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# Application of new strongly convergent iterative methods to split equality problems

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**Abstract**
In this paper, we study the generalized problem of split equality variational inclusion problem. For this purpose, we introduced the problem of finding the zero of a nonnegative lower semicontinuous function over the common solution set of fixed point problem and monotone inclusion problem. We proposed and studied the convergence behaviour of different iterative techniques to solve the generalized problem. Furthermore, we study an inertial form of the proposed algorithm and compare the convergence speed. Numerical experiments have been conducted to compare the convergence speed of the proposed algorithm, its inertial form and already existing algorithms to solve the generalized problem.

**Keywords** Split equality problem · Variational inclusion problem · Fixed point problem · Quasi-nonexpansive mapping

**Mathematics Subject Classification** 47J25 · 47H05 · 47H09 · 49J53

## 1 Introduction

In 1994, Censor and Elfving (Censor and Elfving 1994) first introduced the split feasibility problem (SFP) in finite-dimensional spaces. Such problems arise in signal processing, specif-

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ically in phase retrieval and other image restoration problems. It has been found that the SFP can also be used in different areas such as computer tomography and intensity-modulated radiation therapy (Censor et al. 2005, 2006, 2007).

The split feasibility problem (SFP) is

$$\text{find } x^* \in C \text{ such that } Ax^* \in Q, \tag{1.1}$$

where $C$ and $Q$ are nonempty closed convex subsets of real Hilbert spaces $H_1$ and $H_2$, respectively, and $A : H_1 \to H_2$ is a bounded linear operator. Some works on split feasibility problems in an infinite-dimensional real Hilbert space can be found in Byrne (2002), Censor et al. (2006) and Xu (2006).

In 2012, Censor et al. (2012) introduced the following split variational inequality problem:

$$\text{find } \ x^* \in C \text{ such that } \langle f(x^*), x - x^* \rangle \geq 0 \text{ for all } x \in C,$$

and

$$y^* = Ax^* \in Q \text{ that solves } \langle g(y^*), y - y^* \rangle \geq 0 \text{ for all } y \in Q,$$

where $C$ and $Q$ are nonempty closed convex subsets of real Hilbert spaces $H_1$ and $H_2$, respectively, $A : H_1 \to H_2$ is a bounded linear operator and $f : H_1 \to H_1$, $g : H_2 \to H_2$ are the given operators.

In 2011, Moudafi (2011) extended the split variational inequality problem (Censor et al. 2012) and proposed the following split monotone variational inclusion problem (SMVIP):

$$\text{find } \ x^* \in H_1 \text{ such that } f(x^*) + B_1(x^*) \ni 0,$$

and

$$y^* = Ax^* \in H_2 \text{ that solves } g(y^*) + B_2(y^*) \ni 0, \tag{1.2}$$

where $B_i : H_i \to 2^{H_i}$, for $i = 1, 2$, are multi-valued mappings on the real Hilbert spaces, $A : H_1 \to H_2$ is a bounded linear operator and $f : H_1 \to H_1$, $g : H_2 \to H_2$ are two given single-valued operators. Also, an algorithm for finding the solution of SMVIP (1.2) was introduced and the weak convergence of the proposed algorithm was proved.

In 2014, Kazmi and Rizvi (2014) introduced the split variational inclusion problem **(SVIP)**:

$$\text{find } x^* \in H_1 \text{ such that } B_1(x^*) \ni 0,$$

and

$$y^* = Ax^* \in H_2 \text{ that solves } B_2(y^*) \ni 0, \tag{1.3}$$

where $B_i : H_i \to 2^{H_i}$, for $i = 1, 2$, are multi-valued mappings on the real Hilbert spaces and $A : H_1 \to H_2$ is a bounded linear operator. Problem (1.3) is a special case of split monotone variational inclusion problem. They also proposed strongly convergent iterative method to find the common solution of split variational inclusion problem and fixed point problem.

In 2013, Moudafi (2013) introduced the following split equality problem (SEP):

$$\text{find } x^* \in C \text{ and } y^* \in Q \text{ such that } Ax^* = By^*, \tag{1.4}$$

where $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are two bounded linear operators and $C$, $Q$ are nonempty closed convex subsets of real Hilbert spaces $H_1$, $H_2$, respectively, and $H_3$ is also a Hilbert space. Obviously, if $B = I$ and $H_2 = H_3$, then SEP reduces to SFP.

In 2014, Moudafi (2014) introduced the following split equality fixed point problem (SEFP):

$$\text{find } x^* \in \text{Fix}(R_1) \text{ and } y^* \in \text{Fix}(R_2) \text{ such that } Ax^* = By^*, \tag{1.5}$$

where $A : H_1 \to H_3$, $B : H_2 \to H_3$ are two bounded linear operators, and $R_i : H_i \to H_i$ for $i = 1, 2$ are two nonlinear operators such that $\text{Fix}(R_1) \neq \emptyset$ and $\text{Fix}(R_2) \neq \emptyset$. Also, he proposed iterative method for solving SEFP:

$$\begin{cases} x_{n+1} = R_1(x_n - \gamma_n A^*(Ax_n - By_n)), \\ y_{n+1} = R_2(y_n + \gamma_n B^*(Ax_{n+1} - By_n)) \ \forall n > 0, \end{cases}$$

where $\{\gamma_n\}$ is a positive non-decreasing sequence such that $\gamma_n \in \left(\epsilon, \min\left(\frac{1}{\lambda_A}, \frac{1}{\lambda_B}\right) - \epsilon\right)$ for small enough $\epsilon > 0$, where $\lambda_A$ and $\lambda_B$ denotes the spectral radius of $A^*A$ and $B^*B$, respectively. In this iterative method, computation of the norm of operators used is required, which can be tedious task sometimes.

In 2015, to solve the split equality fixed point problem (1.5) for quasi-nonexpansive mappings, Zhao (2015) proposed the following iteration algorithm which does not require the computation of the operator norms:

$$\begin{cases} u_n = x_n - \gamma_n A^*(Ax_n - By_n), \\ x_{n+1} = \alpha_n u_n + (1 - \beta_n) R_1 u_n, \\ v_n = y_n + \gamma_n B^*(Ax_n - By_n), \\ y_{n+1} = \beta_n v_n + (1 - \beta_n) R_2 v_n, \quad \forall n \geq 0, \end{cases}$$

where the step-size $\gamma_n$ is chosen as follows:

$$\gamma_n \in \left(\epsilon, \frac{\beta_n \parallel Ax_n - By_n \parallel}{\parallel A^*(Ax_n - By_n) \parallel^2 + \parallel B^*(Ax_n - By_n) \parallel^2} - \epsilon\right), \quad n \in \Pi.$$

Otherwise, $\gamma_n = \gamma$ ($\gamma$ being any nonnegative value), where the index set $\Pi = \{n \in \mathbb{N} : Ax_n - By_n \neq 0\}$ and $\alpha_n \subset (\delta, 1 - \delta)$ and $\beta_n \subset (\eta, 1 - \eta)$ for small enough $\delta, \eta \geq 0$.

In 2016, Chang et al. (2016) introduced and studied the split equality variational inclusion problems in the setting of Banach spaces. The split equality variational inclusion problem **(SEVIP)** is defined as follows:

$$\text{find } x^* \in T_1^{-1}(0) \text{ and } y^* \in T_2^{-1}(0) \text{ such that } Ax^* = By^*, \tag{1.6}$$

where $T_i : H_i \to 2^{H_i}$, $i = 1, 2$ are maximal monotone operators, $A : H_1 \to X$ and $B : H_2 \to X$ are bounded linear operators. Here, $H_i$, $i = 1, 2$ are real Hilbert spaces and $X$ is a real Banach space. If we consider $X = H_3$, where $H_3$ is a real Hilbert space, then the main result of Chang et al. (2016) will be as follows.

