Chapter 2

Approximations of Fractional Integrals and Caputo Derivatives with Application in Solving Abel's Integral Equations

2.1 Introduction

Fractional derivatives have gained much attention in recent years and this could be due to its non-local nature compare to the traditional integer order derivatives. Fractional derivatives have played a significant role in analysing the behaviour of the physical phenomena through different domains of the science and engineering. Some of the pioneer contributions in these areas may be considered as biology [136], viscoelasticity [137, 138], bioengineering [139] and more can be found in [2, 5]. Some of recent applications of fractional derivatives in emerging areas could be also noted as mathematical biology [140, 141, 142, 143, 144] and heat and fluid flow [145]. Numerical integration is the basic tool for obtaining the approximate value of the definite integrals where the analytical integrations are difficult to evaluate. Numerical integrations for the fractional integrations also become important in developing the algorithms for solving applied problems defined using fractional derivatives. In recent years, numerical integrations of the fractional integrals and the fractional derivatives have attracted many researchers. The Adams-Bashforth–Moulton method for the fractional differential equations is discussed in [146, 147]. Kumar and Agrawal [148] presented quadratic approximation scheme for fractional differential equations. In [149], Odibat presented a modified algorithm for approximation of fractional integral and Caputo derivatives and also obtained its error estimate. In [150, 151], Agrawal discussed the finite element approximation and fractional power series solution for the fractional variational problems. Pandey and Agrawal [152] discussed a comparative study of different numerical methods such as linear, quadratic and quadratic-linear schemes for solving fractional variational problems defined in terms of the generalized derivatives. Recently, in [153], authors present three schemes for solving fractional integro-differential equations. Reproducing kernel algorithm are discussed for some time fractional partial differential equations in [154, 155]. Some more approximation schemes for solving fractional PDEs are elaborated in detail by Li and Zeng in [156]. In [149], Odibat presented the scheme for approximating the Riemann-Liouville fractional integral and then obtained the approximations for the Caputo derivatives. In this chapter, we focus on the higher order approximations such as quadratic and cubic schemes to approximate the Riemann-Liouville fractional integral and Caputo derivatives. The numerical approximations are based on the idea of dividing the whole interval into a set of small subintervals and between these two successive subintervals the unknown functions are approximated in terms of the quadratic and cubic polynomials. Thus, the numerical scheme presented for the approximation of the Riemann-Liouville fractional integral and Caputo derivatives are named as quadratic and cubic schemes. The error estimates for these approximations are also presented where we observe that the quadratic and cubic approximations achieve high convergence order. To validate these schemes, test examples are considered from the literature [149]. We also show that the obtained results using the proposed schemes preserve the results obtained by Odibat [149]. Further, the presented schemes are applied to solve the Abel's integral equations. The numerical approach for solving Abel's integral equations are recently studied by Jahanshahi et al. [50], using the approximation scheme presented in [149]. Avazzadeh et al. [157], used fractional calculus approach together with Chebyshev polynomials to solve Abel's integral equations.

Saadatmandi and Dehghan [158], applied collocation method to solve Abel's integral equations of first and second kind using shifted Legendre's polynomials. Li and Zhao [62], studied the Abel's type integral equation using the Mikusinski's operator of fractional order. In [159], Badr presented the solution of generalized Abel's integral using Jacobi polynomials. Further, Saleh et al. [64], studied solution of generalized Abel's integral equation using Chebyshev Polynomials. The numerical results presented in [50], are considered here to validate and compare the results obtained by the presented schemes. Numerical simulations validate the presented schemes and show the advantage over existing method [50] .

2.2 Numerical Schemes

Here, two numerical schemes such as Quadratic and Cubic schemes are discussed. First, we divide the domain into several sub domains and then approximate the unknown function into each sub domain. Further, the approximations are obtained using Quadratic and Cubic polynomial approximations of the unknown function into each sub domains.

Here, we follow the simpler notations to the fractional integral and fractional derivatives and denote Riemann-Lowville fractional integral (Eq. (1.1)) and Caputo fractional derivative (Eq. (1.3)) as I-operator and D-operator respectively in the upcoming derivations of the numerical schemes. From Eq. (1.1) and Eq. (1.3), the approximation of the I-operator and D-operator can be expressed as,

$$(I^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau \ \approx \ I(f,h,\alpha), \tag{2.1}$$

$$(D^\alpha f)(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau) d\tau, \ \approx \ D(f,h,\alpha), \tag{2.2}$$

$$(I^\alpha f)(t) = I(f,h,\alpha) + E_I(f,h,\alpha), \tag{2.3}$$

$$(D^\alpha f(t)) = D(f,h,\alpha) + E_D(f,h,\alpha). \tag{2.4}$$

Where, $I(f,h,\alpha)$ and $D(f,h,\alpha)$ denote the approximation of the I-operator and D-operator respectively, and $E_I(f,h,\alpha)$, $E_D(f,h,\alpha)$ represents the error terms of their approximations. Now we present the quadratic and cubic approximation schemes of the I-operator and D-operator respectively as follows:

2.2.1 The Quadratic Scheme ($A1$)

In this subsection, the domain interval $[0,t]$ is distributed into even number of subintervals, $N = 2n$ for $n \geq 1$, equal parts with uniform step size (or time interval) h, where $h = \frac{t}{2n}$ such that the node points are $t_i = ih$, $i = 0, 1, 2, \ldots, 2n$.

$$(I^\alpha f)(t) = \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau) d\tau \ \approx \ IQ(f,h,\alpha), \tag{2.5}$$

$$D^{\alpha} f(t) \approx DQ(f, h, \alpha), \tag{2.6}$$

where, $IQ(f, h, \alpha)$, $DQ(f, h, \alpha)$ represent the quadratic approximation of the I and D-operators respectively and $E_{IQ}(f, h, \alpha)$, $E_{DQ}(f, h, \alpha)$ represent the error terms of the quadratic approximation such that,

$$E_{IQ}(f, h, \alpha) = (I^{\alpha} f)(t) - IQ(f, h, \alpha), \tag{2.7}$$

$$E_{DQ}(f, h, \alpha) = (D^{\alpha} f)(t) - DQ(f, h, \alpha). \tag{2.8}$$

The function $f(\tau)$ is approximated over the interval $[t_{2i}, t_{2i+2}]$ using the following formula [152]:

$$f_{i,2} = \frac{-(\tau - t_{2i+1})}{h}\left[1 - \frac{(\tau - t_{2i+1})}{h}\right] f_{2i} + \left[1 - \left(\frac{\tau - t_{2i+1}}{h}\right)^2\right] f_{2i+1} + \frac{(\tau - t_{2i+1})}{2h}\left[1 + \frac{(\tau - t_{2i+1})}{h}\right] f_{2i+2}. \tag{2.9}$$

In this case, results are presented as following lemmas.

