Chapter 1

Introduction

1.1 Moving Boundary Problem

Problems in which the solution of the differential equation has to satisfy certain conditions given on the boundary of a considered domain are termed as the boundary value problems. In many important circumstances, the boundary of the considered domain is not known beforehand but it has to be determined as a part of the solution of the problem. These problems are called moving boundary problems. These problems arise in many important aspects of science and engineering. Examples include production of ice, solidification of metals, food conservation, crystal growth, casting of metals. All these have either a moving freezing or moving melting phase front which is not known prior. The unknown phase front (moving interface) is called the moving interface (e.g., the liquid-solid interface separating the liquid and solid regions in course of melting or freezing process).

In the moving boundary problems, the solution of partial differential equation (heat equation for phase change problem) in an unknown domain is required. Moreover, the unknown domain has to be calculated which is a part of the solution. A few exact solutions to the moving boundary problems are available in the literature. In case of moving boundary problems involving phase change processes, the similarity exact solutions exist when the location of moving interface changes as the sqaure root of time. Whenever, there is a difficulty for getting the exact solution of the problem then the researcheres try to apply different approximate techniques/numerical approaches for the solution of the problem. The mathematical development of this area proceeded in three main dimensions, numerical methods, approximate methods and the exact solutions with existence and uniqueness.

Moving boundary problems which invlove the phase change phenomena are also known as the Stefan problems. The name Stefan problem is due to the early and formal consideration of the problem by Stefan (1889) who published four research papers on the problems involving unknown moving interface in the same year. The first and third papers demonstrate heat conduction and phase change phenomena while the second and the fourth describe diffusion of material in a reaction zone and evaporation or condensation. Stefan (1891) published his fifth paper on the problem with the same title as his third paper had. As a result of these early works, problem involving phase change and moving interface of separation between phases are now classified as 'Stefan problem'.

This is despite the fact that Lame and Clapeyron (1831) actually were the first to study the problem of determining the thickness of the solid crust generated by the cooling of a liquid under a constant surface temperature. These authors discovered that the location of moving interface is proportional to the square of time and found the exact solution to the one phase Stefan problem. But, they did not determine the numerical value of the proportionality constant as mentioned in Rubinstein (1971). But, nowadays, this solution are usually credited to Franz Neumann because he delivered these solutions in his unpublished lectures in Konigsberg in 1860 (Weber, 1919).

1.2 Historical Background

In the 19th century, the physicist Gabriel Lame (1795-1850) and the mechanical engineer Emile Clapeyron (1799-1864) mathematically coupled the concept of latent heat with the heat conduction equation, during extending Fourier's work on the estimate of the time elapsed since the Earth began to solidify from its initial molten state. They initially assumed the Earth to be in a liquid phase at the melting temperature (a one-phase problem). From the literature survey, it is found that Lame and Claperon (1831) were the first who considered the problem of determing the thickness of the solid crust generated by the cooling of a liquid under a constant surface temperature. They established the exact similarity solution of the problem and explored that the thickness of solid crust is directly proportional to the square root of time but could not determine the proportionality constant as pointed out by Brillouin (1931). Lame and Claperon (1831) were certianly the first authors to derive this exact similarity solution. However, credit goes to Franz Neumann due to his lectures delivered in the beginning of 1860 (Weber, 1919) for this similarity solution and the more general solution derived by Stefan. Almost 60 years later in 1889, this question was picked up and stated in a more formal form by the Austrian physicist and mathematician Joseph Stefan. Stefan published four research papers (Stefan, 1889a, 1889b, 1889c, 1889d) describing mathematical models for real physical problems with a change of phase state. This was the first general study of this type of problem, since then moving boundary problems are called Stefan problems. Among the four published research papers, it was the one about ice formation in the polar seas that has drawn the most attention. The given mathematical solution was actually found earlier by the German physicist and mathematician Franz Ernst Neumann in 1860. It is called the Neumann solution (see Weber 1919).

