# Chapter 1

# Introduction

## 1.1 Motivation

Applications of fractional derivatives and integrals for modelling the memory and hereditary properties of different materials supervised by anomalous diffusion have been pointed out by many researchers. Such effects are neglected in classical integer-order derivatives. So far, there are a large number of research works devoted to integer order differential and integral equation but there are only a few researches available for non –integer order differential and integral equations. Approximation of functions using linear, quadratic and cubic interpolating polynomials provides powerful techniques to the Riemann-Liouville fractional integral (RLFI), Caputo fractional derivatives (CFD), generalized fractional derivatives, and generalized time-fractional derivatives. The basic idea behind the linear, quadratic and cubic interpolation polynomial approximation is to discretize the computational domain into smaller sub-domains and to approximate function by the linear, quadratic and cubic interpolation polynomials. The main objective of this thesis is to analyse an approximate solution of integral and fractional integro-differential equations (FIDEs) based on linear, quadratic and cubic interpolation polynomials and to estimate the convergence as well as an error of approximations for all proposed problems.

The thesis is decomposed into six chapters. In chapter 2, we have presented two methods namely quadratic and cubic approximation for RLFI and CFD. In chapter 3, we have studied two numerical schemes for the generalized Abel's integral equations (GAIEs) based on linear and quadratic interpolation polynomial. A comparative study of three numerical scheme namely Linear, Quadratic and Quadratic-Linear scheme has been presented for the fractional integro-differential equation in chapter 4. In chapter 5, we have extended application of Linear and Quadratic interpolation polynomial for solving Generalized Fractional Integro-Differential Equations. A numerical scheme has been formulated and analyzed for solving a Generalized Time-Fractional Telegraph Type Equation defined in terms of Generalized Time Fractional Derivative in chapter 6.

## 1.2 Basic Definitions and Generalized Fractional Calculus

In this section, we present some basic definitions of fractional integral, fractional derivative and generalized fractional derivative (GFD) and their properties.

**Definition 1.2.1.** *The Riemann-Liouville fractional integral of a function $u$ is defined as*

$$I^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}u(\tau)d\tau, \quad t>0, \tag{1.1}$$

*where $\alpha \in \mathbb{R}^+$ is the order of fractional integral , $u \in L^1[a,b]$.*

**Definition 1.2.2.** *The fractional derivative known as Riemann-Liouville fractional derivative of order* $\alpha > 0$ *is defined as,*

$$(I_\alpha f)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau, \qquad n-1 < \alpha \leq n, \tag{1.2}$$

*where* $n$ *is an integer. Another definition of fractional derivative introduced by Caputo, is defined as below.*

**Definition 1.2.3.** *The definition of Caputo fractional derivatives of function* $u$ *as*

$$D^\alpha u(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} u^{(m)}(\tau) d\tau, \quad t > 0, \tag{1.3}$$

*where* $m-1 \leq \alpha < m$ *is the order of fractional derivatives,* $m \in \mathbb{N}^+$, $u^{(m)} \in L^1[a,b]$.

**Definition 1.2.4.** *The left/ forward generalized fractional integral of order* $\alpha \in \mathbb{R}^+$ *of function* $u(t)$*, with respect to a scale function* $z(t)$ *and weight function* $w(t)$*, is defined as,*

$$I^\alpha_{0+;[z;w]} u(t) = \frac{[w(t)]^{-1}}{\Gamma(\alpha)} \int_0^t \frac{w(\tau) z'(\tau) u(\tau)}{[z(t) - z(\tau)]^{1-\alpha}} d\tau, \tag{1.4}$$

*provided the integral exists.*

**Definition 1.2.5.** *Left/ forward generalized derivatives of order 1 of a function* $u(t)$ *with respect to a scale function* $z(t)$ *and weight function* $w(t)$*, is defined as*

$$D_{[z;w;L]} u(t) = [w(t)]^{-1} \left[ \left( \frac{1}{z'(t)} D_t \right) (w(t) u(t)) \right], \tag{1.5}$$

*provided the right-side of the equation is finite, where* $D_t$ *is the classical first-order derivatives with respect to* $t$.