Lemma 2.2.1. *Suppose that $f \in C^3[0, \delta]$, and the interval $[0, \delta]$ is divided into even number of sub intervals $[t_{2i}, t_{2i+2}]$ such that $t_i = ih$ with $h = \frac{\delta}{2n}$, $i = 0, 1, 2, \ldots, 2n$. Let $f_{i,2}$ is the quadratic polynomial approximation for f to the subintervals $[t_{2i}, t_{2i+2}]$ then the quadratic approximation $IQ(f, h, \alpha)$ of the I-operator is given by,*

$$(i) \quad IQ(f, h, \alpha) = \sum_{i=0}^{n-1} (A_{in} f(t_{2i}) + B_{in} f(t_{2i+1}) + C_{in} f(t_{2i+2})), \tag{2.10}$$

where

$$A_{in} = \frac{2^{\alpha}h^{\alpha}}{\Gamma(\alpha+3)}\{(n-i-1)^{(\alpha+1)}(2-\alpha+4i-4n)$$
$$+(n-i)^{\alpha}(2+\alpha^2+4i^2+i(6-8n)3\alpha(1+i-n)-6n+4n^2)\}, \tag{2.11}$$

$$B_{in} = \frac{2^{(\alpha+2)}h^{\alpha}}{\Gamma(\alpha+3)}\{(n-i-1)^{(\alpha+1)}(\alpha-2i+2n)+(n-i)^{(\alpha+1)}(2+\alpha+2i-2n)\}, \tag{2.12}$$

$$C_{jk} = \frac{2^{\alpha}h^{\alpha}}{\Gamma(\alpha+3)}\{(n-i)^{(\alpha+1)}(2+\alpha+4i-4n)$$
$$+(n-i-1)^{\alpha}(\alpha^2+2i-3\alpha i+4i^2-2n+3\alpha n-8in+4n^2)\}, \tag{2.13}$$

(ii) and the approximation error $E_{IQ}(f,h,\alpha)$ *has the form,*

$$|E_{IQ}(f,h,\alpha)| \leq C_{\alpha}\|f'''\|_{\infty}(t_{2n})^{\alpha}h^3, \tag{2.14}$$

where C_{α} *is a constant depending on* α.

Proof: From the definition of I-operator, we have,

$$IQ(f,h,\alpha) = (I^{\alpha}f)(t_{2i}) = \frac{1}{\Gamma(\alpha)}\int_0^{t_{2i}}(t_{2i}-\tau)^{\alpha-1}f(\tau)d\tau, \tag{2.15}$$

We approximate $f(\tau)$ over the interval $[t_{2i}, t_{2i+2}]$ using the quadratic polynomials [152] as,

$$f_{i,2} = \frac{-(\tau-t_{2i+1})}{h}\left[1-\frac{(\tau-t_{2i+1})}{h}\right]f_{2i}+\left[1-\left(\frac{\tau-t_{2i+1}}{h}\right)^2\right]f_{2i+1}$$
$$+\frac{(\tau-t_{2i+1})}{2h}\left[1+\frac{(\tau-t_{2i+1})}{h}\right]f_{2i+2}. \tag{2.16}$$

Evaluating Eq. (2.15) using Eq. (2.16), the desired approximation of $IQ(f,h,\alpha)$ as given in part (i) of the Lemma 2.2.1 is obtained.

For proof of the part (ii) of the Lemma 2.2.1, we use the following well known result of the interpolation by polynomials.

Theorem 2.2.1. *Let $g_n(t)$ be the polynomial interpolating a function $g \in C^{n+1}[a,b]$ at the nodes $t_0, t_1, t_2, \ldots, t_n$ lying in the interval $[a,b]$. Then for, $t \in [a,b]$, there exist a $\xi_t \in (a,b)$ such that, $E_n(t) = g(t) - g_n(t) = \frac{g^{(n+1)}(\xi_t)}{(n+1)!} \prod_{i=0}^{n}(t-t_i)$.*

From Eq. (2.15) and Eq. (2.16) we have,

$$
\begin{aligned}
|E_{IQ}(f,h,\alpha)| &= |I^\alpha f(t) - IQ(f,h,\alpha)| \\
&= \left| I^\alpha f(t) - \frac{1}{\Gamma(\alpha)} \int_0^{t_{2i}} (t_{2i}-\tau)^{\alpha-1} f(\tau) d\tau \right| \\
&= \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_{2n}} (t_{2n}-\tau)^{\alpha-1} f(\tau) d\tau - \sum_{i=0}^{n-1} \int_{t_{2i}}^{t_{2i+2}} (t_{2n}-\tau)^{\alpha-1} f_{i,2}(\tau) d\tau \right| \\
&= \frac{1}{\Gamma(\alpha)} \left| \sum_{i=0}^{n-1} \int_{t_{2i}}^{t_{2i+2}} (t_{2n}-\tau)^{\alpha-1} (f(\tau) - f_{i,2}(\tau)) d\tau \right|, \qquad (2.17)
\end{aligned}
$$

Using Theorem 2.2.1, and Eq. (2.17) we have,

$$
\begin{aligned}
&\leq \frac{1}{6\Gamma(\alpha)} \|f'''\|_\infty \left| \sum_{i=0}^{n-1} \int_{t_{2i}}^{t_{2i+2}} (t_{2n}-\tau)^{\alpha-1} (\tau - t_{2i})(\tau - t_{2i+1})(\tau - t_{2i+2}) d\tau \right| \\
&\leq \frac{h^3}{9\sqrt{3}\Gamma(\alpha)} \|f'''\|_\infty \left| \sum_{i=0}^{n-1} \int_{t_{2i}}^{t_{2i+2}} (t_{2n}-\tau)^{\alpha-1} d\tau \right| = C_\alpha \|f'''\|_\infty (t_{2n})^\alpha h^3. \qquad (2.18)
\end{aligned}
$$

where C_α is a constant depending on α. The proof is completed.