Neumann's solutions are significant because of two reasons. firstly, in spite of almost one hundred years of research on the Stefan problems, the solutions and numerous generaliztions remain the only physically interesting exact solutions. Few other exact solutions have existence but they do not solve the problems of any genuine practical interest. Secondly, these solutions signify the limit to which extent these have been used in perspective of different physical phenomena, may involve multi-phases or multi-component mixtures and even changes in material properties like diffusivities and densities. The common things to all these different physical situations are a slab geometry and infinite in extension with phase change moving interfaces moving as per law of square root time.

There was no remarkable publication devoted to the problem from 1891 to 1930. In 1931, Leibenzon (1931) presented a method of approximation to solve the Stefan problem arised in mechanics in the oil industry. There are many important books texted on the subject during the development of the problem. One of the informative book on the free and moving boundary problem is the recent text by Crank (1984). This text provides a definitive and broad discussion on the subject and formulates many free and moving boundary problems. This book also comprise many existing approximate and numerical methods for the solution of such types of problems. Crank (1964) generalize the known results and literature mainly having diffusion process with a moving interface. Crank (1964) and carslaw and Jaeger (1965) provides an infortmative content on moving boundary problems. carslaw and Jaeger (1965) explore the subject involving heat conduction with changes of state and provide a compact form of many exact solutions available in the literatures. this text also contains many important references and generalize variuos Neumann,s solutions to the moving boundary problems. Rubinstein (1971) presented the broad discussion of the underlying physical problems. The first four chapters of this book give extremely important background leading to the subject. Flemings (1974) presents the basics of freezing phenomena from a practical as well as physical point of view. This book demonstrates the fundamental scientific laws involving heat conduction, mass transfer and moving boundary leading to the freezing process. Two important books pertaining to numerical approach to moving boundary problems are Lewis and Morgan (1979) and Albrecht et al. (1982). Elliott and Ockendon (1982) is an essential book on numerical methods and it also contains over two hundred relevant references.

There are many important proceedings of conferences on moving boundary problems such as Ockendon and Hodgkins (1975) and Wilson et al. (1975). These works are beneficial to gain some understandings of the range and variety of the contemporary activity in this area. Chalmers (1954) presented a study of melting and solidification with specific reference to the structure of solidified pure metals and alloys. Many relevant review articles in the area of moving boundary problems are well presented. Some are Bankoff (1964), Muehlbauer and Sunderland (1965) and Goodmann (1964). The first two discuss the heat conduction process involving phase change of material, whereas the third describes the applicability of integral methods to solve transient nonlinear heat transfer problem with a change of phase. Furzeland (1977) discusses numerical methods to find the solution of both free and moving boundary problems. Rubinstein (1979) focuses on many unsolved dimensions of moving boundary problems by reviewing the subject. Some Surveys on the subject related to moving boundary problem are well written by Cohen (1971), Boley (1972) and Mori and Araki (1976).

In all aspects of science and engineering, a vast literature on heat conduction with moving boundary problems has progressed. Mathematical development of the field have risen in three main dimensions, approximate techniques, numerial approach and establishment of exact solutions such as existence and uniqueness. One of the most remarkable numerical approaches, to find the solution of Stefan problem, is credited to Moiseennko and Samarskii (1965), who applied the generalized problem setting considered by Oleinik (1960). During the passage of evolution of the subject, many authors utilized the finite element method to solve the Stefan problems (Mori, 1977; Bonnerot and Jamet, 1981; Nochetto et al., 1991).

There are several techniques which have been used for the solution of the moving boundary problems. Numerical methods for moving boundary problems mostly depends on the finite-difference and finite element methods. Various numerical methods for the problem are used by Furzeland (1977), Dalhuijsen and Segal (1986), Voller (1990), Asaithambi (1997), Savovic and Caldwell (2003). Some frequently applied numerical methods are isotherm migration method (Chernousko, 1970; Turland and Wilson, 1977; Durak and Wendroff, 1977) and enthalpy method (Voller, 1987; Voller, 1996; Krabbenhoft et al., 2006; Gudibande and Iyer, 2013).

