Chapter 5

Homotopy Analysis Method for a Fractional Stefan Problem

5.1 Introduction

Classical Stefan problem involves a wide range of mathematical models describing thermal diffusion processes in homogeneous media undergoing a phase change. These problems occur in many natural and industrial process such as solidification/ freezing of liquid material, melting of solid/ice, recrystallization of metals, binary alloy, etc. From mathematical point of view, Stefan problems are interested because of its nonlinearity and presence of unknown domain. Over the past decades, many complicated conditions for the Stefan problems such as nonlinear thermal properties, different kinds of boundary conditions have been discussed and corresponding solutions have been presented (Tao, 1981; Chemiha and Chemiha, 1993; Chemiha and Kovalenko, 2009; Briozzo and Tarzia, 2002; Voller and Falcini, 2013). A special type of Stefan problem is discussed by Swenson et al. (2000) and Voller et al.

(2004) in which the latent heat is space dependent. This type of problem generally appears in the study of the shoreline movement. Rajeev et al. (2009) presented a numerical solution of the same shoreline problem to investigate the dependence of movement of shoreline position on various parameters. Zhou et al. (2014) presented an exact solution of a generalized model of Voller et al. (2004). This model includes a latent heat term which is a power function of position with positive integer exponent. Further, the model discussed in Zhou et al. (2014) is modified by Zhou and Xia (2015) in which latent heat as a power function of position with positive real number exponent is considered and exact solution of the problem is presented.

In the past few decades, fractional calculus has been widely used in various mathematical models to describe anomalous diffusion processes. Classical diffusion equation is based on Fick's law but, it is observed from literature (Miller and Ross, 1993; Podlubny, 1999; Metzler and Klafter, 2000; Piryatinska et al., 2005; Zahran, 2009) that some diffusion processes deviate from Fickian law and violet the assumption of statistical convergence to a Gaussian. These deviations lead a notable feature of the anomalous diffusion which is governed by power law. In anomalous diffusion, the flux at a particular point for the certain moment depends on the distribution of the physical quantities such as temperature or concentration as well as the physical quantities of the other points and their varying history (Gao et al., 2015). Physical basis of anomalous heat diffusion can be seen in (Li and Wang, 2003; Liu et al., 2014).

Stefan problem (a moving boundary problem) with fractional derivatives governing the process of anomalous diffusion in the controlled drug devices is firstly discussed by Liu and Xu (2004). Afterward, many other mathematical models of Stefan problems governed with fractional derivatives have been reported in the literature (Liu and Xu, 2009; Voller, 2010; Das et al., 2011; Singh et al., 2011; Rajeev and Kushwaha, 2013). In general, Stefan problems are tough to get its analytical solution and thus many approximate or numerical techniques have been applied to solve these problems. Gao et al. (2015) discussed a numerical method for the moving boundary problem with space-fractional derivative in drug release devices. Recently, Rajeev et al. (2016) presented an approximate solution by optimal homotopy asymptotic method for fractional mathematical model of a one-dimensional phase change problem (Stefan problem) with latent heat a power function of position. In this chapter, homotopy analysis method is used to find an approximate solution of proposed fractional model. Homotopy analysis method (HAM) is one of the powerful techniques which is applicable for not only linear but also highly nonlinear differential and integral equations. This scheme (HAM) was developed by Liao (Liao, 1997; Liao, 2009). Afterward, many researcher have been successfully used HAM for solving several types of problems (Abbasbandy, 2006; Li et al., 2013; Onyejekwe, 2014; Onyejekwe, 2015) arising in the field of science and engineering. In HAM, one can adjust and control the convergence region of series solution with the help of auxiliary parameter (Gorder and Vajravelu, 2009).

The main aim of this chapter is to discuss an approximate solution of a Stefan problem governed by Caputo fractional derivative. In many complex processes, it is found that mean-square displacement of diffusing particle does not proportional to the first power of time. As a result, anomalous diffusion processes arise. The process of anomalous diffusion occurs in non-locality or memory effect, either in space (fractional Brownian motion) or in time (non-Markovian processes) (Zahram, 2009). By considering these facts, anomalous heat diffusion (non-Fickian) is assumed in the proposed model and the obtained approximate solution of proposed model is compared with existing analytic solution for some particular cases in order to check the accuracy of our solution.