Lemma 2.2.2. *Suppose that $f \in C^{m+3}[0,\delta]$, and the interval $[0,\delta]$ is divided into even number of sub intervals $[t_{2i}, t_{2i+2}]$ such that $t_i = ih$ with $h = \frac{\delta}{2n}$, $i = 0, 1, 2, \ldots, 2n$.*

Let $f_{i,2}$ is the quadratic polynomial approximation for $f^{(m)}$ to the subintervals $[t_{2i}, t_{2i+2}]$ then the quadratic approximation $DQ(f,h,\alpha)$ of the D-operator is given by,

$$(i)\ DQ(f,h,\alpha) = \sum_{i=0}^{n-1}(\mathcal{A}_{in}f^{(m)}(t_{(2i)}) + \mathcal{B}_{in}f^{(m)}(t_{(2i+1)}) + \mathcal{C}_{in}f^{(m)}(t_{(2i+2)})), \quad (2.19)$$

where,

$$\mathcal{A}_{in} = \frac{2^{(m-\alpha)}h^{m-\alpha}}{\Gamma(m-\alpha+3)}[(n-i-1)^{(m-\alpha+1)}(2-m+\alpha+4i-4n) + (n-i)^{(m-\alpha)}$$

$$\{(2+(m-\alpha)^2) + 4i^2 + i(6-8n) + 3(m-\alpha)(1+i-n) - 6n + 4n^2\}], \quad (2.20)$$

$$\mathcal{B}_{in} = \frac{2^{(m-\alpha+2)}h^{m-\alpha}}{\Gamma(m-\alpha+3)}[(n-i-1)^{(m-\alpha+1)}(m-\alpha-2i+2n)$$

$$+ (n-i)^{(m-\alpha+1)}(2+m-\alpha+2i-2n)], \quad (2.21)$$

$$\mathcal{C}_{in} = \frac{2^{(m-\alpha)}h^{m-\alpha}}{\Gamma(m-\alpha+3)}[(n-i)^{(m-\alpha+1)}(2+m-\alpha+4i-4n) + (n-i-1)^{(m-\alpha)}$$

$$\{(m-\alpha)^2 + 2i - 3(m-\alpha)i + 4i^2 - 2n + 3(m-\alpha)n - 8in + 4n^2\}] \quad (2.22)$$

(ii) And the approximation error $E_{DQ}(f,h,\alpha)$ has the form,

$$|E_{DQ}(f,h,\alpha)| \leq C'_\alpha \|f^{(m+3)}\|_\infty t_{2n}^{(m-\alpha)} h^3, \quad (2.23)$$

where C'_α is a constant depending only α.

Proof: The proof of the part (i) and part (ii) of the lemma can be carried out following the similar steps and replacing α to $m-\alpha$ and $f(\tau)$ by $f^{(m)}(\tau)$ as described in the proof of the Lemma 2.2.1.

2.2.2 The Cubic Scheme ($A2$)

Lemma 2.2.3. *Suppose that* $f \in C^4[0,\delta]$, *and the interval* $[0,\delta]$ *is divided into subintervals* $[t_{3i}, t_{3i+3}]$ *such that* $t_i = ih$ *with* $h = \frac{\delta}{3n}$, $i = 0, 1, 2, \ldots, 3n$. *Let* $f_{i,3}$ *is the cubic polynomial approximation for* f *to the subintervals* $[t_{3i}, t_{3i+3}]$ *then the cubic approximation* $IC(f, h, \alpha)$ *of the I-operator is given by,*

$$(i)\ IC(f,h,\alpha) = \sum_{i=0}^{n-1} \left(D_{in} f(t_{3i}) + E_{in} f(t_{3i+1}) + F_{in} f(t_{3i+2}) + G_{in} f(t_{3i+3})\right), \quad (2.24)$$

where,

$$D_{in} = \frac{3^\alpha h^\alpha}{2\Gamma(\alpha+4)} \{2(n-i-1)^{(1+\alpha)} \left(\alpha^2 + \alpha(-4-9i+9n) + 3(1+3i-3n)(2+3i-3n)\right)$$

$$+ (n-i)^\alpha (2\alpha^3 + \alpha^2(12+11i-11n) + \alpha(22+36(i-n)^2 + 55(i-n))$$

$$+ 6(1+3i-3n)(2+3i-3n)(1+i-n))\}, \quad (2.25)$$

$$E_{in} = \frac{3^{\alpha+2} h^\alpha}{2\Gamma(\alpha+4)} \{2(n-i)^{(1+\alpha)} (\alpha^2 + 5\alpha(1+i-n)) + 3(2+3i-3n)(1+i-n)$$

$$- (n-i-1)^{\alpha+1} (\alpha^2 + \alpha(-3-8i+8n) + 6(2+3i-3n)(i-n))\}, \quad (2.26)$$

$$F_{in} = \frac{3^{\alpha+2} h^\alpha}{2\Gamma(\alpha+4)} \{2(n-i-1)^{(\alpha+1)} (\alpha^2 + 5\alpha(n-i)) + 3(1+3i-3n)(i-n)$$

$$- (n-i)^{1+\alpha} (\alpha^2 + \alpha(5+8i-8n) + +6(1+3i-3n)(1+i-n))\}, \quad (2.27)$$

$$G_{in} = \frac{3^\alpha h^\alpha}{2\Gamma(\alpha+4)} \{2(n-i)^{(1+\alpha)} (\alpha^2 + \alpha(5+9i-9n)) + 3(1+3i-3n)$$

$$- (n-i-1)^\alpha (2\alpha^3 + \alpha^2(1-11i+11n) + \alpha(3+36(i-n)^2 + 17(i-n))$$

$$- 6(1+3i-3n)(2+3i-3n)(i-n))\}, \quad (2.28)$$

(ii) and the approximation error $E_{IG}(f,h,\alpha)$ *has the form,*

$$|E_{IC}(f,h,\alpha)| \leq D_\alpha \|f''''\|_\infty (t_{3n})^\alpha h^4, \tag{2.29}$$

where D_α *is a constant depending on* α*.*

Proof: From Eq. (1.1), we have,

$$I^\alpha f(t_{3n}) = \sum_{i=0}^{n-1} \int_{t_{3i}}^{t_{3i+3}} (t_{3n}-\tau)^{\alpha-1} f(\tau) d\tau, \tag{2.30}$$