1.3 The Shifted Chebyshev Polynomials and its Properties

We define the first kind Chebyshev polynomials, denoted by $T_n(t),\ n = 1, 2, 3, ...,$ by following reccurence formula

$$T_{n+1}(t) = 2t\, T_n(t) - T_{n-1}(t), \quad n = 1, 2, 3, ..., \tag{1.1}$$

where $T_0(t) = 1$ and $T_1(t) = t$. These polynomials form an orthogonal set over the interval $[-1, 1]$ with respect to a suitable weight function.

Now, we substitute $t = \left(\frac{2x}{\lambda} - 1\right)$ in the Chebyshev polynomials to use these polynomials on the interval $[0, \lambda]$. This substitution changes the Chebyshev polynomials to the polynomials $T_i\left(\frac{2x}{\lambda} - 1\right)$, $i = 1, 2, 3, ...,$ which are known as the shifted Chebyshev polynomial denoted by $T_{\lambda,i}(x)$. From the recurrence relation (1.1), we can find the desired number of shifted Chebyshev polynomials explicitly.

According to Doha et al. (2011), any square integrable function $f(x)$ on the interval $[0, \lambda]$ can be written as

$$f(x) = \sum_{j=0}^{\infty} c_j T_{\lambda,j}(x) \tag{1.2}$$

where c_j are given by

$$c_j = \frac{1}{h_j} \int_0^{\lambda} f(x) T_{\lambda,j}(x) w_{\lambda}(x) dx, \quad j = 0, 1, 2, ..., \tag{1.3}$$

where h_j and $w_{\lambda}(x)$ are defined by

$$h_0 = \pi, \ h_j = \frac{\pi}{2}, \ j = \ 1, \ 2, \ ..., \text{ and } w_{\lambda}(x) = \frac{1}{\sqrt{\lambda x - x^2}},$$

respectively.

To approximate the function $f(x)$, we can take first $(N+1)$ terms of the series (1.2) and $f(x)$ may be denoted by $f_N(x)$ that is given by

$$f_N(x) = \sum_{j=0}^{N} c_j T_{\lambda,j}(x) = C\phi(x) \tag{1.4}$$

where the vector C and the shifted Chebyshev vector $\phi(x)$ are given by

$$C = [c_0, c_1, ..., c_N]$$

and

$$\phi(x) = [T_{\lambda,0}(x), T_{\lambda,1}(x), ..., T_{\lambda,N}(x)]^T.$$

As given in Atabakzadeh et al. (2013), the first derivative of vector $\phi(x)$ is given by

$$\frac{d\phi(x)}{dx} = D\phi(x) \tag{1.5}$$

where the matrix D is a square matrix of order $(N+1)$ defined by

$$D = (d_{ij}) = \begin{cases} \frac{4i}{\delta_j \lambda}, & j = i-k, \\ 0, & otherwise, \end{cases} \tag{1.6}$$

where $k = 1, 3, ..., N$ if N is odd or $k = 1, 3, ..., N-1$ if N is even and $\delta_0 = 2,\ \delta_k = 1,\ k \geq 1$.

The higher order derivatives of the vector $\phi(x)$ can be given by

$$\frac{d^n\phi(x)}{dx^n} = D^n\phi(x) \tag{1.7}$$

where D^n represents the n-fold multiplication of the matrix D.

1.4 The Shifted Legendre Polynomials and its Operational Matrix of Differentiation

The well-known Legendre polynomials $L_i(x), i = 0, 1, 2, ...,$ are orthogonal on the interval $[-1, 1]$. To make these polynomials orthogonal on the interval $[a, b]$, we define the shifted Legendre polynomials $L_i^*(x)$ as

$$L_i^*(x) = L_i\left(\frac{2x-a-b}{b-a}\right), \quad i = 0, 1, 2, \ldots . \tag{1.8}$$

The shifted Legendre polynomials $L_i^*(x)$ satisfy the following recurrence formula:

$$(i+1)L_{i+1}^*(x) = (2i+1)\left(\frac{2x-a-b}{b-a}\right)L_i^*(x) - iL_{i-1}^*(x), \quad i = 1, 2, 3, ..., \tag{1.9}$$

where $L_0^*(x) = 1$ and $L_1^*(x) = \left(\frac{2x-a-b}{b-a}\right)$.