5.2 Mathematical Model

Here, we consider a space-time fractional model of a one dimensional phase change problem with latent heat a power function of position (Zhou et al., 2009). In this model, anomalous heat diffusion, time dependent surface heat flux and zero phase change temperature are taken into the consideration. Moreover, fractional derivatives in Caputo sense (Voller, 2010; Rajeev et al., 2016) is used for anomalous heat diffusion in the proposed model. The mathematical equations governing the problem are as follow:

$$D_t^{\beta}\, T(x,t) = \upsilon \frac{\partial}{\partial x}\left(D_x^{\alpha}\, T(x,t)\right), \quad 0 < x < s(t),\ t > 0, \tag{5.1}$$

$$T(s(t),t) = 0, \quad t > 0, \tag{5.2}$$

$$k\ D_x^{\alpha} T(0,t) = -b\ t^n, \quad t > 0, \tag{5.3}$$

$$k\ D_x^{\alpha} T(s(t),t) = -\gamma s(t)^{n+1} D_t^{\beta}\, T(s(t),t), \quad t > 0, \tag{5.4}$$

$$s(0) = 0, \tag{5.5}$$

where $T(x,t)$ is the temperature distribution, x is the position coordinate, t is the time, $s(t)$ is the location of phase front, υ is the thermal diffusivity, k is the thermal conductivity, $b\ t^n$ is the time varying surface heat flux, $\gamma s(t)^{n+1}$ is the variable latent heat term, n is an arbitrary non-negative integer, D_t^{β} and D_x^{α} are the Caputo fractional derivatives $(0 < \alpha \leq 1, 0 < \beta \leq 1)$.

The Caputo fractional derivative operator D_t^{α} is defined as given in Section 1.8 of chapter 1.

5.3 Solution of the Problem

In order to apply homotopy analysis method, mentioned in the Section 1.6 of the chapter 1, to Eq. (5.1), we first consider the following linear operator:

$$L[\phi(x,t;p)] = \frac{\partial}{\partial x}(D_x^{\alpha}\phi(x,t;p)) \tag{5.6}$$

and nonlinear operator as

$$N[\phi(x,t;p)] = D_t^{\beta}\phi(x,t;p) - v\frac{\partial}{\partial x}(D_x^{\alpha}\ \phi(x,t;p)) \tag{5.7}$$

Now, we construct the zeroth-order deformation equation (Liao, 2009; Abbasbandy, 2006; Li et al., 2013) for Eq. (5.1) as:

$$(1-p)L[\phi(x,t;p) - T_0(x,t)] = pc_0 H(x,t) N[\phi(x,t;p)] \tag{5.8}$$

where $p \in [0,1]$ is the embedding parameter, $c_0 \neq 0$ is an auxiliary parameter, $\phi(x,t;p)$ is an unknown function, $T_0(x,t)$ is an initial estimate of the solution and $H(x,t)$ is a nonzero auxiliary function. From Eq. (5.8), it can be seen that when the embedding parameter $p = 0$ and $p = 1$ then

$$\phi(x,t;0) = T_0(x,t)$$

$$\phi(x,t;1) = T(x,t),$$

respectively. Consequently, when p raises from 0 to 1, the initial guess $T_0(x,t)$ tends to the solution $T(x,t)$.

According to Liao (2009) and Abbasbandy (2006), we develop the following m-th order deformation equation for Eq. (5.1) as:

$$L[T_m(x,t) - \chi_m T_{m-1}(x,t)] = c_0 \Re_m(\overrightarrow{T}_{m-1}) \tag{5.9}$$

where

$$\Re_m(\overrightarrow{T}_{m-1}) = D_t^{\beta}\left(T_{m-1}(x,t)\right) - v\frac{\partial}{\partial x}\left(D_x^{\alpha}\, T_{m-1}(x,t)\right) \tag{5.10}$$

and

$$\chi_m = \begin{cases} 0, & m \leqslant 1 \\ 1, & m > 1. \end{cases}$$

In order to solve Eq. (5.9), we use initial conditions as given by Abbasbandy (2006), and we take initial guess of $T(x,t)$ as

$$T_0(x,t) = A\left(t^n x^{\alpha} - t^{n+\alpha p}(a_0)^{\alpha}\right) \tag{5.11}$$

where $A = -\frac{b}{k\Gamma(1+\alpha)}$ and $p = \frac{n+\beta}{n+2}$.