We approximate $f(\tau)$ over the interval $[t_{3i}, t_{3i+3}]$ using the cubic polynomial as,

$$\begin{aligned} f_{i,3} = & \left[-\frac{(\tau - t_{3i+1})(\tau - t_{3i+2})(\tau - t_{3i+3})}{6h^3} \right] f_{3i} \\ & + \left[\frac{(\tau - t_{3i})(\tau - t_{3i+2})(\tau - t_{3i+3})}{2h^3} \right] f_{3i+1} \\ & - \left[\frac{(\tau - t_{3i})(\tau - t_{3i+1})(\tau - t_{3i+3})}{2h^3} \right] f_{3i+2} \\ & + \left[\frac{(\tau - t_{3i})(\tau - t_{3i+1})(\tau - t_{3i+2})}{6h^3} \right] f_{3i+3}. \end{aligned} \tag{2.31}$$

Evaluating Eq.(2.30) using Eq.(2.31), the desired approximation of $IC(f,h,\alpha)$ as given in part (i) of the Lemma 2.2.1 is obtained. Proof of the second part of Lemma 2.2.3 is established here using Lemma 2.2.1.

From Eq.(2.30) and Eq.(2.31), we have,

$$|E_{IC}(f,h,\alpha)| = |I^\alpha f(t) - IC(f,h,\alpha)| = \left| I^\alpha f(t_{3n}) - \frac{1}{\Gamma(\alpha)} \int_0^{t_{3n}} (t_{3n}-\tau)^{\alpha-1} f(\tau) d\tau \right|,$$

$$= \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_{3n}} (t_{3n}-\tau)^{\alpha-1} f(\tau) d\tau - \sum_{i=0}^{n-1} \int_{t_{3i}}^{t_{3i+3}} (t_{3n}-\tau)^{\alpha-1} f_{i,3}(\tau) d\tau \right|,$$

$$= \frac{1}{\Gamma(\alpha)} \left| \sum_{i=0}^{n-1} \int_{t_{3i}}^{t_{3i+3}} (t_{3n} - \tau)^{\alpha-1} (f(\tau) - f_{i,3}(\tau)) d\tau \right| \tag{2.32}$$

Using Lemma 2.2.1 and Eq.(2.32), we have,

$$\leq \frac{1}{24\Gamma(\alpha)} \|f''''\|_\infty \left| \sum_{i=0}^{n-1} \int_{t_{3i}}^{t_{3i+3}} (t_{3n} - \tau)^{\alpha-1} (\tau - t_{3i})(\tau - t_{3i+1})(\tau - t_{3i+2})(\tau - t_{3i+3}) d\tau \right|,$$

$$\leq \frac{h^4}{24\Gamma(\alpha)} \|f''''\|_\infty \sum_{i=0}^{n-1} \int_{t_{3i}}^{t_{3i+3}} (t_{3n} - \tau)^{\alpha-1} \tau = D_\alpha \|f''''\|_\infty (t_{3n})^\alpha h^4, \tag{2.33}$$

where D_α is constant depending on α.

This completes the proof.

Lemma 2.2.4. *Suppose that $f \in C^{m+4}[0, \delta]$, and the interval $[0, \delta]$ is divided into subintervals $[t_{3i}, t_{3i+3}]$ such that $t_i = ih$ with $h = \frac{\delta}{3n}$, $i = 0, 1, 2, 3, \ldots, 3n$. Let $f_{i,3}$ is the cubic polynomial approximation for $f^{(m)}$ to the subintervals $[t_{3i}, t_{3i+3}]$ then the cubic approximation $DC(f, h, \alpha)$ of the D-operator is given by,*

$$(i)\ DC(f, h, \alpha) = \sum_{i=0}^{n-1} \left(\mathcal{D}_{in} f^{(m)}(t_{3i}) + \mathcal{E}_{in} f^{(m)}(t_{3i+1}) + \mathcal{F}_{in} f^{(m)}(t_{3i+2}) + \mathcal{G}_{in} f^{(m)}(t_{3i+3}) \right), \tag{2.34}$$

where,

$$\mathcal{D}_{in} = \frac{3^{(m-\alpha)} h^{(m-\alpha)}}{2\Gamma(m - \alpha + 4)} \{2(n - i - 1)^{(m-\alpha+1)}((m - \alpha)^2 + (m - \alpha)(-4 - 9i + 9i)$$

$$+ 3(1 + 3i - 3n)(2 + 3i - 3n)) + (n - i)^{m-\alpha}(2(m - \alpha)^3 + (m - \alpha)^2(12 + 11i - 11n)$$

$$+ (m - \alpha)(22 + 36(i - n)^2 + 55(i - n)) + 6(1 + 3i - 3n)(2 + 3i - 3n)(1 + i - n))\} \tag{2.35}$$

$$\mathcal{E}_{in} = \frac{3^{(m-\alpha+2)}h^{(m-\alpha)}}{2\Gamma(m-\alpha+4)}\{2(n-i)^{(m-\alpha+1)}((m-\alpha)^2 + 5(m-\alpha)(1+i-n)$$

$$+ 3(2+3i-3n)(1+i-n)) - (n-i-1)^{(m-\alpha+1)}((m-\alpha)^2$$

$$+ (m-\alpha)^2(12+11i-11n)$$

$$+ (m-\alpha)(-3-8i+8n) + 6(2+3i-3n)(i-n))\}, \tag{2.36}$$

$$\mathcal{F}_{in} = \frac{3^{(m-\alpha+2)}h^{(m-\alpha)}}{2\Gamma(m-\alpha+4)}\{2(n-i-1)^{(m-\alpha+1)}((m-\alpha)^2 + 5(m-\alpha)(n-i)$$