In this thesis, the following properties of the shifted Legendre polynomials (Abd-Elhameed et al., 2015) are used:

(**1**) Let us first define a space

$$L_0^2[a,b] = \left\{\phi(x) \in L^2[a,b] : \phi(a) = \phi(b) = 0\right\} \tag{1.10}$$

and we select the following basis functions in the Hilbert space $L_0^2[a,b]$:

$$\phi_j(x) = (x-a)(x-b)L_j^*(x), \quad j = 0, 1, 2, \ldots . \tag{1.11}$$

Now, the function $u(x) \in L_0^2[a,b]$ can be written as:

$$u(x) = \sum_{j=0}^{\infty} c_j \phi_j(x), \tag{1.12}$$

where the constants c_j are given below:

$$c_j = \frac{2j+1}{b-a}\int_a^b u(x)\phi_j(x)w(x)dx, \quad j = 0, 1, 2, ..., \tag{1.13}$$

and the weight function $w(x) = \frac{1}{(x-a)^2(x-b)^2}$.

In numerical calculation, the series given in Eq. (1.12) for the function $u(x)$ can be approximated as

$$u(x) \approx u_N(x) = \sum_{j=0}^{N} c_j \phi_j(x) = \mathbf{C}^T \boldsymbol{\phi}(x), \tag{1.14}$$

where $\boldsymbol{C}^T$ represents the transpose of the coefficient vector and $\boldsymbol{\phi}(x)$ is the shifted Legendre vector which are given by

$$\mathbf{C}^T = [c_0, c_1, ..., c_N] \text{ and } \boldsymbol{\phi}(x) = [\phi_0(x), \phi_1(x), ..., \phi_N(x)] \tag{1.15}$$

(2) The derivative of $\boldsymbol{\phi}(x)$ in matrix form is given as:

$$\frac{d\boldsymbol{\phi}(x)}{dx} = D\boldsymbol{\phi}(x) + \boldsymbol{\delta}, \tag{1.16}$$

where $D = (d_{ij})_{0 \le i,j \le N}$ represents the operational matrix of order $(N+1)$ and its elements are given by

$$d_{ij} = \begin{cases} \frac{2}{b-a}(2j+1)(1+2H_i-2H_j), & i > j, \ (i+j) \ \textit{is odd}, \\ 0, \quad \textit{otherwise}, \end{cases} \tag{1.17}$$

and

$$\boldsymbol{\delta} = [\delta_0(x), \delta_1(x), ..., \delta_N(x)]^T, \tag{1.18}$$

$$\delta_i(x) = \begin{cases} a+b-2x, & \textit{when } i \textit{ is even}, \\ a-b, & \textit{when } i \textit{ is odd}. \end{cases} \tag{1.19}$$

In Eq. (1.17), H_i and H_j are harmonic numbers which are defined as

$$H_n = \sum_{k=1}^{n} \frac{1}{k} \quad \textit{with } H_0 = 0. \tag{1.20}$$

(3) The relation between second order derivative of $\boldsymbol{\phi}(x)$ and the operational matrix D is given by

$$\frac{d^2\boldsymbol{\phi}(x)}{dx^2} = D^2\boldsymbol{\phi}(x) + D\boldsymbol{\delta} + \boldsymbol{\delta}', \tag{1.21}$$

where

$$\boldsymbol{\delta}' = [\delta_0'(x), \delta_1'(x), ..., \delta_N'(x)]^T \text{ and } \delta_i'(x) = \begin{cases} -2, & \textit{when } i \textit{ is even}, \\ 0, & \textit{when } i \textit{ is odd}. \end{cases} \tag{1.22}$$