**Definition 1.2.6.** *Left/ forward generalized derivatives of order $m$ of a function $u(t)$ with respect to a scale function $z(t)$ and weight function $w(t)$, is defined as*

$$D^m_{[z;w;L]}u(t) = [w(t)]^{-1}\left[\left(\frac{1}{z'(t)}D_t\right)^m (w(t)u(t))\right], \tag{1.6}$$

*provided the right-side of the equation is finite, where $m \in \mathbb{N}^+$ and $D_t$ is the classical first-order derivatives with respect to $t$.*

**Definition 1.2.7.** *Left/ forward generalized fractional derivatives of order $\alpha \in \mathbb{R}^+$ and type 1 of a function $u(t)$ with respect to a scale function $z(t)$ and weight function $w(t)$, is defined as,*

$$\left(D^\alpha_{0+;[z,w,1]}u\right)(t) = D^m_{[z,w,L]}\left(I^{m-\alpha}_{0+;[z;w]}u\right)(t), \tag{1.7}$$

**Definition 1.2.8.** *Left/ forward generalized fractional derivatives of order $\alpha \in \mathbb{R}^+$ and type 2 of a function $u(t)$ with respect to a scale function $z(t)$ and weight function $w(t)$, is defined as*

$$\left(D^\alpha_{0+;[z;w,2]}u\right)(t) = I^{m-\alpha}_{0+;[z;w]}\left(D^m_{[z,w,L]}u\right)(t), \tag{1.8}$$

*provided the right-side of the Eq.(1.8) is finite, where $m-1 \leq \alpha < m$, and $m \in \mathbb{N}^+$. In the above definition, fractional derivatives of type 2 is also generalized from the Caputo fractional derivatives. We will call them generalized Caputo type fractional derivatives. In all the above definitions, we assume that the weight function $w(t)$ and scale function $z(t)$ are sufficiently smooth such that the integrals and derivatives in definitions are finite. It is to notice that in above definitions, we are only given for left/forward sense of generalized fractional integral and GFD. They can be defined in the right/backward sense. We will not repeat them here, since in this thesis we considered left Caputo type GFD because Caputo derivative appears most frequently* *in depicting real-world models. For $m = 1$, i.e. $0 \leq \alpha < 1$, the new GFD in Eq. (1.8) will be given as,*

$$D_*^\alpha u(t) = \left(D^\alpha_{0+;[z;w;2]}u\right)(t) = \frac{[w(t)]^{-1}}{\Gamma(1-\alpha)} \int_0^t \frac{[w(\tau)u(\tau)]'}{(z(t)-z(\tau))^\alpha} d\tau. \tag{1.9}$$

## 1.3 Review on Fractional Calculus

Fractional calculus is a branch of mathematics which investigates the properties of derivatives and integrals of non-integer order (called fractional derivatives and integrals). Additionally, the theory of fractional calculus includes even complex orders of integro-differential equation and left and right integro-differential equation (analogously to left and right derivatives). The theory of fractional derivative goes back to the Leibniz's note in his list to L'Hospital in 1695, in which the meaning of the half derivatives and integrals of the arbitrary order more or less finished by Liouville, Grunwald, Riemann and Letnikov. In the last few decades, many authors remarked that derivatives and integrals of arbitrary order are very appropriate for the description of various real material, e.g. polymers. It has been proved that new fractional order models are more adequate than integer order models. Laplace, Fourier, Abel, Liouville, Riemann, Grunwald, Letnikov, Levy, Marchaud, Erdelyi, and Riesz have provided important contributions in the field of fractional calculus up to the middle of last century. Initially, the concept of fractional calculus was considered and developed mainly as a purely theoretical field of mathematics, and it has been the subject of specialized conferences and treatises only.

Ross organized the first conference on fractional calculus and its applications at the University of New Haven in June 1974. After a joint collaboration started in 1968, Oldham and Spanier [1] published a book for fractional calculus in 1974. In 1999, Podlubny [2] published a book providing the basic theory of fractional derivatives, fractional-order differential equations, methods of their solution and its applications in the diverse fields of science and engineering. Volumes edited by Carpinteri and Mainardi [3] in 1997 and Hilfer [4] in 2000, the book by Kilbas, Srivastava and Trujillo [5] in 2006, and the book by Sabatier, Agrawal, Tenreiro Machado [6] in 2007 are some of the latest works in the area of fractional models of anomalous kinetics in complex processes. The remarkably comprehensive encyclopedic-type monograph by Samko, Kilbas and Marichev [7] and the book devoted substantially to fractional differential equations by Miller and Ross [8], which was published in 1993 are some remarkable works in this field.