**Theorem 1.1** *Denote $C_1 = H_1$, $Q_1 = H_2$. For given $x_1 \in C_1$ and $y_1 \in Q_1$, let the iterative sequences $\{x_n\}$ and $\{y_n\}$ be generated by*

$$\begin{cases} u_n = J_\lambda^{T_1}(x_n - \gamma_n A^*(Ax_n - By_n)), \\ v_n = J_\lambda^{T_2}(y_n + \gamma_n B^*(Ax_n - By_n)), \\ C_{n+1} \times Q_{n+1} = \{(x, y) \in C_n \times Q_n : \parallel u_n - x \parallel^2 + \parallel v_n - y \parallel^2 \\ \quad \leq \parallel x_n - x \parallel^2 + \parallel y_n - y \parallel^2\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \\ y_{n+1} = P_{Q_{n+1}} x_1. \end{cases} \tag{1.7}$$

*If the solution set* $S := \{(p, q) \in H_1 \times H_2 : (p, q) \in T_1^{-1} \times T_2^{-1}$ *and* $Ap = Bq\}$ *of SEVIP (1.6) is nonempty and the following condition is satisfied*

$$0 < \gamma_n < \frac{2}{\|A\|^2 + \|B\|^2}.$$

*Then the sequence* $\{(x_n, y_n)\}$ *converges strongly to some point* $(x^*, y^*) \in S$*, where* $\|A\|$ *and* $\|B\|$ *are the norms of the operators* $A$ *and* $B$*, respectively.*

The inertial term was first used to define the heavy ball method proposed by Polyak (1964) to minimize the convex smooth function $f$, which is considered as a discretization of time dynamical system, given by

$$\ddot{x}(t) + \alpha_1 \dot{x}(t) + \alpha_2 \nabla f(x(t)) = 0,$$

where $\alpha_1 (> 0)$ and $\alpha_2 (> 0)$ are free model parameters of the equation. Inertial term gives the advantage to use two previous terms to define the next iterate of the algorithm, which in turn increases the convergence speed of the algorithm. This term was further used by Alvarez and Attouch (2001) to define the inertial proximal point algorithm for solving the problem of finding zero of a maximal monotone operator $T$, which is as follows:

$$x_{n+1} = J_{\lambda_n}^T (x_n + \theta_n (x_n - x_{n-1})),$$

where $J_{\lambda_n}^T$ is the resolvent of $T$ with parameter $\lambda_n > 0$ and the inertia is induced by the term $\theta_n (x_n - x_{n-1})$, with $\theta_n \in [0, 1)$. Since their introduction one can notice an increasing interest in inertial algorithms having this particularity, see Bot et al. (2016), Dong et al. (2018), and Moudafi and Oliny (2003).

We consider the following problem:

$$(\mathbf{P}) \qquad \text{find } z^* \in T^{-1}(0) \cap (\cap_{i=1}^m \text{Fix}(R_i)) \text{ such that } F(z^*) = 0, \tag{1.8}$$

where $F : H \to \mathbb{R}$ is a nonnegative lower semicontinuous (l.s.c.) function defined on $H$, $T : H \to 2^H$ is a maximal monotone operator and each $R_i : H \to H, i = 1, 2, \ldots, m$ is a quasi-nonexpansive mapping such that $\cap_{i=1}^m \text{Fix}(R_i) \neq \emptyset$. Throughout the paper, we assume that solution set of the problem **(P)** is denoted by $\Omega$, i.e., $\Omega = \{z \in H : z \in T^{-1}(0) \cap (\cap_{i=1}^m \text{Fix}(R_i))$ and $F(z) = 0\}$.

One can see that problem **(P)** is unification of the following three problems:

(i) finding zero of nonnegative function $F$;
(ii) finding zero of set-valued operator $T$;
(iii) finding common fixed points of operators $R_1, R_2, \ldots, R_m$.

An important particular case of problem (**P**) is split equality variational inclusion fixed point problem which can be expressed as

$$\text{find } x^* \in T_1^{-1}(0) \cap (\cap_{i=1}^m \text{Fix}(M_i))$$

and

$$y^* \in T_2^{-1}(0) \cap (\cap_{i=1}^m \text{Fix}(N_i)) \text{ such that } Ax^* = By^*, \tag{1.9}$$

where $T_i : H_i \to 2^{H_i}$, for $i = 1, 2$ are maximal monotone operators, and $A : H_1 \to H_3$, $B : H_2 \to H_3$ are bounded linear operators. For integers $1 \leq i \leq m$, $M_i : H_1 \to H_1$ and $N_i : H_2 \to H_2$ are two finite families of quasi-nonexpansive mappings.
If we suppose that $M_i = N_i = 0, \forall 1 \leq i \leq m$, then the split equality variational inclusion fixed point problem get converted to split equality variational inclusion problem, which was

studied earlier by Chang et al. (2016) and Chuang (2017). Also, if we assume that $B = I$ and $H_3 = H_2$, then the above problem (1.9) gets converted to split variational inclusion fixed point problem, which was studied by Majee and Nahak (2018).

The main purpose of this paper is to propose three iterative methods for solving problem **(P)** and to study the convergence analysis of the proposed iterative methods in a real Hilbert space setting. Our results unify some known results.

The remaining parts of this paper are organized as follows: some lemmas and definitions required for proving main results are presented in Sect. 2. Three iterative methods for solving problem **(P)** are introduced in Sect. 3. Strong convergence of the proposed iterative methods are also discussed in Sect. 3. The applications of our results are established in Sect. 3 to the split equality variational inclusion fixed point problem and split equality equilibrium fixed point problem are given in Sect. 4. The efficiency of our iterative methods is demonstrated in Sect. 5.

## 2 Preliminaries

Let $R : H \to H$ be a mapping. An element $z \in H$ is said to be a fixed point of $R$ if $z = Rz$. We use $\text{Fix}(R)$ to denote the set of all fixed points of $R$.

**Definition 2.1** A map $R : H \to H$ is called

(i) nonexpansive if

$$\|Rx - Ry\| \leq \| x - y \| \text{ for all} x, y \in H,$$

(ii) quasi-nonexpansive if

$$\text{Fix}(R) \neq \emptyset \text{ and } \|Rx - Rp\| \leq \| x - p \| \quad \text{for all } x \in H \text{ and } p \in \text{Fix}(R).$$

(iii) demi-closed at zero if

$$\lim_{n\to\infty} \|z_n - Rz_n\| = 0 \text{ and } z_n \rightharpoonup z^* \text{ imply that } z^* = Rz^* \text{ for any sequence } \{z_n\} \in H.$$

Throughout this paper, the symbols $\mathbb{N}$ and $\mathbb{R}$ stand for the set of all natural numbers and set of real numbers, respectively. Also, we use the symbol $I$ for the identity operator on $H$.

Let $C$ be a nonempty closed convex subset of $H$. Then for any $x \in H$, there exists a unique nearest point $P_C(x)$ of $C$ such that

$$\|x - P_C(x)\| \leq \|x - y\| \text{ for all } y \in C.$$

The mapping $P_C$ is called the metric projection map from $H$ onto $C$. It is noticeable that the metric projection mapping $P_C$ is nonexpansive mapping from $H$ onto $C$ (see Agarwal et al. 2009 for more details of projection mappings).
The following lemmas will be needed to prove our main results.

**Lemma 2.1** (Agarwal et al. 2009, Proposition 2.10.15) *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$, and $P_C$ be the metric projection mapping, then the following properties hold:*

*(i)* $P_C(x) \in C, \forall x \in H$;
*(ii)* $\langle x - P_C(x), P_C(x) - y\rangle \geq 0, \forall x, y \in C$;
*(iii)* $\|x - y\|^2 \geq \|x - P_C(x)\|^2 + \|y - P_C(x)\|^2, \forall x \in H$ *and* $y \in C$;

*(iv)* $\langle P_C(x) - P_C(y), x - y\rangle \geq \|P_C(x) - P_C(y)\|^2, \forall x, y \in H.$

***Proof*** (iii) Since for any $x \in H$, $y \in C$, we have

$$\begin{aligned}\|x - y\|^2 &= \|x - P_C x + P_C x - y\|^2 \\ &= \|x - P_C x\|^2 + \|P_C x - y\|^2 + 2\langle x - P_C x, P_C x - y\rangle. \end{aligned} \tag{2.1}$$

So, (iii) follows from (ii) and (2.1).

□

**Lemma 2.2** (Zegeye and Shahzad 2011, Lemma 1.1) *Let $H$ be a real Hilbert space. For each $x_1, x_2, \ldots, x_m \in H$ and $\alpha_1, \alpha_2, \ldots, \alpha_m \in [0, 1]$ with $\sum_{i=1}^m \alpha_i = 1$, the equality*

$$\| \alpha_1 x_1 + \cdots + \alpha_m x_m \|^2 = \sum_{i=1}^m \alpha_i \| x_i \|^2 - \sum_{1 \leq i, j \leq m} \alpha_i \alpha_j \| x_i - x_j \|^2$$

*holds.*

**Lemma 2.3** (Xu 2002, Lemma 2.5) *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1 - a_n)s_n + a_n b_n \text{ for all } n \in \mathbb{N},$$

*where $\{a_n\}$ is a sequence in $(0, 1)$ and $\{b_n\}$ is a sequence in $\mathbb{R}$ such that*

*(a)* $\sum_{n=1}^\infty a_n = \infty$ *and*
*(b) either* $\limsup_{n\to\infty} b_n \leq 0$ *or* $\sum_{n=1}^\infty |a_n b_n| < \infty$.

*Then* $\lim_{n\to\infty} s_n = 0$.

