Chapter 4

A Wavelet Based Approach to a Moving Boundary Problem

4.1 Introduction

Moving boundary problems (Stefan problems) involving heat and mass transfer occur in a wide variety of natural and industrial phenomenon. In a moving boundary problem, the location of moving boundary is not known in advance and it depends on the dependence variable which has to calculate. Therefore, the moving boundary becomes a part of solution. Moreover, these problems are nonlinear. Due to these facts, the solutions of these problems are of special interest from mathematical point of view. It is seen from Voller et al. (2004), Rajeev et al. (2009), Zhou et al. (2014) that the moving boundary problems involving variable latent heat terms are of great interest in recent years. In 2015, Zhou and Xia (2015) presented an exact solution for a one phase Stefan problem in which the latent heat is a power function of position with a positive exponent by using theory of the Kummer function.

From literatures (Hill, 1987; Rajeev and Das, 2010), it is found that exact solutions of moving boundary problems are available for restricted cases only. Therefore, various numerical techniques including finite difference method (Rizwan-Uddin, 1999), finite element method (Gawlika and Lew, 2014), integral methods (Sadoun et al., 2006), etc. have been applied for solving these problems. Some recent papers related to numerical solution of moving boundary problems have been reported in (Mitchell and Vynnycky, 2014; Kim, 2014; Blasik and Klimek, 2015).

In the last two decades, many numerical schemes based on different types of wavelets have been reported for solving ordinary and partial differential equations with fixed boundary condition (Chang and Wang, 1983; Razzaghi and Yousefi, 2001; Arsalani and Vali, 2011). Some numerical algorithms based on Chebyshev wavelets for various kind of integral and differential equations have been discussed in (Biazar and Ebrahimi, 2012; Xu and Zhou, 2015; Abd-Elhameed et al., 2013; Heydari et al., 2014). It is seen that numerical schemes based on wavelets are user friendly, accurate and computer intensive. Due to these facts wavelets becomes a notable tool from mathematical point of view. In 2014, Zhou and Xu (2014) presented a numerical solution of convection diffusion equations which is based on operational matrix of integration of second kind Chebyshev wavelets. Recently, Rajeev and Raigar (2015) discussed a numerical solution for a solidification problem in semi-infinite domain which is based on similarity transformation and Operational matrix of differentiation of shifted Chebyshev polynomial of second kind wavelets.

The main aim of this chapter is to present a numerical solution of a Stefan problem involving variable latent heat term which is a power function of position (Zhou et al., 2014). In order to obtain numerical solution, we use similarity transformation and Operational matrix of differentiation of shifted Chebyshev polynomials of second kind wavelets as given in (Rajeev and Raigar, 2015). The obtained numerical results are compared with exact solution for some particular cases to check the accuracy of the solution. The dependence of movement of the moving boundary on various parameters are also analyzed.

4.2 Mathematical Model of the Problem

In this section, we consider a phase change moving boundary problem involving latent heat term which is a power function of position (Zhou et al., 2014; Zhou and Xia, 2015). The governing equations of the problem are formulated as:

$$\frac{\partial T(x,t)}{\partial t} = v\frac{\partial^2 T(x,t)}{\partial x^2}, \quad 0 < x < s(t),\ t > 0, \tag{4.1}$$

$$T(s(t),t) = 0, \quad t > 0, \tag{4.2}$$

$$k\frac{\partial T(0,t)}{\partial x} = -c\, t^{\frac{\alpha-1}{2}}, \quad t > 0, \tag{4.3}$$

$$k\frac{\partial T(s(t),t)}{\partial x} = -\gamma\ s(t)^{\alpha}\ \frac{ds(t)}{dt}, \quad t > 0, \tag{4.4}$$

$$s(0) = 0, \tag{4.5}$$

where $T(x,t)$ is the temperature distribution, v is the diffusivity, $\gamma s(t)^{\alpha}$ is the latent heat term, k is the thermal conductivity, x is the position coordinate, t is the time and $s(t)$ is the position of moving phase front, α is a non – negative real number, γ and c are the constants.

4.3 Solution of the Problem

First, we consider the following similarity transformation (Zhou et al., 2014; Zhou and Xia, 2015):

$$T(x,t) = t^{\frac{\alpha}{2}}\ \eta(\xi), \quad \text{where } \xi = \frac{x}{2\sqrt{vt}} \tag{4.6}$$

and take the movement of phase front as

$$s(t) = 2\lambda\sqrt{vt}, \tag{4.7}$$

where λ is an unknown constant.