From Eqs. (5.9) and (5.10), we have

$$\begin{aligned} T_1(x,t) =& c_0 A\Bigg(\frac{(\alpha+1)\Gamma(n+1)}{\Gamma(n+1-\beta)}\frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}t^{n-\beta} - a_0^{\alpha}\frac{\Gamma(n+\alpha p+1)}{\Gamma(n+\alpha p+1-\beta)} \\ & \times \frac{x^{\alpha+1}}{\Gamma(\alpha+2)}t^{n+\alpha p-\beta}\Bigg) \end{aligned} \tag{5.12}$$

$$T_2(x,t) = T_1(x,t) + c_0^2 A\Bigg(\frac{(\alpha+1)\Gamma(n+1)}{\Gamma(n+1-2\beta)}\frac{x^{2\alpha+2}}{\Gamma(3\alpha+3)}t^{n-\beta} - a_0^{\alpha}\frac{\Gamma(n+\alpha p+1)}{\Gamma(n+\alpha p-2\beta+1)}$$
$$\times\frac{x^{2\alpha+2}}{\Gamma(2\alpha+3)}t^{n+\alpha p-2\beta} - \frac{(\alpha+1)\Gamma(n+1)}{\Gamma(n+1-\beta)}\frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}t^{n-\beta} - \frac{\Gamma(n+\alpha p+1)}{\Gamma(n+\alpha p-\beta+1)}$$
$$\times\frac{x^{\alpha+1}}{\Gamma(\alpha+2)}t^{n-\alpha p-\beta}\Bigg) \tag{5.13}$$

and so on.

Therefore, the approximate solution of temperature distribution is given by

$$T(x,t) = T_0(x,t) + T_1(x,t) + T_2(x,t) + \ldots\,. \tag{5.14}$$

In order to get the expression of $s(t)$ we choose the linear operator and nonlinear operator in Eq. (5.4) as:

$$L\left[\psi(t;p)\right] = D_t^{\beta}\psi(t;p), \tag{5.15}$$

and

$$N\left[\psi(t;p)\right] = kD_t^{\beta}T(\psi(t;p)) + \gamma(\psi(t;p))^{n+1}D_t^{\beta}\psi(t;p), \tag{5.16}$$

respectively.

According to the HAM, we construct the zeroth order deformation equation for Eq. (5.4) as:

$$(1-p)L\left[\psi(t;p) - s_0(t)\right] = p\hbar N[\psi(t;p)], \tag{5.17}$$

Clearly

$$\psi(t;0) = s_0(t) \quad \text{and} \quad \psi(t;1) = s(t). \tag{5.18}$$

As given in Abbasbandy (2006), the mth order deformation equation for Eq. (5.4) will be

$$L\left[s_m(t) - s_0(t)\right] = \hbar N\left[s_{m-1}(t)\right]. \tag{5.19}$$

Now, we take the initial guess for moving interface as follow:

$$s_0(t) = a_0 t^p, \tag{5.20}$$

where $a_0 = \left(\frac{b\Gamma(p-\beta+1)}{\gamma\Gamma(p+1)}\right)^{\frac{1}{n+1}}$ and $p = \left(\frac{n+\beta}{n+2}\right)$.

With the help of Eqs. (5.14), (5.16), the Eq. (5.19) gives various components of $s(t)$ and the approximate solution for $s(t)$ is given by

$$s(t) = s_0(t) + s_1(t) + s_2(t) + ... \, . \tag{5.21}$$

5.4 Comparisons and Discussions

This section contains comparisons of obtained results of temperature distribution and position of moving interface with existing analytical solution (Voller et al., 2004) for the case of integer order derivatives and presented through figures. Dependence of movement of phase front on time for various parameters are also discussed.

Figs. 5.1-5.2 demonstrate the comparisons of proposed approximate solution of temperature distribution with the existing analytical solution at $c_0 = -1, v = 1.5, k = 1.5, \alpha = \beta = 1$ (integer ordered derivatives) for $b = 1.0$ and $b = 2.0$, respectively. The appraisals of obtained approximate solution of location of moving interface $s(t)$ with the analytical solution at $v = 1.5, k = 1.5, b = 1$ and $\hbar = 1$ for integer ordered derivatives i.e., $\alpha = \beta = 1$ are portrayed in Figs. 5.3-5.4 . From the Figs. 5.1- 5.4, it can be seen that our proposed solution to the problem is close to the analytical solution.