**Definition 2.2** (Bauschke and Combettes 2017, *Definition* 16.1) Let $f : H \to (-\infty, \infty]$ be proper. The subdifferential of $f$ is the set-valued operator

$$\partial f : H \to 2^H : x \mapsto \{u \in H | (\forall y \in H) \langle y - x, u\rangle + f(x) \leq f(y)\}.$$

Let $x \in H$. Then $f$ is subdifferentiable at $x$ if $\partial f \neq \emptyset$; the elements of $\partial f$ are the subgradients of $f$ at $x$.

Let $T : H \to 2^H$ be an operator. The domain and graph of $T$ are denoted by $\mathrm{dom}(T)$ and $\mathrm{gra}(T)$, respectively, where $\mathrm{dom}(T) = \{x \in H : Tx \neq \emptyset\}$ and $\mathrm{gra}(T) = \{(x, u) \in H \times H : u \in Tx\}$. A set-valued operator is said to be monotone operator on $H$ if $\langle x - y, u - v\rangle \geq 0$, $\forall (x, u) \in \mathrm{gra}(T)$ and $\forall (y, u) \in \mathrm{gra}(T)$. A monotone operator $T$ on $H$ is said to be maximal if there exists no monotone operator $S : H \to 2^H$ such that $\mathrm{gra}(S)$ properly contains $\mathrm{gra}(T)$. The resolvent of $T$ for $\lambda > 0$ is $J_\lambda^T = (I + \lambda T)^{-1} : H \to H$. If $T$ is a maximal monotone operator then its resolvent is single valued, firmly nonexpansive and maximal monotone operator. Finally, the set $\mathrm{Fix}(J_\lambda^T) = \{x \in H : J_\lambda^T x = x\}$ of fixed points of $J_\lambda^T$ coincides with $T^{-1}(0)$ (Bauschke and Combettes 2017; Ryu and Boyd 2016).

**Lemma 2.4** (Bauschke and Combettes 2017, Example 20.29) *Let $T : H \to H$ be a nonexpansive map on a Hilbert space $H$ and $\alpha \in [-1, 1]$. Then $I + \alpha T$ is maximal monotone operator.*

**Lemma 2.5** (Bauschke and Combettes 2017, Proposition 20.23) *Let $T_1 : H_1 \to 2^{H_1}$ and $T_2 : H_2 \to 2^{H_2}$ be two maximal monotone operator, where $H_1$ and $H_2$ are real Hilbert spaces. Set $H := H_1 \times H_2$ and $T : H \to 2^H : (x, y) \mapsto T_1 x \times T_2 y$. Then $T$ is maximal monotone operator.*

## 3 Main results

In this section, we introduce strongly convergent iterative schemes for finding the solution of problem **(P)**.

Let $H$ be a real Hilbert space, $F : H \to \mathbb{R}$ be a nonnegative lower semicontinuous function and $T : H \to 2^H$ be a maximal monotone operator. Suppose that, for each $i \in \{1, 2, \dots, m\}$, $R_i : H \to H$ be a quasi-nonexpansive mapping. Now, we introduce our iterative algorithms for solving the problem (**P**) as follows:

**Algorithm 3.1** *(1) Initialization: denote $D_1 = H$ and select $z_1 \in D_1$ arbitrarily.*
*(2) Iterative step: select $\{\mu_n\}$ and $\{\delta_{i,n}\}$ as iteration parameters and compute the $(n+1)^{th}$ iteration as follows:*

$$\begin{cases} s_n = J_\lambda^T(z_n - \mu_n d_n), \\ t_n = \delta_{0,n} s_n + \sum_{i=1}^{m} \delta_{i,n} R_i(s_n), \\ D_{n+1} = \{z \in D_n : \|t_n - z\|^2 \leq \|z_n - z\|^2\}, \\ z_{n+1} = P_{D_{n+1}} z_1, \quad n \in \mathbb{N}, \end{cases} \tag{3.1}$$

*where $d_n$ is a search direction, $\lambda > 0$ and $\{\delta_{i,n}\}$ is a sequence such that $\delta_{i,n} \in (0,1)$, $\liminf_n \delta_{i,n} > 0$, $\sum_{i=0}^{m} \delta_{i,n} = 1$. The step size $\mu_n$ is selected as follows:*

$$\mu_n = \begin{cases} \dfrac{\beta_n F(z_n)}{\|d_n\|^2}, & if\ d_n \neq 0 \\ 0, & otherwise, \end{cases} \tag{3.2}$$

*where $\beta_n \in (0, 2)$.*

**Algorithm 3.2** *(1) Initialization: denote $D_1 = H$ and select $z_0, z_1 \in D_1$ arbitrarily.*
*(2) Iterative step: select $\{\mu_n\}$ and $\{\delta_{i,n}\}$ as iteration parameters and compute the $(n+1)^{th}$ iteration as follows:*

$$\begin{cases} w_n = z_n + \alpha_n(z_n - z_{n-1}), \\ s_n = J_\lambda^T(w_n - \mu_n d_n), \\ t_n = \delta_{0,n} s_n + \sum_{i=1}^{m} \delta_{i,n} R_i(s_n), \\ D_{n+1} = \{z \in D_n : \|t_n - z\|^2 \\ \qquad \leq \|z_n - z\|^2 + \alpha_n^2 \|z_n - z_{n-1}\|^2 + 2\alpha_n \langle z_n - z, z_n - z_{n-1}\rangle\}, \\ z_{n+1} = P_{D_{n+1}} z_n, \quad n \in \mathbb{N}, \end{cases} \tag{3.3}$$

*where $d_n$ is a search direction, $\lambda > 0$ and $\{\delta_{i,n}\}$ is a sequence such that $\delta_{i,n} \in (0,1)$, $\liminf_n \delta_{i,n} > 0$, $\sum_{i=0}^{m} \delta_{i,n} = 1$. The step size $\mu_n$ is selected as (3.2). Also, $\alpha_n \in [0, \alpha]$ for some $\alpha \in [0, 1)$ such that $\sum_{n=1}^{\infty} \alpha_n \|z_n - z_{n-1}\| < \infty$.*

**Algorithm 3.3** *(1) Initialization: denote $D_1 = H$ and select $z_1 \in D_1$ arbitrarily.*
*(2) Iterative step: select $\{\mu_n\}$ and $\{\delta_{i,n}\}$ as iteration parameters and compute the $(n+1)^{th}$ iteration as follows:*

$$\begin{cases} s_n = J_\lambda^T(z_n - \mu_n d_n), \\ t_n = \delta_{0,n} s_n + \sum_{i=1}^{m} \delta_{i,n} R_i(s_n), \\ D_{n+1} = \{z \in D_n : \|t_n - z\|^2 \leq \|z_n - z\|^2\}, \\ z_{n+1} = P_{D_{n+1}} z_n, \quad n \in \mathbb{N}, \end{cases} \tag{3.4}$$

*where $d_n$ is a search direction, $\lambda > 0$ and $\{\delta_{i,n}\}$ is a sequence such that $\delta_{i,n} \in (0,1)$, $\liminf_n \delta_{i,n} > 0$, $\sum_{i=0}^{m} \delta_{i,n} = 1$. The step size $\mu_n$ is selected as (3.2).*

***Remark 3.1*** 1. In Algorithm 3.2, we have used two previous terms to define the next iterate of the algorithm, which in turn increases the convergence speed of the algorithm.
2. In Algorithm 3.3, projection of $z_n$ is taken on the set $D_{n+1}$ instead of $z_1$ to calculate the $(n+1)^{th}$ term of the algorithm.
3. By choosing $\alpha_n = 0$, Algorithm 3.2 get converted to Algorithm 3.3.

To establish the strong convergence of Algorithms 3.1, 3.2 and 3.3, we need the following assumptions:

(A0) $\langle d_n, z_n - z\rangle \geq F(z_n)$ for all $n \in \mathbb{N}$ and for all $z \in \Omega$;
(A1) $0 < \mu \leq \mu_n < \bar{\mu}$ for all $n \in \mathcal{I}$;
(A2) $\inf_{n\in\mathcal{I}}[\beta_n(2-\beta_n)] > 0$.

Here $\mathcal{I}$ denotes the index set $\{n \in \mathbb{N} : d_n \neq 0\}$.

***Remark 3.2*** Any vector $d_n \in \partial F(z_n)$is an example of direction satisfying (A0). Since, $F(z) = 0$, we have by definition of the subdifferential of a proper function that

$$0 \geq F(z_n) + \langle d_n, z - z_n\rangle,$$

and thus (A0) is satisfied. On the other hand, from the definition of $\mu_n$ and Assumption (A0), we easily observe if $n \notin \mathcal{I}$, then $d_n = 0$, $F(z_n) = 0$, $\mu_n = 0$, and $s_n = J_\lambda^T z_n$.