With the help of Eqs. (4.6)-(4.7), the Eqs. (4.1)-(4.3) turn into

$$\eta''(\xi) + 2\ \xi\ \eta'(\xi) - 2\alpha\eta(\xi) = 0, \quad 0 < \lambda < 1, \tag{4.8}$$

$$\eta(\lambda) = 0, \tag{4.9}$$

$$\eta'(0) = -\frac{2c\sqrt{v}}{k}, \tag{4.10}$$

respectively and the Stefan condition (4.4) becomes

$$\eta'(\lambda) = -\frac{\gamma}{k}v^{(\alpha+2)/2}(2\lambda)^{\alpha+1}. \tag{4.11}$$

As given in Eq. (1.29), we write $\eta(\xi)$ and ξ in the following truncated series form (Rajeev and Raigar, 2015):

$$\eta(\xi) = \sum_{n=0}^{2^k-1}\sum_{m=0}^{M} C_{nm}\psi_{nm}(\xi) = C\psi(\xi) \tag{4.12}$$

$$\xi = \sum_{n=0}^{2^k-1}\sum_{m=0}^{M} f_{nm}\psi_{nm}(\xi) = F\psi(\xi) \tag{4.13}$$

where

$$C = [c_{0,0}, c_{0,1}, c_{0,2}, ..., c_{0,M}, ..., c_{2^k-1,0}, c_{2^k-1,1}, ..., c_{2^k-1,M}],$$

$$\psi(t) = [\psi_{0,0}, \psi_{0,1}, \psi_{0,2}, ..., \psi_{0,M}, ..., \psi_{2^k-1,0}, \psi_{2^k-1,1}, ..., \psi_{2^k-1,M}]^T,$$

and

$$F = [f_{0,0}, f_{0,1}, f_{0,2}, ..., f_{0,M}, ..., f_{2^k-1,0}, f_{2^k-1,1}, ..., f_{2^k-1,M}]$$

From Eq. (1.35), we have

$$\eta'(\xi) = CD\psi(\xi), \tag{4.14}$$

$$\eta''(\xi) = CD^2\psi(\xi). \tag{4.15}$$

Substituting Eqs. (4.14)-(4.15) into Eq. (4.8), we get

$$CD^2\psi(\xi) + 2F\psi(\xi)CD\psi(\xi) - 2\alpha C\psi(\xi) = 0, \quad 0 < \xi < \lambda, \tag{4.16}$$

Again substituting Eqs. (4.14) and (4.15) into Eqs. (4.9)-(4.11) that give

$$C\psi(\lambda) = 0, \tag{4.17}$$

$$CD\psi(0) = -\frac{2c\sqrt{v}}{k}, \tag{4.18}$$

and

$$CD\psi(\lambda) = -\frac{\gamma}{k}v^{(\alpha+2)/2}(2\lambda)^{\alpha+1} \tag{4.19}$$

In order to get approximate solution of $\eta(\xi)$, we solve Eq. (4.16) at the first $\left(2^k(M+1)-2\right)$ roots of $U^*_{2^k(M+1)}(\xi)$ which gives $\left(2^k(M+1)-2\right)$ equations. Moreover, Eqs. (4.17)-(4.19) generate three more equations in terms of $2^k(M+1)$ constants (c_{nm}) and λ. By applying any appropriate numerical technique, the solution of these equations can be found. With the help of Eq. (4.12), we can obtain an approximate solution of $\eta(\xi)$. After that approximate solutions of $T(x,t)$ and $s(t)$ can be determined with the help of Eqs. (4.6)-(4.7).

4.4 Numerical Comparisons and Discussions

In this section, all numerical computations have been made for temperature distribution $T(x,t)$ and moving interface $s(t)$ by taking M = 2, k = 0. The results are obtained by using MATHEMATICA software and shown through tables and figures. The following matrices are taken in our calculations:

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 8 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 32 & 0 & 0 \end{pmatrix}, \quad C = \sqrt{\frac{\pi}{2}} \begin{pmatrix} c_0 & c_1 & c_2 \end{pmatrix},$$

$$\psi(\xi) = \sqrt{\frac{2}{\pi}} \begin{pmatrix} 2 \\ 8\xi - 4 \\ 32\xi^2 - 32\xi + 6 \end{pmatrix} \text{ and } F = \sqrt{\frac{\pi}{2}} \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}.$$

In order to show the accuracy of our solution, we take two cases $\alpha = 0$ and $\alpha = 1$. It can be seen in (Zhou et al., 2014) that

$$T(x,t) = \frac{c\sqrt{\pi v}}{k} \left(erfc\left(\frac{x}{\sqrt{4vt}} \right) - erfc(\lambda) \right),$$

$$s(t) = 2\ \lambda\sqrt{vt},$$

where λ is the root of following equation:

$$exp(-\lambda^2) = \frac{\gamma\sqrt{v}}{c}\lambda,$$

are the exact solutions of Eqs. (4.1)-(4.5) in case of $\alpha = 0$. When $\alpha = 1$ then Eqs. (4.1)-(4.5) become a limit case of a basic shoreline problem whose analytical solution is discussed by Voller et al. [1]. Table 4.1 shows the absolute errors and relative errors between obtained solution and exact solution (Zhou et al., 2014) for moving interface $s(t)$ at $\alpha = 0, v = 2.0, k = 1.0, \gamma = 20$. Table 4.2 depicts the absolute errors and relative errors between our solution and analytic solution [1] for the case of $\alpha = 1$ at $v = 1.0, k = 1.0, \gamma = 20$. From both the tables, it is clear that our obtained numerical solutions are near to the exact solutions.