The traditional partial differential equations may not be adequate for describing the transport phenomena in complex systems (such as random fractal structures), exhibiting many anomalous features that are qualitatively different from the standard characteristics of regular systems [9]. Various phenomena strongly connected with the interactions within complex and non-homogeneous backgrounds is known as Anomalous diffusion, which can be observed in transport of fluid in porous media, amorphous semiconductors and also in two-dimensional rotating flows. Particularly, in the case of fractals, the spatial complexities of the substrate are the main cause of such anomalies, which imposes geometrical constraints on the transport process on every length scale. In recent years, rigorous analytical as well as numerical work has been made to make some breakthrough interpretations about the unusual transport properties of fractal structures [10]. Special emphasis has been given to understand diffusion phenomena on such spatially correlated media. The laws of Markov diffusion might be altered due to various non-homogeneities of the medium. Particularly, the probability density of the concentration field may have a heavier end than the Gaussian density, resulting in long-range dependence [11]. A key approach to anomalous diffusion is that of a continuous time random walk (CTRW) in which the random motion is performed on a regular lattice, but the length of a jump and the waiting time between two successive jumps are assumed to be random and drawn from a probability density function. Diverse suppositions on this probability density function lead to a variety of fractional differential equations (FDE) such as the fractional heat equation [12], the fractional advection-dispersion equation [13], the fractional kinetic equation [14], and the fractional Fokker-Planck equation (FFPE) [15]. Recently, significant interest in fractional differential equations has been encouraged due to its applications in the numerical analysis and the different areas of physical and chemical processes and engineering, including fractal phenomena [11]. Generalized diffusion equation containing derivatives of fractional order in space, or time, or space-time is the basis for physical-mathematical approach to anomalous diffusion [11]. The rigorous theoretical analysis of fractional differential equations and the development and implementation of efficient and accurate numerical methods are very difficult tasks, in particular for the cases of high dimensions. Therefore, development of new numerical methods and analysis techniques, is needed. The occurrence of fractional derivatives is by no means new. In fact, they are almost as old as integer-order counterparts [1, 2, 7, 8]. Fractional derivatives have been recently applied in the area of system biology [16], medicine [17, 18, 19, 20], physics [11, 14, 21, 22, 23], chemistry and biochemistry [24], hydrology [13, 15, 25], and finance [26, 27, 28].

## 1.4 Integral Equations

Integral equation is an important subject of applied mathematics. Integral equations are used to represent mathematical models for many physical systems, and also occur in formulation of many mathematical problems. We begin with a brief classification of integral equations. We present some of the classical theory for one of the most popular types of integral equations which are called as Fredholm integral equation of second kind. An integral equation is an equation in which an unknown function appears under one or more integral sign.

For Example, for $(x, t) \in [a, b] \times [a, b]$ the equations

$$\int_a^b K(x,t)y(t)dt = f(x), \tag{1.10}$$

$$y(x) - \lambda \int_a^b K(x,t)y(t)dt = f(x), \tag{1.11}$$

$$y(x) = \int_a^b K(x,t)\,[y(t)]^2\,dt = f(x), \tag{1.12}$$

where the function $y(x)$, is unknown function while the function $f(x)$ and $K(x,t)$ are known functions and $\lambda, a$ and $b$ are the constants.

### 1.4.1 Volterra and Fredholm Integral Equations

In 1896, Vito Volterra published nine papers, six of which deal with the solvability of certain functional equations which are now referred to as Volterra integral equations of the first kind. He was 36 years old and had held the chair of rational mechanics at the University of Torino since 1893. In mathematics, the Volterra integral equations are a special type of integral equations. They are divided into two groups referred to as the first and the second kind. A linear Volterra equation of the first kind is

$$f(x) = \int_a^x K(x,t)y(t)dt, \qquad x \in [a, a+A], \tag{1.13}$$

where, $f$ is a given function and $y(t)$ is an unknown function to be solved. A linear Volterra equation of the second kind is

$$y(x) = f(x) + \int_a^x K(x,t)y(t)dt. \tag{1.14}$$

In integral equation the upper limit may be either variable $x$ or fixed. In case of fixed limit, it is called Fredholm otherwise Volterra integral equation.

### 1.4.2 Singular Integral Equations

An integral equation in which the range of integration is infinite, or in which the kernel is discontinuous, is known as a singular integral equation. One of the such example is Abel's integral equation. Abel's integral equation occurs in many branches of scientific fields such as microscopy, seismology, radio astronomy, electron emission, atomic scattering, radar ranging, plasma diagnostics, X-ray radiography, and optical fiber evaluation. Abel's equation is one of the integral equations derived directly from a concrete problem of physics, without passing through a differential equation. The great mathematician Niels Abel, gave the initiative of integral equations in 1823 in his study of mathematical physics. Abel's integral equation can be defined as a singular Volterra integral equation.

### 1.4.3 Abel's Integral Equations

General Abel's integral equation is a type of first and second kind Volterra integral equation and defined by

$$\textit{First kind:} \qquad \int_a^x \frac{\psi(t)}{(x-t)^\mu}dt = f(x), \tag{1.15}$$

$$\textit{Second kind:} \qquad \int_a^x \frac{\psi(t)}{(x-t)^\mu}dt + f(x) = \psi(x), \tag{1.16}$$

where, $0 < \mu < 1$, $\psi$ is unknown function and $f$ is known function.