Before presenting our main results, we need the following proposition:

**Proposition 3.1** *Let $H$ be a real Hilbert space, $F : H \to \mathbb{R}$ be a nonnegative lower semi-continuous function and $T : H \to 2^H$ be a maximal monotone operator. Suppose that for each $i \in \{1, 2, \ldots, m\}$, $R_i : H \to H$ is a quasi-nonexpansive mapping with $I - R_i$ being demi-closed at zero and $\Omega \neq \emptyset$. Assume that (A0) and (A2) hold. Let $\{z_n\}$ be the sequence generated by Algorithms 3.1 or 3.3. Then $\Omega \subseteq D_n$, for all $n \in \mathbb{N}$.*

***Proof*** Let $z$ be any point in $\Omega$. Here $z \in T^{-1}(0) = \mathrm{Fix}(J_\lambda^T) \subset H = D_1$. Hence, $z \in D_1$. If for some $n \geq 2$, $z \in D_n$, we show that $z \in D_{n+1}$. From (3.1), assumption (A2), and the fact that $J_\lambda^T$ is firmly nonexpansive, we have

$$\begin{aligned} \|s_n - z\|^2 &= \left\| J_\lambda^T(z_n - \mu_n d_n) - J_\lambda^T(z) \right\|^2 \\ &\leq \|z_n - \mu_n d_n - z\|^2 \end{aligned}$$

$$
\begin{aligned}
&= \|z_n - z\|^2 + \mu_n^2 \|d_n\|^2 - 2\mu_n \langle z_n - z, d_n \rangle && (3.5)\\
&= \|z_n - z\|^2 + \frac{\beta_n^2 [F(z_n)]^2}{\|d_n\|^2} - 2\langle z_n - z, \frac{\beta_n [F(z_n)]}{\|d_n\|^2} d_n \rangle \\
&= \|z_n - z\|^2 - \frac{\beta_n [F(z_n)]}{\|d_n\|^2} [2\langle z_n - z, d_n \rangle - \beta_n F(z_n)] \\
&\leq \|z_n - z\|^2 - \frac{\beta_n [F(z_n)]}{\|d_n\|^2} [2F(z_n) - \beta_n F(z_n)] \\
&= \|z_n - z\|^2 - \beta_n (2 - \beta_n) \frac{[F(z_n)]^2}{\|d_n\|^2} && (3.6)\\
&\leq \|z_n - z\|^2 . && (3.7)
\end{aligned}
$$

From (3.1) and Lemma 2.2, we have

$$
\begin{aligned}
\|t_n - z\|^2 &= \| \delta_{0,n} s_n + \sum_{i=1}^{m} \delta_{i,n} R_i(s_n) - z \|^2 \\
&= \| \delta_{0,n}(s_n - z) + \sum_{i=1}^{m} \delta_{i,n} (R_i(s_n) - z) \|^2 \\
&\leq \delta_{0,n} \|s_n - z\|^2 + \sum_{i=1}^{m} \delta_{i,n} \| (R_i(s_n) - R_i z) \|^2 \\
&\quad - \sum_{1 \leq i \leq m} \delta_{0,n} \delta_{i,n} \|s_n - R_i(s_n)\|^2 \\
&\leq \delta_{0,n} \|s_n - z\|^2 + \sum_{i=1}^{m} \delta_{i,n} \| s_n - z \|^2 - \sum_{1 \leq i \leq m} \delta_{0,n} \delta_{i,n} \|s_n - R_i(s_n)\|^2 \\
&= \|s_n - z\|^2 - \delta_{0,n} \sum_{1 \leq i \leq m} \delta_{i,n} \|s_n - R_i(s_n)\|^2 && (3.8)\\
&\leq \|s_n - z\|^2 && (3.9)\\
&\leq \|z_n - z\|^2. && (3.10)
\end{aligned}
$$

Hence, $z \in D_{n+1}$ and so $\Omega \subseteq D_{n+1}, \forall n \geq 1$. □

Now, we are ready to establish the strong convergence of Algorithm 3.1 for solving problem (**P**).

**Theorem 3.1** *Let $H$ be a real Hilbert space, $F : H \to \mathbb{R}$ be a nonnegative lower semicontinuous function and $T : H \to 2^H$ be a maximal monotone operator. Suppose that for each $i \in \{1, 2, \ldots, m\}$, $R_i : H \to H$ is a quasi-nonexpansive mapping with $I - R_i$ being demiclosed at zero and $\Omega \neq \emptyset$. Assume that (A0)–(A2) hold. Let $\{z_n\}$ be the sequence generated by Algorithm 3.1. Then the sequence $\{z_n\}$ converges strongly to some point $z^* \in \Omega$.*

***Proof*** Since $D_n, n \geq 1$ is a nonempty closed convex subset of $H$, sequence $\{z_n\}$ is well defined.

We proceed the proof in the following steps:

**Step 1**: $\{z_n\}$ is Cauchy sequence.
By Proposition 3.1, we get $\Omega \subseteq D_{n+1}, \quad \forall n \geq 0, \quad D_{n+1} \subseteq D_n$ and $z_{n+1} = P_{D_{n+1}} z_1$.

Note that for any $z \in \Omega$,

$$\|z_{n+1} - z_1\| \leq \|z - z_1\|.$$

Hence, $\{z_n\}$ is a bounded sequence. Moreover, it follows from (3.1) that

$$\|z_n - z_1\| \leq \|z_{n+1} - z_1\|, \; \forall n \geq 1.$$

So, $\{\| z_n - z_1 \|\}$ is a convergent sequence.

Note that $z_k = P_{D_k} z_1, \forall k \geq 1$. By the definition of projection and by item (iii) of Lemma 2.1, we have

$$\begin{aligned} \| z_n - z_k \|^2 + \| z_k - z_1 \|^2 &= \| z_n - P_{D_k} z_1 \|^2 + \| P_{D_k} z_1 - z_1 \|^2 \\ &\leq \| z_n - z_1 \|^2, \end{aligned}$$

and so,

$$\lim_{n,k\to\infty} \| z_n - z_k \|^2 \leq \lim_{n\to\infty} \| z_n - z_1 \|^2 - \lim_{k\to\infty} \| z_k - z_1 \|^2 = 0,$$

which proves that $\{z_n\}$ is a Cauchy sequence in $H$.

Without loss of generality, we can assume that $z_n \to z^*$.

**Step 2**: $z^* \in \Omega$.
Since $z_{n+1} \in D_{n+1}$, it follows from (3.1) that

$$\| t_n - z_{n+1} \| \leq \| z_n - z_{n+1} \| .$$

Hence, $\lim_{n\to\infty} \| t_n - z_{n+1} \| = 0$ and so, $t_n \to z^*$.

Since for $z \in \Omega$, from (3.7) and (3.10), we have $\|t_n - z\|^2 \leq \|s_n - z\|^2 \leq \|z_n - z\|^2$; hence, the sequences $\{\|s_n - z\|\}$, $\{\|t_n - z\|\}$ and $\{\|z_n - z\|\}$ have same limit. From (3.8), we have

$$\|t_n - z\|^2 \leq \|s_n - z\|^2 - \delta_{0,n} \sum_{1\leq i\leq m} \delta_{i,n} \|s_n - R_i(s_n)\|^2.$$

Let $\nu_i = \inf_{n\in\mathbb{N}} \delta_{i,n}, \; \forall i \in \{0, 1, \ldots, m\}$. Hence,

$$\nu_0 \sum_{1\leq i\leq m} \nu_i \|R_i(s_n) - s_n\|^2 \leq \|s_n - z\|^2 - \|t_n - z\|^2 \to 0, \text{ as } n \to \infty, \tag{3.11}$$

which implies that $\|R_i(s_n) - s_n\| \to 0$, as $n \to \infty$. From (3.6), we have

$$\lim_{n\to\infty} \beta_n(2 - \beta_n) \frac{[F(z_n)]^2}{\|d_n\|^2} \leq \lim_{n\to\infty} \|z_n - z\|^2 - \lim_{n\to\infty} \|s_n - z\|^2 = 0. \tag{3.12}$$

Hence,

$$\lim_{n\to\infty} \frac{[F(z_n)]^2}{\|d_n\|^2} = 0. \tag{3.13}$$

Also, since $0 < \mu \leq \mu_n = \beta_n \frac{F(z_n)}{\|d_n\|^2}$, for all $n \in \mathbb{N}$, $0 \leq \mu_n \|d_n\| = \beta_n \frac{F(z_n)}{\|d_n\|}$. Hence, from (3.13) and (A2), $\mu_n \|d_n\| \to 0$. So, $\|d_n\| \to 0$ as $\mu_n \geq \mu > 0$ and accordingly

$$F(z_n) = \frac{F(z_n)}{\|d_n\|} \|d_n\| \to 0 \text{ as } n \to \infty.$$

So, $F(z^*) = 0$, as F is a positive lower semicontinuous function and $z_n \to z^*$. Also,

$$\lim_{n\to\infty} \| z_n - s_n \| \leq \lim_{n\to\infty} \|z_n - t_n\| + \lim_{n\to\infty} \|t_n - s_n\| = 0. \tag{3.14}$$