The dependence of movement of interface $s(t)$ for different value of v at $\alpha = 0$ and $\alpha = 1$ are presented through Fig. 4.1 and Fig. 4.2, respectively. From both the figures, it is seen that movement of interface increases as the value of v decreases and consequently, the freezing/ melting process becomes fast if we decrease the diffusivity for $\alpha = 0$ as well as $\alpha = 1$. Fig. 4.3 and Fig. 4.4 show variations of trajectories of interface $s(t)$ for different values of c at $\alpha = 0$ and $\alpha = 1$, respectively. It is clear from Figs. 4.3 and 4.4 that the freezing/ melting process becomes fast if we increase the value of c for both the cases $\alpha = 0$ as well as $\alpha = 1$. Figs. 4.1-4.4 show that the movement of interface increases as the value of α increases, i.e. the freezing/melting process becomes fast with the increment in the value of α.

c	t	$s_N(t)$	$s_E(t)$	$Absolute Error$	$Relative Error$
	0.1	0.0152591	0.0158064	5.4×10^{-4}	3.4×10^{-2}
	0.2	0.162679	0.160212	2.40×10^{-3}	1.50×10^{-2}
0.5	0.3	0.113274	0.109579	3.60×10^{-3}	3.30×10^{-2}
	0.4	0.064106	0.059189	4.90×10^{-3}	8.30×10^{-2}
	0.5	0.015176	0.009037	6.10×10^{-3}	6.70×10^{-2}
	0.1	0.641957	0.637125	4.80×10^{-3}	7.50×10^{-3}
	0.2	0.542968	0.533223	9.70×10^{-3}	1.80×10^{-2}
1.0	0.3	0.444652	0.430042	1.40×10^{-2}	3.30×10^{-2}
	0.4	0.347007	0.327569	1.90×10^{-2}	5.90×10^{-2}
	0.5	0.250031	0.225792	2.40×10^{-2}	1.00×10^{-1}
	0.1	1.213060	1.202430	1.00×10^{-2}	8.80×10^{-3}
	0.2	1.064920	1.043280	2.10×10^{-2}	2.00×10^{-2}
1.5	0.3	0.918012	0.885505	3.20×10^{-2}	3.60×10^{-2}
	0.4	0.772339	0.729075	4.30×10^{-2}	5.90×10^{-2}
	0.5	0.627896	0.573966	5.30×10^{-2}	9.30×10^{-2}

TABLE 4.1: Comparison between exact value $s_E(t)$ and numerical value $s_N(t)$ of moving interface at $\alpha = 0, v = 2.0, k = 1.0, \gamma = 20$.

c	t	$s_N(t)$	$s_E(t)$	$AbsoluteError$	$RelativeError$
0.5	0.1	0.212321	0.211090	1.20×10^{-3}	5.80×10^{-3}
	0.2	0.162679	0.160212	2.40×10^{-3}	1.50×10^{-2}
	0.3	0.113274	0.109579	3.60×10^{-3}	3.30×10^{-2}
	0.4	0.064106	0.059189	4.90×10^{-3}	8.30×10^{-2}
	0.5	0.015176	0.009037	6.10×10^{-3}	6.70×10^{-2}
1.0	0.1	0.641957	0.637125	4.80×10^{-3}	7.50×10^{-3}
	0.2	0.542968	0.533223	9.70×10^{-3}	1.80×10^{-2}
	0.3	0.444652	0.430042	1.40×10^{-2}	3.30×10^{-2}
	0.4	0.347007	0.327569	1.90×10^{-2}	5.90×10^{-2}
	0.5	0.250031	0.225792	2.40×10^{-2}	1.00×10^{-1}
1.5	0.1	1.213060	1.202430	1.00×10^{-2}	8.80×10^{-3}
	0.2	1.064920	1.043280	2.10×10^{-2}	2.00×10^{-2}
	0.3	0.918012	0.885505	3.20×10^{-2}	3.60×10^{-2}
	0.4	0.772339	0.729075	4.30×10^{-2}	5.90×10^{-2}
	0.5	0.627896	0.573966	5.30×10^{-2}	9.30×10^{-2}

TABLE 4.2: Comparison between exact value $s_E(t)$ and numerical value $s_N(t)$ of moving interface at $\alpha = 1, v = 1.0, k = 1.0, \gamma = 20$.