$$+ 3(1+3i-3n)(i-n)) - (n-i)^{(m-\alpha+1)}((m-\alpha)^2$$

$$+ (m-\alpha)(5+8i-8n) + 6(2+3i-3n)(1+i-n))\}, \tag{2.37}$$

$$\mathcal{G}_{in} = \frac{3^{(m-\alpha)}h^{(m-\alpha)}}{2\Gamma(m-\alpha+4)}\{2(n-i)^{(m-\alpha+1)}((m-\alpha)^2 + (m-\alpha)(5+9i-9m)$$

$$+ 3(1+3i-3n)(2+3i-3n)) - (n-i-1)^{(m-\alpha)}(2(m-\alpha)^3$$

$$+ (m-\alpha)^2(1-11i-11n) + (m-\alpha)(3+36(i-n)^2$$

$$+ 17(i-n)) - 6(1+3i-3n)(2+3i-3n)(i-n))\}, \tag{2.38}$$

(ii) and the approximation error $E_{DC}(f,h,\alpha)$ takes the form,

$$|E_{DC}(f,h,\alpha)| \leq D_\alpha^{'} \|f^{(m+4)}\|_\infty t_{3n}^{(m-\alpha)} h^4, \tag{2.39}$$

where $D_\alpha^{'}$ is a constant depending only on α.

Proof: The proof of the part (i) and part (ii) of the above lemma can be carried out using the similar steps and replacing α to $m-\alpha$ and $f(\tau)$ by $f^{(m)}(\tau)$ as described in the proof of Lemma 2.2.3.

2.3 Results and Discussions

Here, we consider the example as illustrated by Odibat [149] with $f(\tau) = \sin\tau$ in the I-operator for the comparison purpose. Lemmas 2.2.1, 2.2.2, 2.2.3 and 2.2.4 are applied for the approximation of the I and D-operators for different values of the fractional order α and numerical results are obtained. The numerical results using Quadratic and Cubic approximation schemes for the I-operator is calculated for different values of the step size and fractional order α , and are placed in the Tables 2.1-2.3. For the comparison purpose, the similar values of the parameters such as fractional order α and step size are chosen as presented in [149]. It is clear from the Tables 2.1-2.3, that the scheme $A1$ works well and achieves the better accuracy compare to the linear scheme presented in [149]. From, Tables 2.1-2.3, it can be seen that the errors are getting reduced as we increase the number of subintervals. The convergence order of the scheme $A1$ for the results discussed in Tables 2.1-2.3 are presented in Tables 2.4-2.6 respectively. From Tables 2.4-2.6, it can be seen that the scheme $A1$ achieves the convergence order more than 3. Further, we observe that the scheme $A2$ works well and achieves the better accuracy compared to the scheme [149] and the scheme $A1$. The results of scheme $A2$ are presented in Tables 2.7-2.9. Table 2.10 represents the convergence order of the scheme $A2$ for a particular case considered in Table 2.8. Schemes $A1$ and $A2$ are also applied to approximate the Caputo derivative (D-operator). We consider the test function $f(\tau) = \sin\tau$, the fractional order $\alpha = 0.5$ and vary the step size to generate the numerical results. Numerical results using schemes $A1$ and $A2$ for approximations of D-operators are showed in Table 2.11 and table 2.12 respectively. In the tables, MAE denotes the maximum absolute error and the convergence order (CO) is calculated as: Convergence order =lg[MAE(h)/MAE(h/2)]/lg(2).

TABLE 2.1: Numerical results obtained using scheme $A1$ for I-operator, $I^{\alpha}f(t)(1)$ for $f(t) = \sin t$ and $\alpha = 0.5$.

n	h	$IQ(f,h,0.5)$	$E_{IQ}(f,h,0.5)$	$E_{IL}(f,h,0.5)$[149]
10	0.05	0.6696838267942012	4.32783×10^{-7}	1.30405×10^{-4}
20	0.025	0.6696842212539105	3.83238×10^{-8}	3.32769×10^{-5}
40	0.0125	0.6696842561832945	3.39437×10^{-9}	8.4373×10^{-6}
80	0.00625	0.6696842592769013	3.00762×10^{-10}	2.1301×10^{-6}

TABLE 2.2: Numerical results obtained using scheme $A1$ for I-operator, $I^{\alpha}f(t)(1)$ for $f(t) = \sin t$ and $\alpha = 1$.

n	h	$IQ(f,h,1)$	$E_{IQ}(f,h,1)$	$E_{IL}(f,h,1)$ [149]
10	0.05	0.4596977100983376	1.59665×10^{-8}	9.57743×10^{-5}
20	0.025	0.4596976951295424	9.97682×10^{-10}	2.39428×10^{-5}
40	0.0125	0.4596976941942119	6.23516×10^{-11}	5.9856×10^{-6}
80	0.00625	0.4596976941357571	3.89683×10^{-12}	1.4964×10^{-6}

TABLE 2.3: Numerical results obtained using scheme $A1$ for I-operator, $I^{\alpha}f(t)(1)$ for $f(t) = \sin t$ and $\alpha = 1.5$.

n	h	$IQ(f,h,1.5)$	$E_{IQ}(f,h,1.5)$	$E_{IL}(f,h,1.5)$ [149]
10	0.05	0.2823225014367666	1.21065×10^{-7}	5.89010×10^{-5}
20	0.025	0.2823223880461549	7.67480×10^{-9}	1.47111×10^{-5}
40	0.0125	0.2823223808560551	4.84695×10^{-10}	3.6767×10^{-6}
80	0.00625	0.2823223804019575	3.05977×10^{-11}	9.191×10^{-7}

TABLE 2.4: Convergence order using scheme $A1$ for I-operator, $I^{\alpha}f(t)(1)$ for $f(t) = \sin t$ and $\alpha = 0.5$.

$h = \frac{1}{2n}$	MAE ($A1$)	CO
1/10	4.89232×10^{-6}	
1/20	4.32783×10^{-7}	3.49733
1/40	3.83238×10^{-8}	3.49733
1/80	3.39437×10^{-9}	3.49702
1/160	3.00762×10^{-10}	3.49645

TABLE 2.5: Convergence order using scheme $A1$ for I-operator, $I^{\alpha}f(t)(1)$ for $f(t) = \sin t$ and $\alpha = 1$.