Now,

$$\begin{aligned}
\| z_n - J_\lambda^T z_n \| &\leq \| z_n - s_n \| + \| s_n - J_\lambda^T z_n \| \\
&= \| z_n - s_n \| + \| J_\lambda^T (z_n - \mu_n d_n) - J_\lambda^T z_n \| \\
&\leq \| z_n - s_n \| + \| \mu_n d_n \| \\
&= \| z_n - s_n \| + \left\| \frac{\beta_n F(z_n)}{\| d_n \|^2} d_n \right\| \\
&= \| z_n - s_n \| + \left\| \frac{\beta_n F(z_n)}{\|d_n\|^2} \right\| \|d_n\| \\
&= \| z_n - s_n \| + \left\| \frac{\beta_n F(z_n)}{\|d_n\|} \right\|.
\end{aligned}$$

So, from (3.13) and (3.14), we get that

$$\| z_n - J_\lambda^T z_n \| \to 0 \ \text{ as } n \to \infty.$$

Thus, we have $z^* = J_\lambda^T z^*$.

**Step 3**: Next, we show that $z^* \in \text{Fix}(R_i)$. Since $\lim_{n\to\infty} \|s_n - R_i(s_n)\| = 0$ and $s_n \to z^*$. Using the fact that $I - R_i$ is demi-closed, we get $z^* \in \text{Fix}(R_i)$ (for each $i = 1, 2, \ldots, m$). Hence, $z^* \in \text{Fix}(R_i)$, for each $i = 1, 2, \ldots, m$.
Therefore, we conclude that $z^* \in \Omega$ and $z_n \to z^*$. □

We now study the convergence analysis of Algorithm 3.2 for solving problem (**P**).

**Theorem 3.2** *Let $H$ be a real Hilbert space, $F : H \to \mathbb{R}$ be a nonnegative lower semicontinuous function and $T : H \to 2^H$ be a maximal monotone operator. Suppose that for each $i \in \{1, 2, \ldots, m\}$, $R_i : H \to H$ is a quasi-nonexpansive mapping with $I - R_i$ being demi-closed at zero and $\Omega \neq \emptyset$. Assume that (A0)–(A2) hold. Let $\{z_n\}$ be the sequence generated by Algorithm 3.2. Then the sequence $\{z_n\}$ converges strongly to some point $z^* \in \Omega$.*

***Proof*** We proceed the proof in the following steps:

**Step 1**: $\Omega \subseteq D_{n+1}$
For any $z \in \Omega$, we have $z \in T^{-1}(0) = \text{Fix}(J_\lambda^T) \subset H = D_1$. Hence, $z \in D_1$. If for some $n \geq 2$, $z \in D_n$, we show that $z \in D_{n+1}$. From (3.3), and (3.2), we have

$$\begin{aligned}
\|s_n - z\|^2 &= \|J_\lambda^T (w_n - \mu_n d_n) - J_\lambda^T (z)\|^2 \\
&\leq \|w_n - \mu_n d_n - z\|^2 \\
&= \|z_n + \alpha_n (z_n - z_{n-1}) - \mu_n d_n - z\|^2 \\
&= \|z_n - \mu_n d_n - z\|^2 + \alpha_n^2 \|z_n - z_{n-1}\| + 2\langle z_n - \mu_n d_n - z, \alpha_n (z_n - z_{n-1})\rangle \\
&= \|z_n - z\|^2 + \mu_n^2 \|d_n\|^2 - 2\langle z_n - z, \mu_n d_n\rangle + \alpha_n^2 \|z_n - z_{n-1}\|^2 \\
&\quad + 2\alpha_n \langle z_n - z, z_n - z_{n-1}\rangle - 2\langle \mu_n d_n, \alpha_n (z_n - z_{n-1})\rangle \\
&= \|z_n - z\|^2 + \alpha_n^2 \|z_n - z_{n-1}\|^2 + 2\alpha_n \langle z_n - z, z_n - z_{n-1}\rangle \\
&\quad + \mu_n^2 \|d_n\|^2 - 2\langle \mu_n d_n, z_n - z + \alpha_n (z_n - z_{n-1})\rangle \qquad (3.15) \\
&= \|z_n - z\|^2 + \alpha_n^2 \|z_n - z_{n-1}\|^2 + 2\alpha_n \langle z_n - z, z_n - z_{n-1}\rangle
\end{aligned}$$

$$-\frac{\beta_n[F(z_n)]}{\|d_n\|^2}[2\langle w_n - z, d_n\rangle - \beta_n F(z_n)]$$
$$\leq \|z_n - z\|^2 + \alpha_n^2\|z_n - z_{n-1}\|^2 + 2\alpha_n\langle z_n - z, z_n - z_{n-1}\rangle. \tag{3.16}$$

From (3.8), we have

$$\|t_n - z\|^2 \leq \|s_n - z\|^2 - \delta_{0,n}\sum_{1\leq i\leq m}\delta_{i,n}\|R_i(s_n) - s_n\|^2 \tag{3.17}$$
$$\leq \|s_n - z\|^2. \tag{3.18}$$

From (3.16) and (3.18), we obtain

$$\|t_n - z\|^2 \leq \|z_n - z\|^2 + \alpha_n^2\|z_n - z_{n-1}\|^2 + 2\alpha_n\langle z_n - z, z_n - z_{n-1}\rangle.$$

By the definition of $D_{n+1}$, we get $z \in D_{n+1}$ and so $\Omega \subseteq D_{n+1}$, $\forall n \geq 1$.

Since $D_n, n \geq 1$ is a nonempty closed convex subset of $H$, therefore sequence $\{z_n\}$ is well defined sequence.

**Step 2**: $\{z_n\}$ is Cauchy sequence.
By Proposition 3.1, we get $\Omega \subseteq D_{n+1}$, $\forall n \geq 0$, $D_{n+1} \subseteq D_n$ and, from (3.3), $z_{n+1}= P_{D_{n+1}}z_n$.

Note that for any $z \in \Omega$,

$$\|z_{n+1} - z_1\| \leq \|z - z_1\|.$$

Hence, $\{z_n\}$ is a bounded sequence. Moreover, it follows from (3.3) that

$$\|z_n - z_1\| \leq \|z_{n+1} - z_1\|,\ \forall n \geq 1.$$

So, $\{\| z_n - z_1 \|\}$ is a convergent sequence.

Note that $z_k = P_{D_k}z_{k-1}, \forall k \geq 1$. By the definition of projection and by item (iii) of Lemma 2.1, we have

$$\begin{aligned}\| z_n - z_k \|^2 + \| z_k - z_1 \|^2 &= \| z_n - P_{D_k}z_{k-1} \|^2 + \| P_{D_k}z_{k-1} - z_1 \|^2\\ &\leq \| z_n - z_1 \|^2,\end{aligned}$$

and so,

$$\lim_{n,k\to\infty} \| z_n - z_k \|^2 \leq \lim_{n\to\infty} \| z_n - z_1 \|^2 - \lim_{k\to\infty} \| z_k - z_1 \|^2 = 0,$$

which proves that $\{z_n\}$ is a Cauchy sequence in $H$.

Without loss of generality, we can assume that $z_n \to z^*$.