1.5 Some Properties of Shifted Second Kind Chebyshev Wavelets

In wavelets theory, we expand functions in terms of trigonometric polynomials as well as wavelets. Wavelets are a family of functions which are generated in the form of dilation and translation of a fixed function and this fixed function is known as mother wavelet. The following family of continuous wavelets can be established if the dilation parameter and the translation parameter changes continuously (Abd-Elhammed et al., 2013; Heydari et al., 2014; Zhou and Xu, 2014):

$$\psi_{a,b}(t) = |a|^{-\frac{1}{2}}\psi\left(\frac{t-b}{a}\right), \quad a, b \in \mathbb{R}, a \neq 0, \tag{1.23}$$

where a and b are the dilation parameter and the translation parameter, respectively. Chebyshev wavelets of second kind $((\psi_{nm})(t) = \psi(k, n, m, t))$ contain four arguments k, n, m, t, where k is any positive integer, $n = 0, 1, 2, ..., 2^k - 1, m$ is the order of second kind Chebyshev polynomials $(m = 0, 1, 2, ..., M)$ and t is the normalized time.

Second kind Chebyshev wavelets are defined on the interval $[0, 1]$ as:

$$\psi_{nm}(t) = \begin{cases} \frac{2^{\frac{k+3}{2}}}{\sqrt{\pi}} U_m^*(2^k t - n), & t \in \left[\frac{n}{2^k}, \frac{n+1}{2^k}\right], \\ 0, & otherwise. \end{cases} \tag{1.24}$$

where $U_m^*(x)$ is shifted second kind Chebyshev polynomials which are defined on $[0, 1]$ as:

$$U_m^*(x) = U_m(2x - 1) \tag{1.25}$$

in which $U_m(x)$ denotes the second kind Chebyshev polynomials defined on $[-1, 1]$ and

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad x = \cos\theta. \tag{1.26}$$

In this thesis, the following properties of second kind Chebyshev wavelets (Abd-Elhammed et al., 2013; Heydari et al., 2014; Zhou and Xu, 2014; Rajeev and Raigar, 2015) are used:

(1) In terms of shifted second kind Chebyshev wavelets, a function $f(t)$ defined over $[0, 1]$ may be written as:

$$f(t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C_{nm}\psi_{nm}(t) \tag{1.27}$$

where

$$C_{nm} = < f(t), \psi_{nm}(t) > = \int_0^1 \sqrt{1 - t^2}\, f(t)\psi_{nm}(t)dt \tag{1.28}$$

If the infinite series of Eq. (1.27) is truncated, then the function $f(t)$ can be written as:

$$f(t) \approx \sum_{n=0}^{2^k-1} \sum_{m=0}^{M} C_{nm}\psi_{nm}(\xi) = C\psi(\xi) \tag{1.29}$$

where C is $1 \times 2^k(M+1)$ and $\psi(t)$ is $2^k(M+1) \times 1$ matrices which are defined as:

$$C = [c_{0,0}, c_{0,1}, c_{0,2}, ..., c_{0,M}, ..., c_{2^k-1,0}, c_{2^k-1,1}, ..., c_{2^k-1,M}], \tag{1.30}$$

and

$$\psi(t) = [\psi_{0,0}, \psi_{0,1}, \psi_{0,2}, ..., \psi_{0,M}, ..., \psi_{2^k-1,0}, \psi_{2^k-1,1}, ..., \psi_{2^k-1,M}]^T, \tag{1.31}$$

respectively.

(2) First derivative of the second kind Chebyshev wavelets vector $\psi(t)$ can be written as:

$$\frac{d\psi(t)}{dt} = D\psi(t) \tag{1.32}$$

where D is $2^k(M+2)$ square operational matrix of derivative of shifted second kind Chebyshev wavelets (Abd-Elhameed, 2013) and is defined as:

$$D = \begin{bmatrix} G & 0 & \dots & 0 \\ 0 & G & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & G \end{bmatrix} \tag{1.33}$$

where G is an $(M+1)$ square matrix and its $(r,s)th$ element is given by

$$G_{r,s} = \begin{cases} 2^{k+2}s, & r \geq 2,\ r > s,\ (r+s) \text{ is odd}, \\ 0, & \text{otherwise}. \end{cases} \tag{1.34}$$

(3) For the nth derivative of second kind Chebyshev wavelets vector, we have following equation:

$$\frac{d^n\psi(t)}{dt^n} = D^n\psi(t), \quad n = 1, 2, ..., \tag{1.35}$$

where D^n is the operational matrix of differentiation for the nth derivative and it is the nth power of the matrix D which is given in Eq. (1.33).