$h = \frac{1}{2n}$	MAE ($A1$)	CO
1/10	2.55692×10^{-7}	
1/20	1.59665×10^{-8}	4.00129
1/40	9.97682×10^{-10}	4.00032
1/80	6.23516×10^{-11}	4.00008
1/160	3.89683×10^{-12}	4.00005

It is noticed that exact value of the fractional integral $I^{\alpha}\sin t$ is calculated using the formula stated in Odibat [149] as, $I^{\alpha}\sin t = t^{\alpha}\sum_{i=0}^{\infty}\frac{(-1)^i t^{2i+1}}{\Gamma(\alpha+2i+2)}, \quad t > 0$, and value at $t = 1$, is used to compute the error.

Table 2.6: Convergence order using scheme $A1$ for I-operator, $I^{\alpha} f(t)(1)$ for $f(t) = \sin t$ and $\alpha = 1.5$.

$h = \frac{1}{2n}$	MAE ($A1$)	CO
1/10	1.90186×10^{-6}	
1/20	1.21065×10^{-7}	3.97356
1/40	7.67480×10^{-9}	3.97951
1/80	4.84695×10^{-10}	3.98498
1/160	3.05977×10^{-11}	3.98558

Table 2.7: Numerical results obtained using scheme $A2$ for I-operator, $I^{\alpha} f(t)(1)$ for $f(t) = \sin t$ and $\alpha = 0.5$.

n	h	$IC(f, h, 0.5)$	$E_{IC}(f, h, 0.5)$	$E_{IL}(f, h, 0.5)$ [149]
10	1/30	0.6696842705520611	1.09744×10^{-8}	1.30405×10^{-4}
20	1/60	0.6696842602530784	6.75415×10^{-10}	3.32769×10^{-5}
40	1/120	0.6696842596262024	4.85388×10^{-11}	8.4373×10^{-6}
80	1/240	0.6696842596896193	1.11956×10^{-10}	2.1301×10^{-6}

Table 2.8: Numerical results obtained using scheme $A2$ for I-operator, $I^{\alpha} f(t)(1)$ for $f(t) = \sin t$ and $\alpha = 1$.

n	h	$IC(f, h, 1)$	$E_{IC}(f, h, 1)$	$E_{IL}(f, h, 1)$ [149]
10	1/30	0.4596977012278377	7.09598×10^{-9}	9.57743×10^{-5}
20	1/60	0.4596976945752709	4.43411×10^{-10}	2.39428×10^{-5}
40	1/120	0.4596976941318603	6.23516×10^{-11}	5.9856×10^{-6}
80	1/240	0.4	3.89683×10^{-12}	1.4964×10^{-6}

TABLE 2.9: Numerical results obtained using scheme $A2$ for I-operator, $I^{\alpha} f(t)(1)$ for $f(t) = \sin t$ and $\alpha = 1.5$.

n	h	$IC(f, h, 1.5)$	$E_{IC}(f, h, 1.5)$	$E_{IL}(f, h, 1.5)$ [149]
10	1/30	0.2823223847105428	4.33918×10^{-9}	5.89010×10^{-5}
20	1/60	0.2823223806431222	2.71762×10^{-10}	1.47111×10^{-5}
40	1/120	0.282322380395719	2.43592×10^{-11}	3.6767×10^{-6}
80	1/240	0.2823223804266486	5.52888×10^{-11}	9.191×10^{-7}

TABLE 2.10: Convergence order using scheme $A2$ for I-operator, $I^{\alpha} f(t)(1)$ for $f(t) = \sin t$ and $\alpha = 1$.

$h = \frac{1}{3n}$	MAE ($A2$)	CO
1/15	1.13626×10^{-7}	
1/30	7.09598×10^{-9}	4.00115
1/60	4.43411×10^{-10}	4.00029
1/120	2.77117×10^{-11}	4.00008
1/240	1.73189×10^{-12}	4.00008

TABLE 2.11: Numerical results obtained using scheme $A1$ for the D-operator, $D^{\alpha} f(\tau)(1)$ for $f(t) = \sin t$ and $\alpha = 0.5$.

k	h	$DQ(f, h, 0.5)$	$E_{DQ}(f, h, 0.5)$	$E_{DL}(f, h, 0.5)$ [149]
10	0.05	0.846057377964953	5.91241×10^{-7}	1.706097×10^{-4}
20	0.025	0.84605684138235	5.46582×10^{-8}	4.30544×10^{-5}
40	0.0125	0.846056791702752	4.9786×10^{-9}	1.08365×10^{-5}
80	0.00625	0.846056787174921	4.50768×10^{-10}	2.7222×10^{-6}

Table 2.12: Numerical results obtained using scheme $A2$ for the D-operator, $D^{\alpha}f(\tau)(1)$ for $f(t) = \sin t$ and $\alpha = 0.5$.

k	h	$DC(f, h, 0.5)$	$E_{DC}(f, h, 0.5)$	$E_{DL}(f, h, 0.5)$ [149]
10	1/30	0.846056800339386	1.36152×10^{-8}	1.706097×10^{-4}
20	1/60	0.846056787554831	8.30678×10^{-10}	4.30544×10^{-5}
40	1/120	0.846056786769922	4.57691×10^{-11}	1.08365×10^{-5}
80	1/240	0.846056786169989	5.54164×10^{-10}	2.7222×10^{-6}

2.4 Application: Solving Abel's Integral Equation

To establish the application of the Quadratic and Cubic schemes for the D-operator as discussed in Section 2.2, we go through Abel integral equation of the first kind,

$$f(t) = \int_0^t \frac{g(\tau)}{(t-\tau)^{\alpha}} d\tau, \qquad 0 < \alpha < 1, \quad 0 \leq t \leq \delta, \tag{2.40}$$

where, $f \in C^1[a, b]$ is given function satisfying $f(0) = 0$ and $g(\tau)$ is unknown function. The solution to Eq.(2.40) can be obtained as,

$$g(t) = \frac{sin(\alpha t)}{\pi} \int_0^t \frac{f'(\tau)}{(t-\tau)^{1-\alpha}} d\tau. \tag{2.41}$$

The solution given by Eq.(2.41) can also be presented in terms of the I and D-operators, using definition (Eq.(1.3)) as follows,

$$f(t) = \Gamma(1-\alpha) I^{1-\alpha} g(t). \tag{2.42}$$

Using the property D-operator is left inverse of I-operator and simplifying Eq.(2.42), it follows that,

$$g(t) = \frac{1}{\Gamma(1-\alpha)} D^{1-\alpha} f(t). \tag{2.43}$$

Now, we apply Lemma 2.2.2 and Lemma 2.2.4 to Eq.(2.43) to get the approximate solution of the Abel's integral equation given by Eq.(2.40).