**Step 3**: $z^* \in \Omega$.
Since $\{z_n\}$ is a Cauchy sequence, we have

$$\|w_n - z_n\| = \alpha_n\|z_n - z_{n-1}\| \to 0,\ \text{as } n \to \infty. \tag{3.19}$$

From (3.19), we get

$$\|w_n - z_{n+1}\| \leq \|w_n - z_n\| + \|z_{n+1} - z_n\| \to 0,\ \text{as } n \to \infty. \tag{3.20}$$

From (3.16), we have

$$\|s_n - z\|^2 - \|z_n - z\|^2 \leq \alpha_n^2\|z_n - z_{n-1}\|^2 + 2\alpha_n\langle z_n - z, z_n - z_{n-1}\rangle \to 0,\ \text{as } n \to \infty. \tag{3.21}$$

From (3.15) and (3.3), we deduce

$$\begin{aligned}\|s_n - z\|^2 &\leq \|z_n - z\|^2 + \alpha_n^2\|z_n - z_{n-1}\|^2 + 2\alpha_n\langle z_n - z, z_n - z_{n-1}\rangle \\ &\quad + \mu_n^2\|d_n\|^2 - 2\langle \mu_n d_n, z_n - z + \alpha_n(z_n - z_{n-1})\rangle \\ &\leq \|z_n - z\|^2 + \alpha_n^2\|z_n - z_{n-1}\|^2 + 2\alpha_n\langle z_n - z, z_n - z_{n-1}\rangle \\ &\quad + \beta_n^2\frac{[F(z_n)]^2}{\|d_n\|^2} - 2\mu_n F(z_n) \\ &\leq \|z_n - z\|^2 + \alpha_n^2\|z_n - z_{n-1}\|^2 + 2\alpha_n\langle z_n - z, z_n - z_{n-1}\rangle \\ &\quad - \beta_n(2-\beta_n)\frac{[F(z_n)]^2}{\|d_n\|^2}, \end{aligned} \tag{3.22}$$

which implies that

$$\begin{aligned}\beta_n(2-\beta_n)\frac{[F(z_n)]^2}{\|d_n\|^2} &\leq \|z_n - z\|^2 - \|s_n - z\|^2 + \alpha_n^2\|z_n - z_{n-1}\|^2 \\ &\quad + 2\alpha_n\langle z_n - z, z_n - z_{n-1}\rangle \\ &\to 0, \text{ as } n \to \infty. \end{aligned} \tag{3.23}$$

Hence, $\lim_{n\to\infty}\frac{[F(z_n)]^2}{\|d_n\|^2} = 0$. Also, since $0 < \mu \leq \mu_n = \beta_n\frac{F(z_n)}{\|d_n\|^2}$, for all $n$. So, $0 \leq \mu_n\|d_n\| = \beta_n\frac{F(z_n)}{\|d_n\|}$ which implies that $\mu_n\|d_n\| \to 0$. So, $\|d_n\| \to 0$ as $\mu_n \geq \mu > 0$ and accordingly

$$F(z_n) = \frac{F(z_n)}{\|d_n\|}\|d_n\| \to 0, \text{ as } n \to \infty.$$

Since $F$ is a positive lower semicontinuous function and $z_n \to z^*$, it follows that $F(z^*) = 0$. Also,

$$\begin{aligned}\|t_n - s_n\| &= \|\delta_{0,n}s_n + \sum_{i=1}^m \delta_{i,n}R_{i,n}(s_n) - s_n\| \\ &\leq \delta_{0,n}\|s_n - s_n\| + \sum_{i=1}^m \delta_{i,n}\|R_{i,n}(s_n) - s_n\|.\end{aligned}$$

So, $\lim_{n\to\infty}\|t_n - s_n\| \to 0$. Since $z_{n+1} \in D_{n+1} \subset D_n$, from (3.20), we obtain

$$\begin{aligned}\|w_n - s_n\| &\leq \|w_n - z_{n+1}\| + \|t_n - s_n\| + \|t_n - z_{n+1}\| \\ &\leq \|w_n - z_{n+1}\| + \|t_n - s_n\| \\ &\quad + \sqrt{\|z_n - z_{n+1}\|^2 + \alpha_n^2\|z_n - z_{n-1}\|^2 + 2\alpha_n\langle z_n - z_{n+1}, z_n - z_{n-1}\rangle} \\ &\leq \|w_n - z_{n+1}\| + \|t_n - s_n\| \\ &\quad + \sqrt{\|z_n - z_{n+1}\|^2 + \alpha_n^2\|z_n - z_{n-1}\|^2 + 2\alpha_n\|z_n - z_{n+1}\|\|z_n - z_{n-1}\|} \\ &\to 0, \text{ as } n \to \infty. \end{aligned} \tag{3.24}$$

From (3.19) and (3.24), we have

$$\begin{aligned}\| z_n - J_\lambda^T z_n \| &\leq \|z_n - s_n\| + \|s_n - J_\lambda^T z_n\| \\ &= \|z_n - s_n\| + \| J_\lambda^T(w_n - \mu_n d_n) - J_\lambda^T z_n \|\end{aligned}$$

$$
\begin{aligned}
&\le \|z_n - w_n\| + \|w_n - s_n\| + \|\alpha_n(z_n - z_{n-1})\| + \|\mu_n d_n\| \\
&= \|z_n - w_n\| + \|w_n - s_n\| + \|\alpha_n(z_n - z_{n-1})\| + \left\| \frac{\beta_n F(z_n)}{\|d_n\|^2} d_n \right\| \\
&\le \|z_n - w_n\| + \|w_n - s_n\| + \|\alpha_n(z_n - z_{n-1})\| + \left\| \frac{\beta_n F(z_n)}{\|d_n\|} \right\| \\
&\to 0, \text{ as } n \to \infty.
\end{aligned}
$$

So, we have $z^* = J_\lambda^T z^*$. As in Theorem 3.1, we can see that $z^* \in \text{Fix}(R_i)$, for each $i = 1, 2, \ldots, m$. Therefore, we conclude that $z^* \in \Omega$ and $z_n \to z^*$. □

Now with $\alpha_n = 0$, we obtain the following result by Theorem 3.2.

**Theorem 3.3** *Let $H$ be a real Hilbert space, $F : H \to \mathbb{R}$ be a nonnegative lower semicontinuous function and $T : H \to 2^H$ be a maximal monotone operator. Suppose that for each $i \in \{1, 2, \ldots, m\}$, $R_i : H \to H$ is a quasi-nonexpansive mapping with $I - R_i$ is demiclosed at zero and $\Omega \neq \emptyset$. Assume that (A0)–(A2) hold. Let $\{z_n\}$ be the sequence generated by Algorithm 3.3. Then the sequence $\{z_n\}$ converges strongly to some point $z^* \in \Omega$.*

***Remark 3.3*** The value of $\|z_n - z_{n-1}\|$ is known before the value of $\alpha_n$. Indeed, the parameters $\alpha_n$ can be chosen such that $0 \le \alpha_n \le \alpha'_n$, where

$$
\alpha'_n = \begin{cases} \min\left\{\frac{\omega_n}{\|z_n - z_{n-1}\|}, \alpha\right\} & \textit{if } z_n \neq z_{n-1}, \\ \alpha & \textit{otherwise}, \end{cases} \tag{3.25}
$$

where $\{\omega_n\}$ is a positive sequence such that $\sum_{n=1}^{\infty} \omega_n < \infty$.

# 4 Applications

## 4.1 Split equality variational inclusion fixed point problem

Here, we investigate the split equality variational inclusion fixed point problems as an application.

Let $H_1$, $H_2$ and $H_3$ be Hilbert spaces. In particular, take $H = H_1 \times H_2$ and for any $(x, y) \in H_1 \times H_2$, the operators $T$, $F$ and $R_i$ are defined by

$$
\begin{cases} T(x, y) := T_1(x) \times T_2(y), \\ F(x, y) := \dfrac{1}{2}\|Ax - By\|^2, \\ R_i(x, y) := M_i(x) \times N_i(y), \quad \text{for each } i = 1, 2, \ldots, m, \end{cases} \tag{4.1}
$$

where $T_i : H_i \to 2^{H_i}$, for $i = 1, 2$ are maximal monotone operators and $A : H_1 \to H_3$, $B : H_2 \to H_3$ are bounded linear operators. For integers $1 \le i \le m$, $M_i : H_1 \to H_1$ and $N_i : H_2 \to H_2$ are two finite families of set-valued quasi-nonexpansive operators such that

$$
\bigcap_{i=1}^{m} \text{Fix}(M_i) \neq \emptyset \quad \textit{and} \quad \bigcap_{i=1}^{m} \text{Fix}(N_i) \neq \emptyset.
$$

With the above setting, Problem (**P**) becomes

$$\text{(SEVIFP)} \qquad \text{find } x \in \bigcap_{i=1}^{m} \mathrm{Fix}(M_i) \bigcap T_1^{-1}(0) \text{ and } y \in \bigcap_{i=1}^{m} \mathrm{Fix}(N_i) \bigcap T_2^{-1}(0) \quad \text{such that } Ax = By.$$

We assume that the search direction $d_n$ coincides with the gradient $\nabla F(z_n)$ of the function $F$. So, we have the following result:

