Chapter 6

A Moving Boundary Problem Governed by the Fractional Derivatives

6.1 Introduction

Phase change processes arise in many practical applications in natural science, engineering and physics. Almost every industrial product involves melting/solidification at some stage. Example includes metal casting, production of steel, crystal growth, thermal welding, food conservation, soil freezing and many more (Benham et al., 2016; Vynnycky et al., 2015; Benes et al., 2015; Fang et al., 2015). The mathematical models which describe these physical phenomena are commonly known as a moving boundary problems or Stefan problems (Crank, 1984; Hill, 1987). Stefan problems or moving boundary problems are the special type of boundary value problems in which one part of boundary is not known beforehand but it must be determined as a part of solution to the problem. As the location of the moving interface is not known in advance, and must be solved for along with the temperature distribution, Stefan problems are nonlinear (Rubinstein, 1971; Gupta, 2003).

In recent decades, the use of fractional derivatives increases to formulate many physical phenomena because they interpret the behaviour of phenomena more accurately (Podlubny, 1999). Therefore, the fractional differential equations attract more attention of researchers working in different fields of physical sciences (Metzler and Klafter, 2000; Metzler and Klafter, 2004; Mainardi et al., 2001). In the classical Stefan problems, it is seen that the position of moving boundary is constant multiple of the square root function of time. But in the presence of heterogeneous medium where anomalous diffusion occurs, the moving interface may not proportional to the square root of time. Therefore, we need to modify the moving boundary problem by inclusion of fractional derivatives to model the physical problems involving anomalous diffusion (Voller et al., 2004; Zhou and Xia, 2015; Zhou et al., 2014).

The presence of moving interface makes the moving boundary problems non-linear. Hence, it is very difficult to get the closed form solution to a general Stefan problem. Only in few simple cases, the exact solutions to the Stefan problems are available in the literatures (Voller et al., 2004; Zhou and Xia, 2015; Zhou et al., 2014). In most of the cases, we have to obtain the approximate/numerical solutions to the Stefan problems (Rajeev et al., 2009; Gupta et al., 2010). With the rapid evolvement of non-linear science, the physicists and engineers have been considering the analytical techniques for nonlinear problems for the last two decades (He, 2003; He, 2005). It is well-known that the perturbation techniques give the most useful tools for solving non-linear problems of engineering. In recent years, the homotopy perturbation method has been successfully applied to get the approximate solutions to those problems of which the exact solutions are very difficult to be found (He, 2006; Li et al., 2009; Rajeev and Kushwaha, 2013; Das and Gupta, 2011). This method is the combination of an important notion of topology which is called homotopy and the perturbation methods. Homotopy perturbation techniques empower us to obtain the approximate solutions of wide range of problems occurring in different field.

In this chapter, we use the homotopy perturbation method to obtain an approximate solution to the moving boundary problem. In the mathematical formulation section, we describe the problem in detail and give the definitions and fundamental results required in the problem. Many existing Stefan problems are particular case of our problem discussed in this chapter.

6.2 Mathematical Model of the Problem

Motivated by the work of Zhou and Xia (2015), we present a time fractional model of melting /freezing process with some modification in the heat flux. Here, we assume anomalous heat diffusion in the domain $0 < x < s(t)$ and time varying nature of heat flux. The mathematical model of the process is as follow:

$$D_t^{\alpha} T(x,t) = v \frac{\partial^2 T(x,t)}{\partial x^2}, \quad 0 < x < s(t),\ t > 0, \tag{6.1}$$

$$k \frac{\partial T}{\partial x}(0,t) = -c\, t^{\frac{p-\alpha}{2}}, \quad t > 0, \tag{6.2}$$

$$T(s(t),t) = 0, \quad t > 0, \tag{6.3}$$

and the additional conditions on the moving boundary are

$$k \frac{\partial T}{\partial x}(s(t),t) = -\gamma s(t)^p D_t^{\alpha} s(t), \quad t > 0, \tag{6.4}$$