1.6 Homotopy Analysis Method

The homotopy analysis method is an approximate analytic approach applied to attain the series solutions of various problems such as algebraic equations, ordinary differential equations, partial differential equations and differential integral equation. This method is introduced by Liao (1992) in his Ph.D. dessertation. The homotopy analysis method uses an important notion, the homotopy from the topology to find the series solution of the considered problem. Later in 1997, this method is modified by Liao (1999) to introduce a nonzero auxiliary parameter, c_0, termed as the convergence control parameter to construct a homotopy for the problem. The convergence control parameter gives a simple way to check and enforce convergence of the solution series.

In order to describe the basic idea of homotopy analysis method, let us consider the following differential equation

$$N[u(x,t)] = 0, \tag{1.36}$$

where N is a nonlinear operator, x and t are the independent variables and $u(x,t)$ is an unknown function. For simplicity, we ignore all boundary or initial conditions, which can treated in a similar way. By means of generalizing the traditional homotopy method, Liao (1999) constructs the so-called zero-order deformation equation

$$(1-q)L[\phi(x,t;q) - u_0(x,t)] = qc_0H(x,t)N[\phi(x,t;q)] \tag{1.37}$$

where $q \in [0, 1]$ is the embedding parameter, L is an auxiliary linear operator, $c_0 \neq 0$ is an auxiliary parameter, $\phi(x, t; q)$ is an unknown function, $u_0(x, t)$ is an initial approximation and $H(x, t)$ represents a nonzero auxiliary function.
It is clear that when the parameter $q = 0$ and $q = 1$, the Eq. (1.37) comes out

$$\phi(x, t; 0) = u_0(x, t), \quad \phi(x, t; 1) = u(x, t),$$

respectively. This implies that when q varies fron 0 to 1, the solution increases from initial approximatiom $u_0(x, t)$ to the solution $u(x, t)$. Assuming that the unknown function $\phi(x, t; q)$ has Taylor series expansion with respect to q, we have

$$\phi(x, t; q) = u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t) q^m, \tag{1.38}$$

where

$$u_m(x, t) = \frac{1}{m!} \frac{\partial^m \phi(x, t; q)}{\partial q^m} \bigg|_{q=0}.$$

The convergence of the series (1.38) depends upon the auxiliary parameter c_0. When the series (1.38) converges for $q = 1$, one obtain

$$\phi(x, t; 1) = u(x, t) = u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t), \tag{1.39}$$

which must satisfy the original equation (1.36), as proven by Liao (2009). Consider the vector

$$\vec{u_n} = (u_0(x, t), u_1(x, t), ..., u_n(x, t)).$$

Differentiating the zeroth-order deformation equation (1.37) m-times with respect to q and then dividing them by $m!$ and finally putting $q = 0$, we get the m-th order deformation equation

$$L\left[u_m(x,t) - \chi_m u_{m-1}(x,t)\right] = c_0 \Re_m(\vec{u_{m-1}}), \tag{1.40}$$

where

$$\Re_m(\vec{u_{m-1}}) = \frac{1}{m!}\frac{\partial^{m-1} N[\phi(x,t;q)]}{\partial q^{m-1}}\bigg|_{q=0},$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases}$$

It should be noted that $u_m(x,t)$ for $m \geq 1$ is obtained by the linear equation (1.40) after imposing the boundary conditions that comes out from the original problem, which can be solved easily. If the Eq. (1.36) has unique solution, then this technique will provide the unique solution. When Eq. (1.36) does not admit unique solution, the homotopy analysis method will produce a single solution from many other existing solutions.