### 1.4.4 Generalized Abel's Integral Equations

The Generalized Abel's Integral Equation is defined by ,

$$\int_a^x \frac{\psi(t)}{(x-t)^\mu}dt + \int_x^b \frac{\psi(t)}{(t-x)^\mu}dt = \xi(x), \tag{1.17}$$

Or,

$$\int_a^x \frac{\psi(t)}{(x^\alpha-t^\alpha)^\mu}dt + \int_x^b \frac{\psi(t)}{(t^\alpha-x^\alpha)^\mu}dt = \xi(x), \tag{1.18}$$

where, $0 < \mu < 1$, $\psi$ is unknown function, $\xi$ is known function and $\alpha \in \mathbb{N}$. Above Eq.(1.17) have been studied in the chapter three .

## 1.5 Literature Review on Abel's and Generalized Abel's Integral Equations

Abel's equations are identified with a widespread variety of physical real world problems like heat transfer [29], the propagation of nonlinear waves [30], nonlinear diffusion [31] and applications in the area of neutron transport and traffic theory. Abel inversion is broadly utilized by various researchers in the field of plasma physics to get the electronic density from phase-shift maps got by laser interferometry [32] or radial emission patterns from observed plasma radiances [33, 34]. Photoion and photoelectron imaging in molecular dynamics [35], assessment of mass density and velocity laws of stellar winds in astrophysics [36, 37], and atmospheric radio occultation signal analysis [38, 39] are additional fields which often necessitate the numerical solution of Abel's integral equations. Concentration on Abel's integral equations has been increased in several methodologies including numerical investigation along with their numerous applications [40, 41, 42, 43, 44, 45, 46, 47, 48, 49]. In [50], Jahanshahi solved Abel's integral equation numerically by approximating Riemann-Liouville fractional integrals and Caputo derivatives. Integrable solutions of Abel's integral equation under certain restrictions using different integral operators has been studied by Tamarkin [51] in 1930. In [52], authors obtained a stable numerical solution of Abel's integral equation by using concept of an almost Bernstein operational matrix. Using orthogonal polynomials, Minerbo and Levy [53] studied a numerical solution of Abel's integral equation. Iterative schemes [54] have been proved to be rather stable but are time-consuming in nature. Fourier-Hankel transform based inversion techniques are studied in [55, 56]. In [57], Kumar et al. suggested a simple algorithm for the analytical solution of Abel's integral equation via Laplace transform. Stable inversion of Abel's integral equation of first kind by means of orthogonal polynomials is presented in [58]. In [59], stable solution of a deconvolution problem of the Abel's integral equation based on Jacobi-Legendre polynomial is discussed by Ammari and Karoui. Chebyshev polynomials based methods on getting the approximate solution of Abel inversion have been presented in [60, 61]. Recently, Mikusinski's operator of fractional order is used to solve integral equation of Abel's type by Li and Zhao [62]. Sumner [63] studied Abel's integral equation using convolutional transform. Saleh et al. [64] investigated the numerical solution of Abel's integral equation by Chebyshev polynomials.

## 1.6 Literature Review on Fractional Integro Differential Equations

The fractional integro-differential equation is an equation in which fractional derivative and integral terms appear for the unknown function to be determined. Fractional integro-differential equation (FIDEs) arises in numeral fields such as fluid dynamics, biological models and chemical kinetics [65, 66]. Generally speaking, it is very difficult to obtain analytical solutions of most FIDEs. Therefore, it is very important to find numerical solutions of such FIDEs. In [67], Zhu and Fan proposed a numerical technique to solve fractional order Volterra integro-differential equations using Chebyshev wavelet of second kind. Khader and Sweilam discussed the numerical solutions for system of FIDEs based on Chebyshev pseudo-spectral method in [68]. Recently, numerous numerical methods to solve FIDEs have been given including variational iteration method [69, 70], Adomian's decomposition method [71], homotopy perturbation method [70, 72, 73, 74], homotopy analysis method [75] and collocation method [76, 77]. In [69], author presented approximate solution for seepage flow with fractional derivatives in porous media. Swielam et al. [70] studied solution for system of FIDEs using variational iteration and homotopy perturbation methods. In [71], Hashim proposed a solution for linear and non-linear boundary value problems of fourth order integro-differential equations using Adomian's decomposition method. He's homotopy perturbation method [74] has been applied to many areas such as heat radiation [72], reaction-diffusion [73] etc. In [75], authors considered approximate analytical solutions of linear and non-linear fractional initial value problems using homotopy analysis method. Khader presented numerical treatment for the solutions of fractional diffusion equation [76] and fractional Riccati differential equation [77] based on collocation method.