$$s(0) = 0. \tag{6.5}$$

The operator D_t^α in Eqs. (6.1) and (6.4) represents the fractional time derivative of order α in the Caputo sense which is defined by Caputo (1967)

$$D_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\tau)}{(t-\tau)^\alpha} d\tau, & 0 < \alpha < 1, \\ f'(t), & \alpha = 1, \end{cases} \tag{6.6}$$

for any $f(t) \in C^1[0, \infty)$, where Γ is the gamma function defined by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, \quad x > 0. \tag{6.7}$$

$T(x,t)$ denotes the temperature at position x and time t in liquid region, $s(t)$ is the location of solid-liquid interface at time t, $c\ t^{\frac{p-\alpha}{2}}$ is the time varying surface heat flux ($c > 0$ for melting, $c < 0$ for freezing), γx^p is the variable latent heat per unit volume ($\gamma > 0$ for melting, $\gamma < 0$ for freezing), p is a non-negative integer, k and v denote the thermal conductivity and thermal diffusivity, respectively. We take phase change temperature zero in the process.

6.3 Solution to the Problem

According to the method discussed in the Section 1.7 of chapter 1, to solve the problem (6.1)-(6.5), we construct the following homotopy:

$$(1-q)v\frac{\partial^2 \phi}{\partial x^2} + q\left(v\frac{\partial^2 \phi}{\partial x^2} - D_t^\alpha \phi\right) = 0, \tag{6.8}$$

which provides

$$v\frac{\partial^2 \phi}{\partial x^2} - qD_t^\alpha \phi = 0, \tag{6.9}$$

where $q \in [0,1]$ is an embedding parameter. It is clear from Eq. (6.9) that when $q = 0$, we get $\frac{\partial^2 \phi}{\partial x^2} = 0$ which is easy to solve with boundary conditions (6.2)-(6.5). The basic assumption is that the solution can be written as a power series in q

$$\phi(x,t;q) = \sum_{n=0}^{\infty} T_n(x,t) q^n, \qquad s(t;q) = \sum_{n=0}^{\infty} s_n(t) q^n \tag{6.10}$$

It is clear from the above that the solution to problem (6.1)-(6.5) can be obtained by putting $q = 1$ in Eq. (6.10), i.e.

$$T(x,t) = \sum_{n=0}^{\infty} T_n(x,t), \qquad s(t) = \sum_{n=0}^{\infty} s_n(t) \tag{6.11}$$

Substituting Eq. (6.10) into Eq. (6.9), we have the following equation

$$v \sum_{n=0}^{\infty} \frac{\partial^2 T_n(x,t)}{\partial x^2} q^n = \sum_{n=0}^{\infty} D_t^{\alpha} T_n(x,t) \; q^n \tag{6.12}$$

Now, substituting Eq. (6.10) in the boundary condition (6.3), we get

$$\sum_{n=0}^{\infty} T_n \left(\sum_{n=0}^{\infty} s_n(t) q^n, t \right) q^n = 0 \tag{6.13}$$

It is obvious that the embedding parameter q occurs in both forms explicit as well as implicit in Eq. (6.13). The implicit form is in first argument of $T_n(x,t)$. In order to compare the coefficients of various powers of q, we need the explicit involvement of q in Eq. (6.13). To do this we can expand $T_n(x,t)$ in a suitable neighbour of a point (s_0, t). Expanding $T_n(x,t)$ in terms of x, we have

$$T_i(x,t) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n T_i}{\partial x^n}\bigg|_{(s_0,t)} (x - s_0)^n, \qquad i = 0, 1, 2, \ldots\ . \tag{6.14}$$

Putting the value of $T_i(x,t)$ from Eq. (6.14) into the Eq. (6.13), we obtain

$$\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{q^l}{m!}\left(\sum_{n=1}^{\infty}s_n(t)q^n\right)^m\frac{\partial^m T_l}{\partial x^m}\bigg|_{(s_0,t)}=0. \tag{6.15}$$