Lemma 2.4.1. *Let $0 < t < \delta$ and suppose that the interval $[0, \delta]$ is subdivided into n sub intervals $[t_{2i}, t_{2i+2}]$, $i = 1, 2, 3, \ldots, n-1$ of length $h = \frac{\delta}{2n}$ by using the nodes $t_i = ih,\ i = 0, 1, \ldots, 2n$. Then the approximate solution $g(t)$ to the solution $g(t)$ of the Abel integral equation given by Eq.(47) can be expressed using scheme A1 as,*

$$\tilde{g}(t) = \sum_{i=0}^{n-1} \left(\mathbb{A}_{in} f'(t_{2i}) + \mathbb{B}_{in} f'(t_{2i+1}) + \mathbb{C}_{in} f'(t_{2i+2}) \right), \tag{2.44}$$

where,

$$\mathbb{A}_{in} = \frac{2^{\alpha} h^{\alpha}}{\Gamma(\alpha+3)\Gamma(1-\alpha)} \{(n-i-1)^{(\alpha+1)}(2-\alpha+4i-4n)$$

$$+ (n-i)^{\alpha}(2+\alpha^2+4i^2+i(6-8n)+3\alpha(1+i-n)-6n+4n^2\}, \tag{2.45}$$

$$\mathbb{B}_{in} = \frac{2^{(\alpha+2)} h^{\alpha}}{\Gamma(\alpha+3)\Gamma(1-\alpha)} \{(n-i-1)^{(\alpha+1)}(\alpha-2i+2n) + (n-i)^{(\alpha+1)}(2+\alpha+2i-2n)\}, \tag{2.46}$$

$$\mathbb{C}_{in} = \frac{2^{\alpha} h^{\alpha}}{\Gamma(\alpha+3)\Gamma(1-\alpha)} \{(n-i)^{(\alpha+1)}(2+\alpha+4i-4n)$$

$$+ (n-i-1)^{\alpha}(\alpha^2+2i-3\alpha i+4i^2-2n+3\alpha n-8in+4n^2)\} \tag{2.47}$$

Moreover, if $f \in C^4[0,t]$*, then* $g(t) = \tilde{g}(t) - \frac{1}{\Gamma(1-\alpha)}E(t)$ *with*

$$|E(t)| \leq S_\alpha \|f''''\|_\infty t^\alpha h^3, \tag{2.48}$$

where S_α *is the constant depending only on* α *and* $\|f''''\|_\infty = max_{x\in[0,t]}|f''''(x)|$.

Proof: The solution of the Abel's integral equation (Eq.(2.40)) represented by Eq.(2.43) in the form of D-operator can be express as,

$$g(t) = \frac{sin(\alpha\pi)\Gamma(\alpha)}{\pi} D^{1-\alpha} f(t). \tag{2.49}$$

The results can be obtained using Lemma 2.2.2 to Eq.(2.49) with some simple calculation. To validate the proposed approximation, an illustrative example from [50] is considered and the approximate solution is obtained.

Lemma 2.4.2. *Let* $0 < t < \delta$ *and suppose that the interval* $[0,\delta]$ *is subdivided into* n *sub intervals* $[t_{3i}, t_{3i+3}]$*,* $i = 0,1,2,3,\dots,n-1$ *of length* $h = \frac{\delta}{3n}$ *by using the nodes* $t_i = ih, \quad i = 0,\dots,2n$*. Then the approximate solution* $g(t)$ *to the solution* $g(t)$ *of the Abel integral equation given by Eq.(2.43) can be expressed using scheme A2 as,*

$$\tilde{g}(t) = \sum_{i=0}^{n-1} \left(\mathbb{D}_{in} f'(t_{3i}) + \mathbb{E}_{in} f'(t_{3i+1}) + \mathbb{F}_{in} f'(t_{3i}+2) + \mathbb{G}_{in} f'(t_{3i+3})\right), \tag{2.50}$$

where,

$$\begin{aligned}\mathbb{D}_{in} = \frac{3^\alpha h^\alpha}{2\Gamma(1-\alpha)\Gamma(\alpha+4)}\{&2(n-i-1)^{(1+\alpha)}(\alpha^2+\alpha(-4-9i+9n)\\ &+3(1+3i-3n)(2+3i-3n)) + (n-i)^\alpha(2\alpha^3+\alpha^2(12+11i-11n)\\ &+\alpha(22+36(i-n)^2+55(i-n)) + 6(1+3i-3n)(2+3i-3n)(1+i-n))\}\end{aligned} \tag{2.51}$$

$$\mathbb{E}_{in} = \frac{3^{\alpha+2}h^{\alpha}}{2\Gamma(1-\alpha)\Gamma(\alpha+4)}\{2(n-i)^{(1+\alpha)}(\alpha^2+5\alpha(1+i-n)$$

$$+3(2+3i-3n)(1+i-n)) - (n-i-1)^{(\alpha+1)}(\alpha^2+\alpha(-3-8i+8n)$$

$$+6(2+3i-3n)(i-n))\}, \tag{2.52}$$

$$\mathbb{F}_{in} = \frac{3^{\alpha+2}h^{\alpha}}{2\Gamma(1-\alpha)\Gamma(\alpha+4)}\{2(n-i-1)^{(\alpha+1)}(\alpha^2+5\alpha(n-i)$$