## 1.7 Finite Difference Method

Analytical solutions of partial differential equations provide closed-form expressions, which depict the variation of the dependent variables in the domain. However, it is hard to find analytical solutions of a large number of PDEs. To cope with this problem, numerical solutions play a very important role for such PDEs. The numerical solutions based on finite differences provide values at discrete points in the domain, known as grid points. The finite difference approximations for derivatives are one of the most straightforward and most seasoned numerical techniques to understand differential equations. It was at that point known by L. Euler (1707-1783) ca. 1768, in one-dimensional space and was presumably reached out to two-dimensional space by C. Runge (1856-1927) ca. 1908. The approach of finite difference methods in numerical applications started in the mid-1950s, and their advancement was fortified by the rise of PCs that offered a helpful system for managing complex problems of science and engineering. Theoretical outcomes have been acquired amid the most recent five decades in regards to the accuracy, stability, and convergence of the finite difference method for the partial differential equation.

## 1.8 Literature Review on Numerical Methods for Fractional Partial Differential Equations

Recently, many researchers have solved fractional partial differential equations with the help of numerical as well as analytical methods. Problems related with fractional order partial differential equations are also solved using several numerical techniques, but their convergence has rarely been discussed. We have existence and uniqueness theorem for integer order partial differential equations. However, there does not exist any analytical method for the existence and uniqueness of the solution for the fractional partial differential equations in general; the existing methods may obtain only approximate solutions. It is not encouraged to search a strong solution of fractional order partial differential equations due to the complexity and non-local nature of fractional order derivative. Due to frequent appearance of fractional order ordinary and partial differential equations in various areas like viscoelasticity, fluid mechanics, biology, physics, and engineering, etc., it always has been in the center of attraction for researchers. A natural extension of integer-order calculus is fractional calculus which provides a fundamental idea to construct different mathematical tools for many modeling processes arises from real life problems. Some scholars have provided many contributions in this area. Due to their work, there are several Fractional Derivatives (FDs) being used nowadays, such as Riemann–Liouville Derivative (RLD), Grünwald–Letnikov derivative, and Caputo derivative, see [1, 2, 5, 78] for a concrete comprehension of fractional derivatives. In recent years, many researchers have contributed to the development of fractional derivatives by introducing new results on the analysis of fractional order differential equations [79, 80, 81, 82, 83]. Fractional Diffusion Equations (FDEs) can be obtained by introducing fractional derivatives in place of integer order derivatives. Recently, a new Generalized Fractional Derivative (GFD) is discussed in [83], which in special case reduces to other fractional derivatives such as Riemann–Liouville, Caputo, Riesz, Hadamard, Erdéldi–Kober fractional derivatives. Agrawal [83] discussed the solutions of integral equations in terms of the new GFD. During past two decades, this area has attracted many researchers for investigation of its applications in various fields of science and engineering [84, 85, 86] such as particle diffusion [87], chemistry [88], biology [89], economics and finance [90, 91]. Numerical methods for solving fractional partial differential equations have been studied by many researchers [92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105].

Authors studied numerical solutions of fractional partial differential equations using finite difference methods [93, 95, 96, 97, 98, 99, 100, 101, 102, 103], finite element method [94], spectral methods [104, 105] and so on.

Ma and Sun [106], presented a Legendre-Petrov-Galerkin and Chebyshev collocation methods for FPDEs. In [107], a pseudo-spectral method is implemented with the split-step technique to solve KdV equation.
Advection-diffusion equations arise while dealing with many physical processes. In physical, chemical, and biological sciences involving dispersion or diffusion, these equations are broadly used to model a range of problems (see [108, 109, 110, 111, 112, 113, 114, 115, 116, 117]). In recent years, the number of contributions on the research of potentially useful tools for solving FDEs and its applications has been increased. In which, most of the FDEs have been solved by variational iteration method [118], Laplace transforms method [119], multigrid method [120], finite element methods [121] and finite difference schemes [96, 122, 123]. We refer the readers to go through references [121, 123, 124, 125, 126, 127, 128, 129] for detail review of other analytical and numerical methods. Among all these discussed methods, the finite difference method is very widespread for solving FPDEs. Nowadays, the initial value problem, as well as a boundary value problem for FPDEs, have been studied extensively [13, 130, 131, 132, 133, 134, 135]. Since the investigation of FDEs is very much useful for many real-world applications, hence further study is required.

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