 for varying the step size $h = 1/10$ and $h = 1/20$ respectively. Further, maximum absolute errors and the convergence orders obtained for Example 3.3.2 are shown in Tabels 3.7 and 3.8 for the schemes $N1$ and $N2$.

Note: Comparison with the existing methods for GAIEs.

The above example is taken from [166] where the solution of GAIE is obtained via operational matrix of the Bernstein polynomials. In the formation of operational matrix, the continuous integral term is calculated. The proposed method is based on discretization of the integral which converts the integral into summation. In fact, it reduces the absolute error on the boundary in comparison to [166]. The obtained maximum absolute error for Example 3.3.1 and Example 3.3.2 is of orders up to 2 and 3 respectively. In addition, the numerical convergence order validates the theoretical results.

3.4 Conclusions

We studied two approximation schemes such as Linear scheme and Quadratic scheme for GAIEs. The approximation schemes are first presented for the T-operator and then is used to solve the GAIEs. The error convergences of the schemes are obtained. The proposed schemes successfully achieve accuracy in the numerical solutions of the examples. Further, it is observed that the proposed schemes work well and produce accurate numerical results. The numerical error convergence for the examples are also obtained and presented through tables. It is observed that the scheme $N2$ performs comparatively better than the scheme $N1$ and $N2$ achieves the convergence order very close to three. In a special case, the schemes presented here can also be applied to solve first types of Abel's integral equations.

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