The trajectories of phase front $s(t)$ at $\gamma = 20, v = 1.5, k = 1.5, b = 1$ and $\hbar = 1$ for different value of n $(n = 1, 2, 3)$ are presented in Figs 5.5-5.7. Fig 5.5 is plotted for integer ordered derivatives (Classical or Fickian diffusion) and Figs 5.6-5.7 are plotted for fractional ordered derivatives (non-Fickian). From Figs 5.5-5.7, it is clear that the movement of phase front increases as the value of positive exponent (n) increases for both the Fickian and non-Fickian diffusion after a fixed time. As a result of this, the melting/freezing process becomes fast with the increment in n for large time.

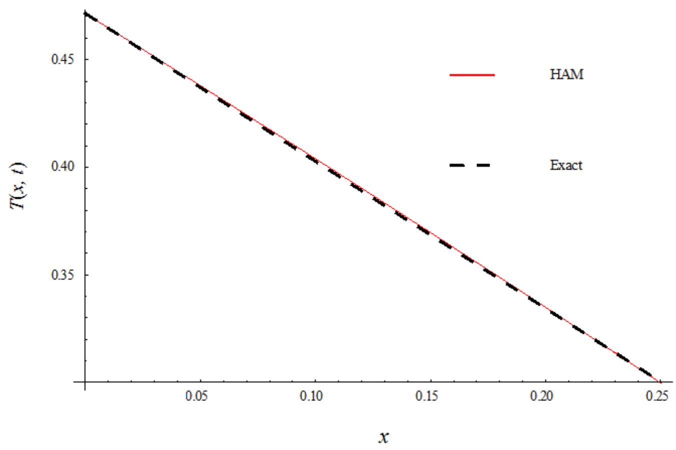

FIGURE 5.1: Plot of $T(x,t)$ vs x at $\gamma = 20$ and $n = 0$.

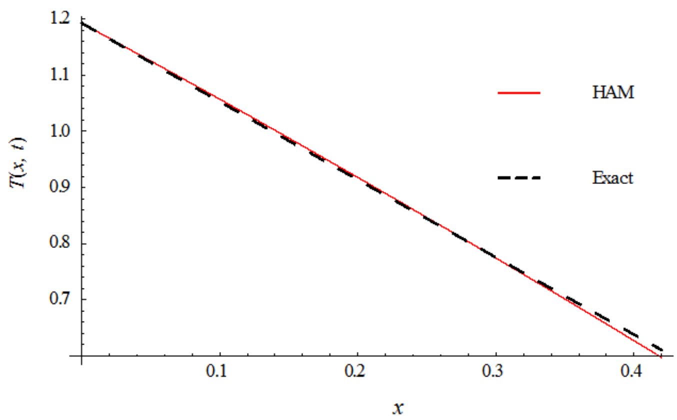

FIGURE 5.2: Plot of $T(x,t)$ vs x at $\gamma = 25$ and $n = 0$.

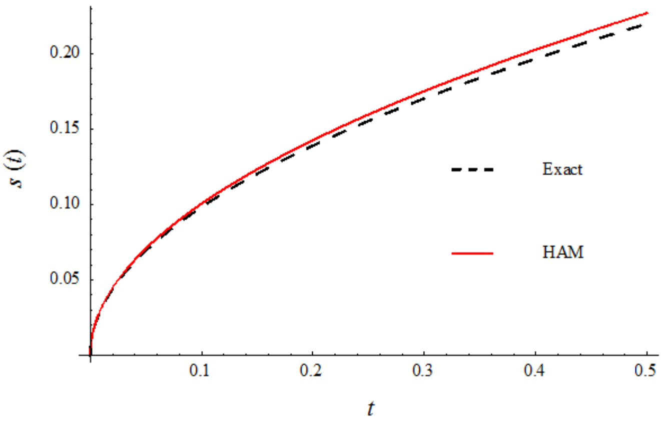

FIGURE 5.3: Plot of $s(t)$ vs t at $\gamma = 20$ and $n = 0$.