1.7 Homotopy Perturbation Method

In many important physical situations, the closed form solutions to the problems are not available and it also very difficult to find the closed form solution to the problem even in very simple cases. Therefore, in recent years, many approximate approaches were utilized to find the solutions of wide variety of linear and nonlinear problems. Homotopy Perturbation method introduced by He (1999) is one of the approximate method for solving linear and nonlinear differential as well as integral equations. this method, which couples the traditional perturbation method and homotopy in topology, deforms difficult problem continuously to a simple problem which is easily solvable.

To describe the basic idea of this method, let us consider the following differential equation

$$L(u)+N(u)=f(r), \quad r\in\Omega \tag{1.41}$$

with the boundary conditions

$$B\left(u,\frac{\partial u}{\partial n}\right)=0, \quad r\in\Gamma, \tag{1.42}$$

where L is a linear operator,N is a nonlinear operator, B is a boundary operator, Γ is the boundary of the domain Ω and $f(r)$ is a known analytic function.

By the homotopy perturbation method, He (2003, 2004, 2005) construct a homotopy $V(r,p):\Omega\times[0,1]\longrightarrow R$ for Eq. (1.41) as:

$$H(v,p)=(1-p)\left[L(v)-L(u_0)\right]+p\left[L(v)+N(v)-f(r)\right]=0, \tag{1.43}$$

or

$$H(v,p)=L(v)-L(u_0)+pL(u_0)+p\left[N(v)-f(r)\right]=0, \tag{1.44}$$

where $r\in\Omega$, $p\in[0,1]$ is an embedding parameter and u_0 is an initial approximation which verifies the boundary conditions.

It is clear from the Eq. (1.44) that when $p=0$, it becomes

$$L(v)-L(u_0)=0,$$

and if $p=1$, then Eq. (1.44) comes out as

$$L(v)+N(v)=f(r).$$

This shows that when the embedding parameter p changes from 0 to 1, $v(r,p)$ varies from $u_0(x)$ to $u(x)$.

The basic assumption is that the solution of Eq. (1.44) can be written as a power series

$$v = v_0 + pv_1 + p^2 v_2 + \dots \,. \tag{1.45}$$

Hence, the approximate solution of the Eq. (1.44) can be obtained as

$$u = \lim_{p \to 1} \; v = v_0 + v_1 + v_2 + \dots \,. \tag{1.46}$$

1.8 Caputo Fractional Derivative

Fractional derivative are used in many area of research. There are several fractional derivatives such as Riemann-Liouville fractional derivative, Caputo fractional derivative, Riesz fractional derivative, Gruenwald-Letnikov fractional derivative which are used very frequently in many aspects of research. Caputo fractional derivative is utilized in the last two chapters of this thesis. The caputo fractional derivative suits the most for the problem of real applications (Podlubny (1999), Rajeev et al. (2013)). This fractional derivative is first introduced by Italian mathematician M. Caputo in 1967 (Caputo (1967)).

Definition 1.8.1. *Let $\alpha > 0$, $x > a$, $\alpha, a, x \in \mathbb{R}$. The Caputo fractional derivative of order α, denoted by D_x^α, of the function $f(x)$ is defined by*

$${}_aD_x^\alpha f(x) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(\tau)}{(x-\tau)^{\alpha-n+1}} d\tau, & n-1 < \alpha < n, \;\; n \in \mathbb{N} \\ \frac{d^n f(x)}{dx^n}, & \alpha = n, \;\; n \in \mathbb{N}. \end{cases} ,$$

where a is the base point of the caputo fractional derivative, $\Gamma(.)$ *denotes the gamma function which is defined as*

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, \quad x > 0, \ x \in \mathbb{R}.$$

This definition is extended over the real numbers $\mathbb{R}$ excluding zero and negative integers by using the following property of gamma function

$$\Gamma(x+1) = x\Gamma(x), \quad x > 0.$$

In this thesis, the following properties of Caputo fractional derivative are used

(1) $D_x^\alpha C = 0, \quad C$ is a constant,

(2) $D_x^\alpha x^p = \frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)} x^{p-\alpha}$,

(3) $D_x^\alpha \left(D_x^\beta f(x)\right) = D_x^{\alpha+\beta} f(x)$.

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