Similarly, the Stefan condition (6.4) becomes

$$\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{q^l}{m!}\left(\sum_{n=1}^{\infty}s_nq^n\right)^m\frac{\partial^m T_l}{\partial x^m}\bigg|_{(s_0,t)}=-\frac{\gamma}{k}\left(\sum_{m=1}^{\infty}s_mq^m\right)^p\left(\sum_{n=0}^{\infty}q^nD_t^{\alpha}s_n\right). \tag{6.16}$$

Comparing the terms with equal powers of q in Eqs. (6.12), (6.15) and (6.16), we obtain a series of equations of the form

Coefficients of q^0 :

$$\begin{aligned}
&\frac{\partial^2 T_0}{\partial x^2}=0\\
&\frac{\partial T_0}{\partial x}\bigg|_{(0,t)}=-\frac{\gamma}{k}t^{\frac{p-\alpha}{2}}\\
&T_0(s_0,t)=0\\
&\frac{\partial T_0}{\partial x}\bigg|_{(s_0,t)}=-\frac{\gamma}{k}s_0(t)^pD_t^{\alpha}s_0(t)\\
&s_0(0)=0
\end{aligned} \tag{6.17}$$

Coefficients of q^1 :

$$\frac{\partial^2 T_0}{\partial x^2} = 0$$

$$\left.\frac{\partial T_0}{\partial x}\right|_{(0,t)} = -\frac{\gamma}{k} t^{\frac{p-\alpha}{2}}$$

$$T_0(s_0, t) = 0 \tag{6.18}$$

$$\left.\frac{\partial T_0}{\partial x}\right|_{(s_0,t)} = -\frac{\gamma}{k} s_0(t)^p D_t^\alpha s_0(t)$$

$$s_0(0) = 0$$

and so on.

By taking the first three equations of Eq. (6.17), we get

$$T_0(x,t) = \frac{c}{k} t^{\frac{p-\alpha}{2}} (s_0 - x). \tag{6.19}$$

Considering Eq. (6.19) and the last two equations of Eq. (6.17), we have

$$s_0(t) = a_0 t^\beta, \tag{6.20}$$

where $a_0 = \left(\frac{c}{\gamma} \frac{\Gamma(1+\beta-\alpha)}{\Gamma(1+\beta)}\right)^{\frac{1}{p+1}}$ and $\beta = \left(\frac{p+\alpha}{2(p+1)}\right)$.

Substituting $T_0(x,t)$ and $s_0(t)$ into Eq. (6.18) and using the same process, we get the following expression of $T_1(x,t)$ and $s_1(t)$

$$T_1(x,t) = \frac{A_0}{2}(x^2 - s_0^2) t^{\frac{p-3\alpha}{2}+\beta} + \frac{B_0}{6}(s_0^3 - x^3) t^{\frac{p-3\alpha}{2}} + \frac{c}{k} s_1 t^{\frac{p-\alpha}{2}}, \tag{6.21}$$

where

$$A_0 = \frac{ca_0}{vk}\frac{\Gamma\left(\frac{p-\alpha}{2}+\beta+1\right)}{\Gamma\left(\frac{p-3\alpha}{2}+\beta+1\right)}, \qquad B_0 = \frac{c}{vk}\frac{\Gamma\left(\frac{p-\alpha}{2}+1\right)}{\Gamma\left(\frac{p-3\alpha}{2}+1\right)} \tag{6.22}$$

and $s_1(t)$ is given by

$$s_1(t) = a_1 t^{\delta}, \tag{6.23}$$

where

$$a_1 = -\frac{k}{\gamma}\frac{\left(A_0 - \frac{a_0 A_0}{2}\right)}{a_0^{p-1}\left(\frac{p\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} + \frac{\Gamma(1+\delta)}{\Gamma(1+\delta-\alpha)}\right)}, \qquad \delta = \frac{p-\alpha}{2} + (2-\alpha)\beta. \tag{6.24}$$

Sequentially, $T_2(x,t), s_2(t); T_3(x,t), s_3(t); ...,$ can be obtained.