$$+3(1+3i-3n)(i-n)) - (n-i)^{(1+\alpha)}(\alpha^2+\alpha(5+8i-8n)$$

$$+6(1+3i-3n)(1+i-n))\}, \tag{2.53}$$

$$\mathbb{G}_{in} = \frac{3^{\alpha+2}h^{\alpha}}{2\Gamma(1-\alpha)\Gamma(\alpha+4)}\{2(n-i)^{(1+\alpha)}(\alpha^2+\alpha(5+9i-9n)$$

$$+3(1+3i-3n)(2+3i-3n)) - (n-i-1)^{\alpha}(2\alpha^3+\alpha^2(1-11i+11n)$$

$$\alpha(3+36(i-n)^2+17(i-n)) - 6(1+3i-3n)(2+3i-3n)(i-n))\}, \tag{2.54}$$

$$|E(t)| \leq S_{\alpha}\|f^{(5)}\|_{\infty}t^{\alpha}h^4, \tag{2.55}$$

where T_{α} is the constant depending only on α and $\|f'''''\|_{\infty} = max_{x\in[0,t]}|f'''''(x)|$.

Proof: The proof can be acquired using some simple calculations to Eq. (2.43) together with the scheme $A2$ as discussed in Lemma 2.2.4. The results can be obtained using Lemma 2.2.4 to Eq.(2.49) with some simple calculation. To validate the proposed approximation, an illustrative example from [50] is considered and numerical results are presented.

Example 2.4.1. *Consider the Abel's integral equation [50], $e^t - 1 = \int_0^t \frac{g(\tau)}{(t-\tau)^{1/2}}d\tau$. The exact solution for this problem is given by, $g(t) = \frac{e^t}{\sqrt{\pi}}erf(\sqrt{t})$, where $erf(x)$ is error fnction, that is, $erf(t) = \frac{2}{\sqrt{\pi}}\int_0^t e^{-\tau^2}d\tau$.*

Example 2.4.1 is solved using the Lemma 2.4.1 and Lemma 2.4.2 and the obtained approximate results are presented in Tables 2.13-2.14 respectively. For solving this problem, the number of subintervals is considered as 10 and 100 and in each case the errors are obtained. From the Tables 2.13-2.14, it is clear that the error obtained by the proposed scheme is comparatively better even with the less number of subintervals than the method presented in [50].

TABLE 2.13: Comparison of the exact solution, approximate solution using Lemma 2.4.1 and respective errors for $n = 10, 100$.

t_i	Exact solution	Approx. sol.$n = 10$	Error $n = 10$	Error $n = 100$	Error [50] for $n = 100$
0.1	0.2152905021493694	0.2152905022928531	1.434×10^{-10}	1.72×10^{-10}	3.75×10^{-8}
0.2	0.3258840763232928	0.3258840781067156	1.783×10^{-9}	1.99×10^{-12}	2.61×10^{-8}
0.3	0.427565657562311	0.4275656656608028	8.098×10^{-9}	5.18×10^{-13}	2.14×10^{-7}

TABLE 2.14: Comparison of the exact solution, approximate solution using Lemma 2.4.2 and respective errors for $n = 10, 100$.

t_i	Exact solution	Approx. sol. $n = 10$	Error $n = 10$	Error $n = 100$	Error [50] for $n = 100$
0.1	0.2152905021493694	0.2152905021496013	2.31787×10^{-13}	4.34356×10^{-15}	3.75×10^{-8}
0.2	0.3258840763232928	0.325884076331575	8.28221×10^{-12}	6.19821×10^{-10}	2.61×10^{-8}
0.3	0.427565657562311	0.4275656576182749	5.5964×10^{-11}	7.65846×10^{-10}	2.14×10^{-7}

Example 2.4.2. *Consider the following Abel integral equation [50], such that,* $t = \int_0^t \frac{g(\tau)}{(t-\tau)^{4/5}} d\tau$, *having the exact solution,* $g(t) = \frac{5}{4}\frac{\sin(\frac{\pi}{5})}{\pi} t^{4/5}$.

Lemma 2.4.1 and Lemma 2.4.2 are applied to solve the considered integral equation and the obtained numerical results are presented in Tables 2.15-2.16 respectively. The numerical results are obtained using the values of $n = 5, 10$ and the results are presented. Absolute errors for each values of the subinterval n are also presented. Numerical results show that the presented schemes works well and produce the approximate solution to high accuracy.

TABLE 2.15: Comparison of the exact solution, approximate solution using Lemma 2.4.1 and respective errors for $n = 5, 10$.

t_i	Exact solution	Approx. sol. $n = 5$	Error $n = 5$	Error $n = 10$	Error [50] for $n = 10$
0.4	0.1123639036486324	0.1123639036486326	2.77556×10^{-16}	2.58127×10^{-15}	1×10^{-10}
0.5	0.1343243751756705	0.1343243751756709	3.60822×10^{-16}	3.13638×10^{-15}	1×10^{-10}
0.6	0.1554174667790617	0.155417466779062	3.33067×10^{-16}	3.60822×10^{-15}	$\leq 10^{-11}$

TABLE 2.16: Comparison of the exact solution, approximate solution using Lemma 2.4.2 and respective errors for $n = 5, 10$.

t_i	Exact solution	Approx. sol. $n = 5$	Error $n = 5$	Error $n = 10$	Error [50] for $n = 10$
0.4	0.1123639036486324	0.1123639036486273	5.06539×10^{-15}	2.35562×10^{-13}	1×10^{-10}
0.5	0.1343243751756705	0.1343243751756645	5.9952×10^{-15}	2.03615×10^{-13}	1×10^{-10}
0.6	0.1554174667790617	0.1554174667790548	6.93889×10^{-15}	1.7035×10^{-13}	$\leq 10^{-11}$

2.5 Conclusions

We studied two approximation schemes namely Quadratic and Cubic schemes for Riemann-Liouville and Caputo derivatives. The error convergences for the presented schemes are obtained. The presented schemes are successfully validated on test cases. It is clear that the presented schemes show the advantages over the scheme discussed in [149]. Further, the presented schemes are applied to solve Abel's integral equation. The numerical results obtained by the presented schemes are appreciable as compare to the schemes presented in [50]. The schemes presented in the chapter could be considered as the higher order approximation methods for the approximations of the fractional integrals and fractional derivatives.

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