**Theorem 4.1** *Let $H_1$, $H_2$ and $H_3$ be real Hilbert spaces, $T_i : H_i \to 2^{H_i}$, for $i = 1, 2$ be maximal monotone operators, $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be bounded linear operators and for positive integers $1 \leq i \leq m$, $M_i : H_1 \to H_1$ and $N_i : H_2 \to H_2$ be two finite families of quasi-nonexpansive operators with $I - M_i$ and $I - N_i$ are demi-close at zero. Let $A^*$, $B^*$ be the adjoint of $A$, $B$, respectively. Denote $C_1 = H_1$, $Q_1 = H_2$. For a given $x_1 \in C_1$ and $y_1 \in Q_1$, let the iterative sequences $\{x_n\}$ and $\{y_n\}$ be generated by*

$$\begin{cases} u_n = J_\lambda^{T_1}(x_n - \mu_n A^*(Ax_n - By_n)), \\ p_n = \delta_{0,n} u_n + \sum_{i=1}^{m} \delta_{i,n} M_i(u_n), \\ v_n = J_\lambda^{T_2}(y_n + \mu_n B^*(Ax_n - By_n)), \\ q_n = \delta_{0,n} v_n + \sum_{i=1}^{m} \delta_{i,n} N_i(v_n), \\ C_{n+1} \times Q_{n+1} \\ \quad = \{(x, y) \in C_n \times Q_n : \| p_n - x \|^2 + \| q_n - y \|^2 \leq \| x_n - x \|^2 + \| y_n - y \|^2\}, \\ x_{n+1} = P_{C_{n+1}} x_n, \\ y_{n+1} = P_{Q_{n+1}} y_n, \end{cases} \tag{4.2}$$

*for all $n \in \mathbb{N}$, where $\{\delta_{i,n}\}$ is a sequence such that $\delta_{i,n} \in (0, 1)$, $\sum_{i=0}^{m} \delta_{i,n} = 1$. The step size $\mu_n$ is chosen in such a way that*

$$\mu_n = \begin{cases} \dfrac{\beta_n F(x_n, y_n)}{\|\nabla F(x_n, y_n)\|^2}, & if\ \nabla F(x_n, y_n) \neq 0 \\ 0, & otherwise, \end{cases} \tag{4.3}$$

*where $\beta_n \in (0, 2)$ and $\inf_{n \in \mathbb{N}}[\beta_n(2 - \beta_n)] > 0$.*
*If the solution set $\Omega_1 := \{(p, q) \in H_1 \times H_2 : p \in \bigcap_{i=1}^{m} \mathrm{Fix}(M_i) \bigcap T_1^{-1}(0), q \in \bigcap_{i=1}^{m} \mathrm{Fix}(N_i) \bigcap T_2^{-1}(0) and\ Ap = Bq\}$ is nonempty, then there exists $(x^*, y^*) \in \Omega_1$ such that $x_n \to x^*$ and $y_n \to y^*$.*

***Proof*** Let the operators $T$, $F$ and $R_i$ be defined by (4.1). From Lemma 2.5, T is a maximal monotone operator. Here, function $F$ is of class $C^1$ and for every $(x, y) \in H_1 \times H_2$, we have $\nabla F(x, y) = (A^*(Ax - By), -B^*(Ax - By))$. Here, $R_i$ is a quasi-nonexpansive mapping such that $I - R_i$ is demiclosed at 0, for each $i = 1, 2, \ldots, m$.
Conditions (A0) and (A1) follow from Definition 2.2 and the fact that $d_n = \nabla F(x, y) = (A^*(Ax - By), -B^*(Ax - By))$. Hence, from Theorem 3.3, we conclude the proof. □

### 4.2 Split equality equilibrium fixed point problem

Let $C$ be a nonempty closed and convex subset of a real Hilbert space $H$ and $f : C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problem for $f$ is to find $x^* \in C$ such that

$$f(x^*, y) \geq 0, \quad \forall y \in C. \tag{4.4}$$

The solution set of equilibrium problem is denoted by EP($f$).

Recently, many authors (see, e.g. Colao et al. 2011; Eslamian 2013; Takahashi and Takahashi 2007) have studied strong convergence of iterative schemes for finding a common solution of an equilibrium problem and fixed point problem for a nonlinear mapping.

Let us assume that the bifunction $f$ satisfies the following conditions:

$(B1)$ $f(x, x) = 0, \quad \forall x \in C,$
$(B2)$ $f$ is monotone, i.e., $f(x, y) + f(y, x) \leq 0, \forall x, y \in C,$
$(B3)$ $\lim_{t\to 0} f(tz + (1-t)x, y) \leq f(x, y)$, for each $x, y, z \in C,$
$(B4)$ for each $x \in C$, $y \mapsto f(x, y)$ is convex and lower semicontinuous.

Further, we quote the following lemma:

**Lemma 4.1** (Takahashi et al. 2010, Theorem 4.2) *Let $C$ be a nonempty closed and convex subset of a Hilbert space $H$ and let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying $(B1)-(B4)$. Let $\Phi_f$ be a set-valued mapping of $H$ into itself defined by*

$$\Phi_f(x) = \begin{cases} \left\{z \in C : f(z, y) + \frac{1}{\lambda}\langle y - z, z - x\rangle \geq 0\right\}, & \forall x \in C \\ \emptyset, & \forall x \notin C. \end{cases} \tag{4.5}$$

*Then $EP(f) = \Phi_f^{-1}(0)$ and $\Phi_f$ is a maximal monotone operator with* dom $\Phi_f \subset C$. *Furthermore, for any $x \in H$ and $\lambda > 0$, the resolvent $G_\lambda^f$ of $f$ coincides with the resolvent of $\Phi_f$, where*

$$G_\lambda^f x = \left\{z \in C : f(z, y) + \frac{1}{\lambda}\langle y - z, z - x\rangle \geq 0 \ \forall y \in C\right\}.$$

The so-called *Split equality equilibrium fixed point problem* with respect to bifunction $f$ and $g$ is to find $x \in C$ and $y \in Q$ such that

$$\text{(SEEFP)} \quad \text{find } x \in \bigcap_{i=1}^m \text{Fix}(M_i) \bigcap EP(f) \text{ and } y \in \bigcap_{i=1}^m \text{Fix}(N_i) \bigcap EP(g) \text{ such that } Ax = By.$$

Using Lemma 4.1 and Theorem 4.1, we have the following result.

**Theorem 4.2** *Let $H_1$, $H_2$ and $H_3$ be real Hilbert spaces, $C$ and $Q$ be two nonempty closed convex subset of $H_1$ and $H_2$, respectively, and $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be bounded linear operators. Let $f : C \times C \to \mathbb{R}$ and $g : Q \times Q \to \mathbb{R}$ be two bifunctions satisfying $(B1)-(B4)$. Suppose that for each $i \in \{1, 2, \ldots, m\}$, $M_i : H_1 \to H_1$ and $N_i : H_2 \to H_2$ be quasi-nonexpansive operators with $I - M_i$ and $I - N_i$ are demi-close at zero. For a given $x_1 \in C_1$ and $y_1 \in Q_1$, let the iterative sequences $\{x_n\}$ and $\{y_n\}$ be generated by*

$$\begin{cases} u_n = G_\lambda^f(x_n - \mu_n A^*(Ax_n - By_n)), \\ p_n = \delta_{0,n}u_n + \sum_{i=1}^m \delta_{i,n}M_i(u_n), \\ v_n = G_\lambda^g(y_n + \mu_n B^*(Ax_n - By_n)), \\ q_n = \delta_{0,n}v_n + \sum_{i=1}^m \delta_{i,n}N_i(v_n), \\ C_{n+1} \times Q_{n+1} \\ \quad = \{(x, y) \in C_n \times Q_n : \| p_n - x \|^2 + \| q_n - y \|^2 \le \| x_n - x \|^2 + \| y_n - y \|^2\}, \\ x_{n+1} = P_{C_{n+1}}x_n, \\ y_{n+1} = P_{Q_{n+1}}y_n, \end{cases} \tag{4.6}$$

*for all $n \in \mathbb{N}$. Let the sequences $\{\delta_{i,n}\}$ and $\{\mu_n\}$ satisfy the condition of Theorem 4.1. If the solution set $\Omega_2 := \{(p, q) \in H_1 \times H_2 : p \in \bigcap_{i=1}^m \mathrm{Fix}(M_i) \bigcap EP(f), q \in \bigcap_{i=1}^m \mathrm{Fix}(N_i) \bigcap EP(g)$ and $Ap = Bq\}$ is nonempty, then there exists $(x^*, y^*) \in \Omega_2$ such that $x_n \to x^*$ and $y_n \to y^*$.*

## 5 Numerical experiments

In this section, we discuss some examples in support of Theorems 3.1, 3.2, 3.3, 4.1 and 4.2. We have implemented our code in Python 2.7 (Anaconda) on a personal Dell computer with Intel(R)Core(TM) i5-7200U CPU 2.50 GHz and RAM 8.00 GB.

### 5.1 Test problem for problem (P)

***Example 5.1*** Let $H = \mathbb{R}^N$, $N \in \mathbb{N}$, be a real Hilbert space. Let $z = (x_1, x_2, \dots, x_N)$ and $F : H \to \mathbb{R}$ be a function defined by $F(z) = \|z\|^2$. Let $L : H \to H$ be an operator defined by

$$L[x_1, \dots, x_N] = \begin{bmatrix} \frac{1}{2N} & 0 & \cdots \\ \vdots & \ddots & 0 \\ 0 & 0 & \frac{1}{2N} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}.$$

Note that $L$ is a nonexpansive operator. Hence, by Lemma 2.4, $T = (I + \frac{1}{2}L)$ is a maximal monotone operator.