Substituting $T_0(x,t), T_1(x,t), s_0(t)$ and $s_1(t)$ into Eq. (6.11), we have the first order approximate solution to the problem (6.1)-(6.5)

$$T(x,t) = \frac{c}{k}t^{\frac{p-\alpha}{2}}(s_0 - x) + \frac{A_0}{2}(x^2 - s_0^2)t^{\frac{p-3\alpha}{2}+\beta} + \frac{B_0}{6}(s_0^3 - x^3)t^{\frac{p-3\alpha}{2}} + \frac{c}{k}s_1 t^{\frac{p-\alpha}{2}}, \tag{6.25}$$

$$s(t) = s_0(t) + s_1(t). \tag{6.26}$$

6.4 Numerical Comparison and Analysis

We have obtained an approximate analytical solution of a Stefan problem of which exact solution is not known till now. But, if we consider some particular value of parameters then we have exact solution of the problem in that cases (Voller et al., 2004; Zhou et al., 2014; Roscani and Marcus, 2014). In order to check the accuracy of the proposed approximate solution, we consider the following cases:

6.4.1 For integer order

6.4.1.1 When $\alpha = 1$ and $p = 0$.

In this case, we can observe that

$$T(x,t) = \frac{c\sqrt{\pi v}}{k}\left[erf\left(\frac{x}{\sqrt{4vt}}\right) - erf(\lambda)\right] \tag{6.27}$$

and

$$s(t) = 2\lambda\sqrt{vt}, \tag{6.28}$$

where λ is the root of the following transcendental equation

$$e^{-\lambda^2} = \frac{\lambda\sqrt{v}}{c}\lambda, \tag{6.29}$$

are exact solution to (6.1)-(6.5) as given in [15]. We have shown the comparison of our obtained result with this exact solution in Table 6.1 at $\alpha = 1$. It is obvious from table that our result is very close to exact solution in this case.

6.4.1.2 When $\alpha = 1$ and $p = 1$.

In this case, it can be seen that

$$T(x,t) = \frac{2c}{v}\left[\lambda\left(\frac{e^{-\frac{x^2}{4vt}} + \sqrt{\frac{\pi}{v}}\frac{x}{2\sqrt{t}}erf\left(\frac{x}{2\sqrt{vt}}\right)}{e^{-\frac{-\lambda^2}{v}} + \sqrt{\frac{\pi}{v}}\lambda erf\left(\frac{x}{2\sqrt{vt}}\right)}\right) - \frac{x}{2\sqrt{t}}\right], \tag{6.30}$$

$$s(t) = 2\lambda\sqrt{vt}, \tag{6.31}$$

where λ is the solution of the following non-linear equation

$$\frac{e^{-\frac{x^2}{4vt}} + \sqrt{\frac{\pi}{v}} \frac{x}{2\sqrt{t}} erf\left(\frac{x}{2\sqrt{vt}}\right)}{e^{-\frac{-\lambda^2}{v}} + \sqrt{\frac{\pi}{v}} \lambda erf\left(\frac{x}{2\sqrt{vt}}\right)} - \frac{1}{\lambda} + \frac{2\gamma\lambda}{c} = 0, \tag{6.32}$$

is the exact solution to the problem (6.1)-(6.5) according to Voller et al. (2004). Table 6.2 depicts the absolute and relative errors between our obtained results and the exact solution (Voller et al., 2004) for location of moving interface at $\alpha = 1.0$. From the table it is apparent that our result is very close to exact solution. It is also clear from the table that as the values of c increase, the accuracy in the location of moving interface decreases.