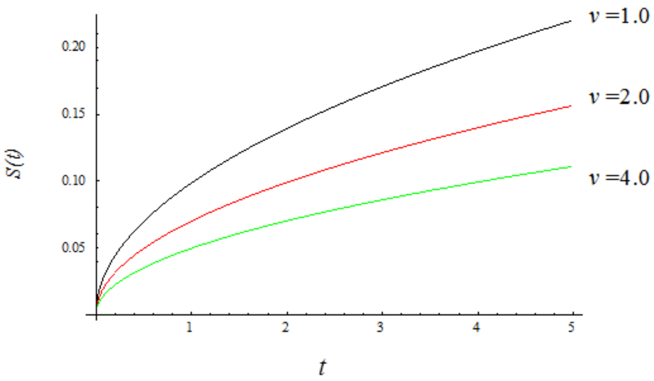

FIGURE 4.1: Plot of $s(t)$ vs t at $\alpha = 0, c = 1.0, k = 1.0, \gamma = 20$.

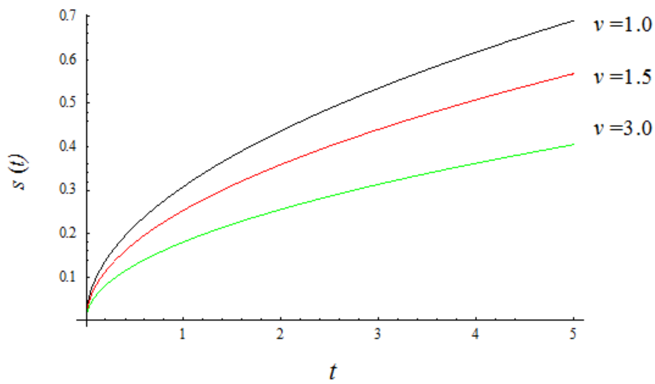

FIGURE 4.2: Plot of $s(t)$ vs t at $\alpha = 1, c = 1.0, k = 1.0, \gamma = 20$.

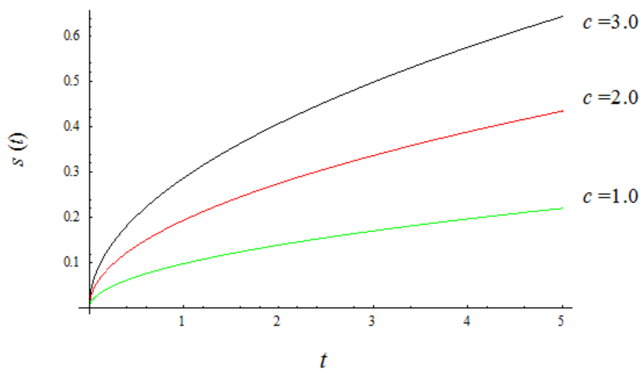

FIGURE 4.3: Plot of $s(t)$ vs t at $\alpha = 0, v = 1.0, k = 1.0, \gamma = 20$.

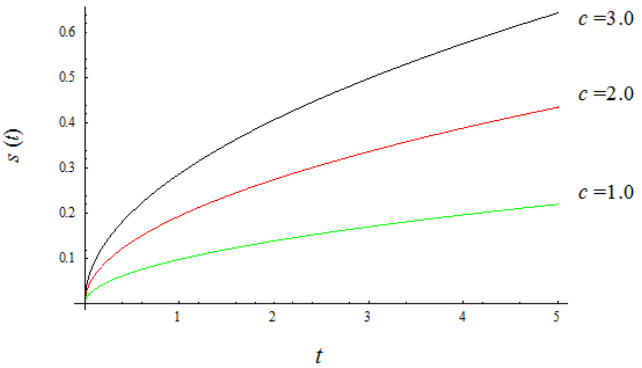

FIGURE 4.4: Plot of $s(t)$ vs t at $\alpha = 1, v = 1.0, k = 1.0, \gamma = 20$.

4.5 Conclusion

In this chapter, an approach based on operational matrix of differentiation of shifted second kind Chebyshev wavelets is successfully applied to a moving boundary problem involving variable latent heat term (a power function of position). It is seen that this approach provides a simple and accurate algorithm for the moving boundary problems in order to understand the physical behavior of dependent variables involved in the process. Moreover, the proposed method is computer intensive. Therefore, this approach is helpful for the researchers who are working in this area. It is also found that the movement of interface increases if we increase the values of α and constant c. However, the movement of interface decreases with the increment in the value of diffusivity.

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