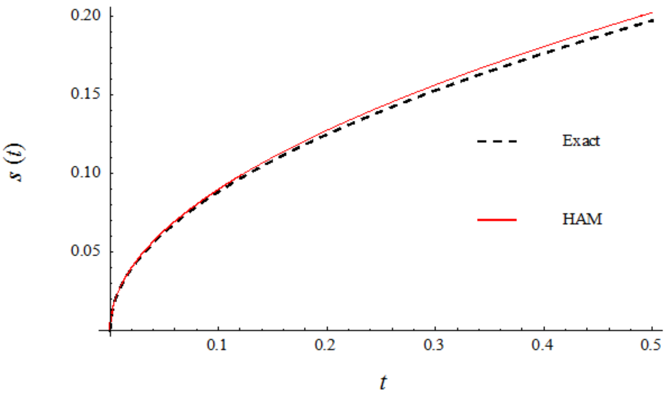

FIGURE 5.4: Plot of $s(t)$ vs t at $\gamma = 25$ and $n = 0$.

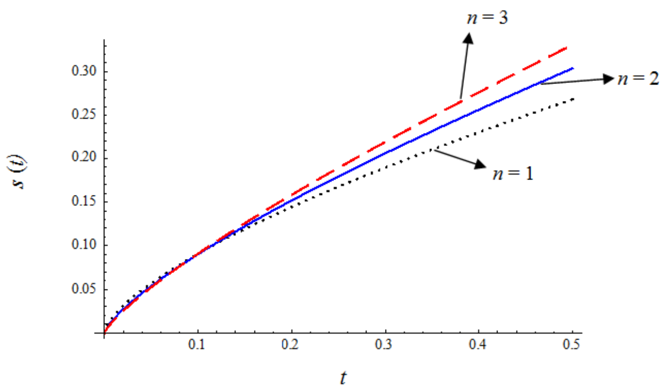

FIGURE 5.5: Dependence of path of $s(t)$ on n at $\alpha = 1, \beta = 1$.

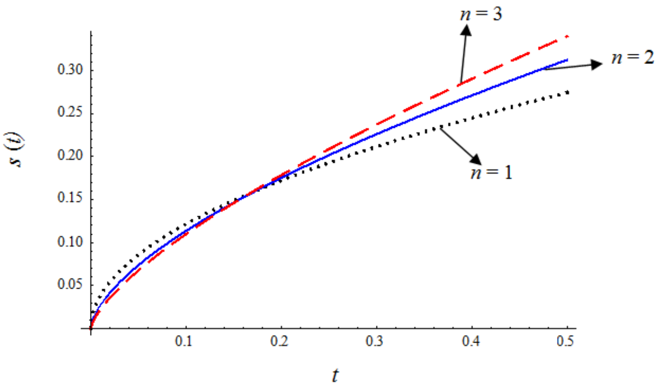

FIGURE 5.6: Dependence of path of $s(t)$ on n at $\alpha = 0.5, \beta = 0.2$.

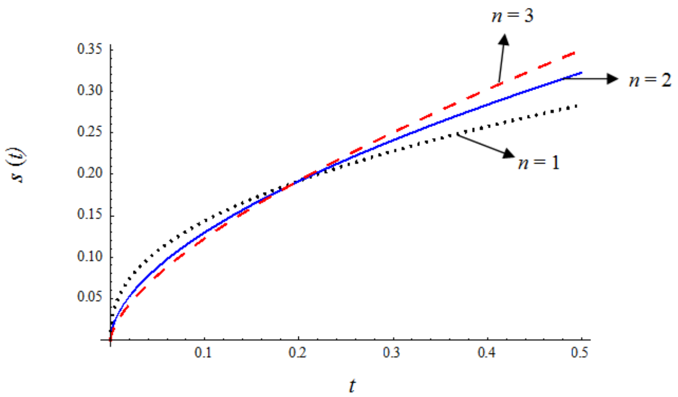

FIGURE 5.7: Dependence of path of $s(t)$ on n at $\alpha = 0.5, \beta = 0.25$.

5.5 Conclusion

In this work, we have successfully applied the homotopy analysis method to find an approximate analytical solution of a Stefan problem governed by space-time fractional derivatives. For some limit cases, we have shown that our proposed solution to the problem is near to the analytical solution of Voller et al. (2004). It is seen that homotopy analysis method provides a simple way to adjust and control the convergence region for the obtained series solution by selecting a proper value of auxiliary parameter. Moreover, it is observed that homotopy analysis method is an efficient and accurate technique to find approximate solutions of different types of Stefan problems which arise in science and engineering. From this chapter, it is also clear that the melting/freezing process becomes fast as the value of n increases after a fixed time.

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