6.4.2 For fractional order

If we take $p = 0$ in Eqs. (6.2) and (6.4), then our problem converts into the following problem

$$D_t^{\alpha} T(x,t) = v \frac{\partial^2 T(x,t)}{\partial x^2}, \quad 0 < x < s(t),\ t > 0,\ 0 < \alpha < 1, \tag{6.33}$$

$$kT_x(0,t) = -\frac{c}{t^{\frac{\alpha}{2}}}, \quad t > 0, \tag{6.34}$$

$$T(s(t),t) = 0, \quad t > 0, \tag{6.35}$$

$$kT_x(s(t),t) = -\gamma D_t^{\alpha} s(t), \quad t > 0, \tag{6.36}$$

$$s(0) = 0. \tag{6.37}$$

According to Roscani and Marcus (2014), the exact solution to the problem (6.33)-(6.37) is

$$\begin{aligned} T(x,t) =& \frac{c}{k}\sqrt{v}\,\Gamma\left(1-\frac{\alpha}{2}\right)\left[1-W\left(-\lambda,-\frac{\alpha}{2},1\right)\right] \\ &-\frac{c}{k}\sqrt{v}\,\Gamma\left(1-\frac{\alpha}{2}\right)\left[1-W\left(-\frac{x}{\sqrt{v}\,t^{\frac{\alpha}{2}}},-\frac{\alpha}{2},1\right)\right] \end{aligned} \tag{6.38}$$

$$s(t) = \sqrt{v}\ \lambda\ t^{\frac{\alpha}{2}}, \tag{6.39}$$

where $W(x,\rho,1-\rho) = \sum_{n=0}^{\infty} \frac{x^n}{n!\Gamma(n\rho+1-\rho)}$ denotes Wright function and λ is the unique solution to the equation

$$\frac{c\ \gamma}{\sqrt{v}\ k^2}\frac{\Gamma\left(1-\frac{\alpha}{2}\right)^2}{\Gamma\left(1-\frac{\alpha}{2}\right)}W\left(-\lambda,-\frac{\alpha}{2},1-\frac{\alpha}{2}\right) = \lambda. \tag{6.40}$$

For case 2, we have presented Figs. 6.1, 6.2 and 6.3 at the fixed value of $c = 2.5$ and $p = 0$ for different values of α (0.75, 0.50, 0.25, respectively) to check the accuracy of our result with exact solution (Roscani and Marcus, 2014) for $s(t)$. It is clear that when value of α tends to 1, accuracy in the value of $s(t)$ increases. Fig. 6.4 depicts that the dependence of moving interface on p at fixed value of α and $c = 2.5$. Clearly, the velocity of phage front is highly affected by the variation of p and it increases as the value of p increases. Moreover, the melting /freezing process becomes fast with the increment in the value of p.

c	t	$s_E(t)$	$s_{HPM}(t)$	*Absolute error*	*Relative error*
1.0	0.1	0.0525833	0.0525827	6.0×10^{-7}	1.1×10^{-5}
	0.2	0.0743640	0.0743631	9.0×10^{-7}	1.2×10^{-7}
	0.3	0.0910770	0.0910759	1.1×10^{-6}	1.2×10^{-5}
	0.4	0.1051670	0.1051650	2.0×10^{-6}	2.0×10^{-5}
	0.5	0.1175800	0.1175780	2.0×10^{-6}	1.7×10^{-5}
1.5	0.1	0.0786506	0.0786452	5.4×10^{-6}	6.7×10^{-5}
	0.2	0.1112290	0.1112210	8.0×10^{-6}	7.1×10^{-5}
	0.3	0.1362270	0.1362180	9.0×10^{-6}	6.6×10^{-5}
	0.4	0.1573010	0.1572900	1.1×10^{-5}	7.0×10^{-5}
	0.5	0.1758680	0.1758560	1.2×10^{-5}	6.8×10^{-5}
2	0.1	0.1044550	0.1044330	2.2×10^{-5}	2.1×10^{-4}
	0.2	0.1477220	0.1476910	3.1×10^{-5}	2.0×10^{-4}
	0.3	0.1809220	0.1808840	3.8×10^{-5}	2.1×10^{-4}
	0.4	0.2089100	0.2088670	4.3×10^{-5}	2.0×10^{-4}
	0.5	0.2335690	0.2335200	4.9×10^{-5}	2.0×10^{-4}

TABLE 6.1: Error between exact and approximate solution for $s(t)$ at $\alpha = 1,\ v = 3,\ k = 3,\ \gamma = 12$ and $p = 0$.

c	t	$s_E(t)$	$s_{HPM}(t)$	*Absolute error*	*Relative error*
1.0	0.1	0.113631	0.113546	8.5×10^{-5}	7.8×10^{-4}
	0.2	0.160699	0.160578	1.2×10^{-4}	7.5×10^{-4}
	0.3	0.196815	0.196667	1.4×10^{-4}	7.5×10^{-4}
	0.4	0.227262	0.227091	1.7×10^{-4}	7.5×10^{-4}
	0.5	0.254087	0.253896	1.9×10^{-4}	7.5×10^{-4}
1.5	0.1	0.138115	0.137886	2.3×10^{-4}	1.6×10^{-3}
	0.2	0.195324	0.195000	3.2×10^{-4}	1.6×10^{-3}
	0.3	0.239222	0.238825	3.9×10^{-4}	1.6×10^{-3}
	0.4	0.276229	0.275771	4.6×10^{-4}	1.6×10^{-3}
	0.5	0.308834	0.308322	5.1×10^{-4}	1.6×10^{-3}
2	0.1	0.158313	0.157856	4.6×10^{-4}	2.9×10^{-3}
	0.2	0.223889	0.223242	6.5×10^{-4}	2.9×10^{-3}
	0.3	0.274206	0.273415	7.9×10^{-4}	2.9×10^{-3}
	0.4	0.316626	0.315712	9.1×10^{-4}	2.9×10^{-3}
	0.5	0.353999	0.352977	1.0×10^{-3}	2.9×10^{-3}

TABLE 6.2: Error between exact and approximate solution for $s(t)$ at $\alpha = 1,\ v = 2,\ k = 2,\ \gamma = 15$ and $p = 1$.

Fig. 6.5 presents the evolution of movement of interface at fixed value of p and $c = 2.5$. From this figure it is confirmed that as the values of α increase, the evolution of phase front shifts towards standard solution (case of integer order derivative). It can also be seen from the figure that the velocity of phase front increases with the increment in the value of α for larger value of time.

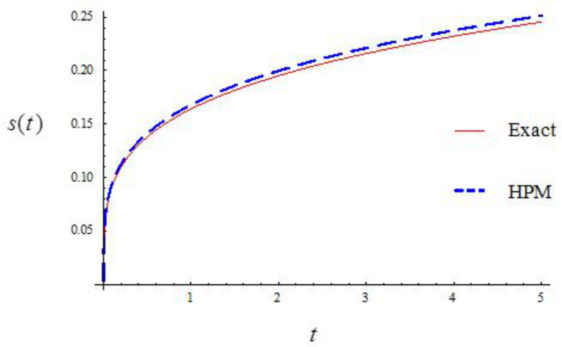

FIGURE 6.1: Plot of $s(t)$ vs. t at $\alpha = 0.75,\ v = 2,\ \gamma = 0.05,\ k = 1.0$.

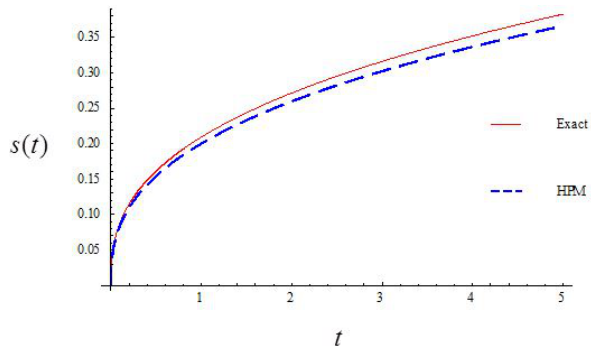

FIGURE 6.2: Plot of $s(t)$ vs. t at $\alpha = 0.50,\ v = 2,\ \gamma = 0.05,\ k = 1.0.$.

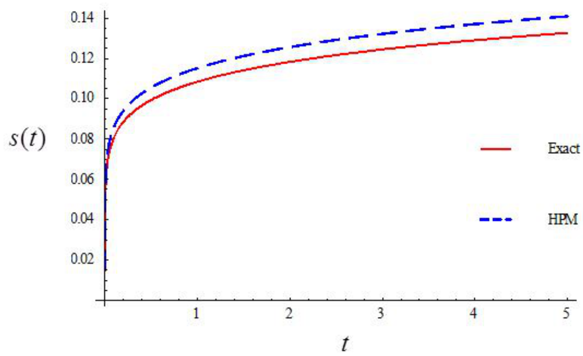

FIGURE 6.3: Plot of $s(t)$ vs. t at $\alpha = 0.25,\ v = 2,\ \gamma = 0.05,\ k = 1.0.$.

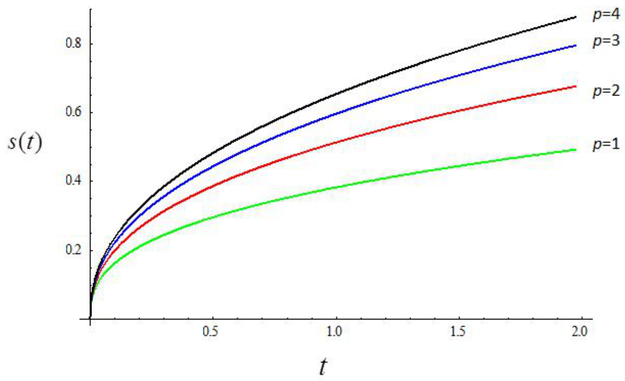

FIGURE 6.4: Plot of $s(t)$ vs. t at $\alpha = 0.25,\ v = 2,\ \gamma = 0.05,\ k = 1.0.$

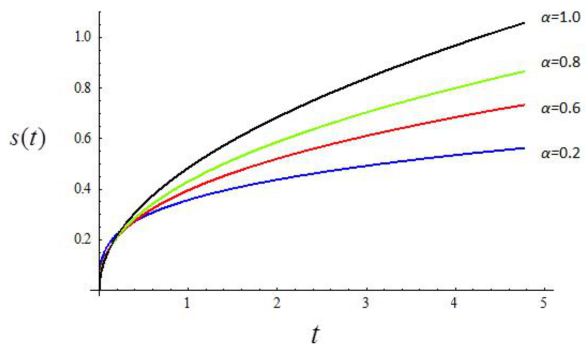

FIGURE 6.5: Plot of $s(t)$ vs t at $p = 1.0,\ v = 2.0,\ \gamma = 0.005,\ k = 1.0.$

6.5 Conclusion

We solve a time fractional Stefan problem with latent heat a power function of position. In this study, we consider a Stefan problem with fractional derivative for describing the process of melting/freezing. We have successfully applied homotopy perturbation method to the problem to find an approximate solution. We have seen that our results are sufficiently near to the exact solution for integer order derivative and fractional order derivative for some special cases of the problem. This shows the validity of solution by this method to the Stefan problem. We have also seen that the velocity of phase front highly depends on α and p. Further, it is found that the evolution of velocity of phase front increases as the value of α and p increases. Moreover, when the value of α increases from 0 to 1, the solution of moving interface for fractional case shifts towards the solution for integer order derivative as time proceeds.

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