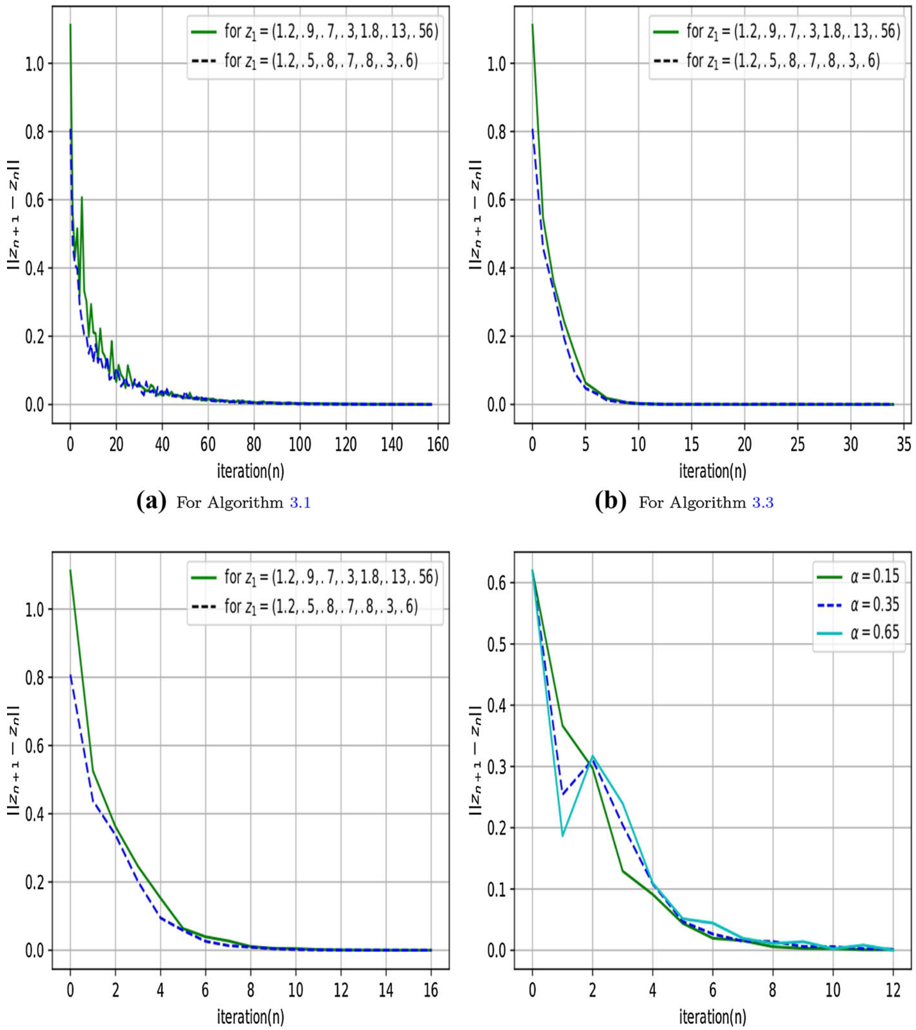
For $i = 1, 2, \dots, m$, $R_i : H \to H$ is defined by

$$R_i(x_1, x_2, \dots, x_N) = (R_{i_1}(x_1), R_{i_2}(x_2), \dots, R_{i_N}(x_N)),$$

where

$$R_{i_j}(x_j) = \begin{cases} 0, & if\ x_j = 0, \\ \frac{x_j}{i+1} \sin \frac{1}{x_j}, & if\ x_j \neq 0, \end{cases} \tag{5.1}$$

for $j = 1, 2, \dots, N$. Here, each $R_{i_j}$ is quasi-nonexpansive operator with $\mathrm{Fix}(R_{i_j}) = \{0\}$. Also suppose that $\lambda = 2.5$, $\alpha = 0.3$, $\omega_n = \frac{1}{n^2}$, $\beta_n = \frac{n}{n+1}$, $\delta_{i,n} = \frac{1}{m+1}$, $\forall i = 0, 1, 2, \dots, m$ and search direction $d_n = \nabla F(z_n)$. Observe that all the assumptions of Theorems 3.1, 3.2




**(a)** For Algorithm 3.1

**(b)** For Algorithm 3.3

**(c)** For Algorithm 3.2

**(d)** For Algorithm 3.2 for different values of $\alpha$

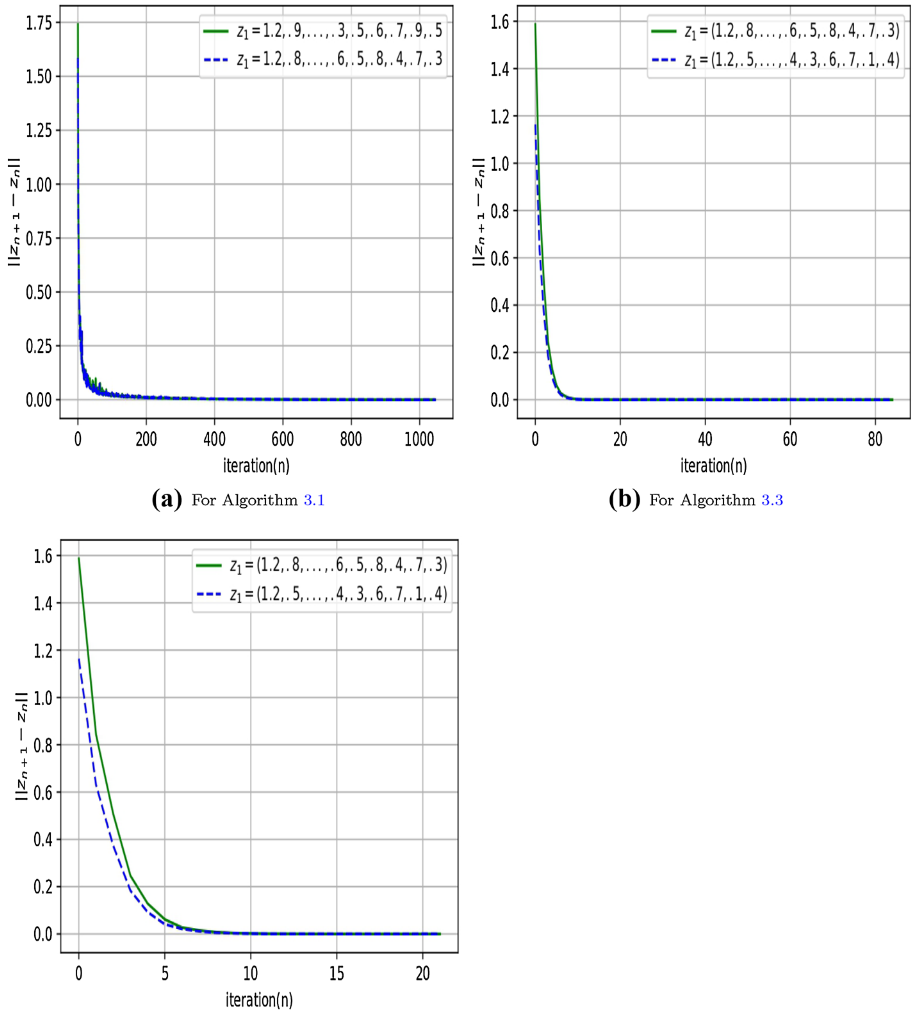
**Fig. 1** Convergence of sequence $\{\|z_{n+1} - z_n\|\}$ for Example 5.1 for $z_1 \in \mathbb{R}^7$

and 3.3 are satisfied. Consequently, we conclude that sequence $\{z_n\}$ converges strongly to $z^* = (0, 0) \in \Omega$.

For stopping criteria $\|z_{n+1} - z_n\| < \epsilon = 10^{-4}$, Figures 1a–c and 2a–c show the convergence of sequence $\{\|z_{n+1} - z_n\|\}$ for different values of $z_1 \in \mathbb{R}^7$ and $z_1 \in \mathbb{R}^{25}$ using Algorithms 3.1, 3.3, and 3.2, respectively. Figure 1d shows the convergence of sequence $\{\|z_{n+1} - z_n\|\}$ for different values of $\alpha \in [0, 1)$ and $z_0 = z_1 = (.23, .4, .6, .52, .7, .8, .7) \in \mathbb{R}^7$ using Algorithm 3.2. From Table 1, we observe the following:

(i) For $z_1 = u_1 = (1.2, .9, .7, .3, 1.8, .13, .56) \in \mathbb{R}^7$, Algorithms 3.1, 3.2, and 3.3 approximate the solution after 181, 19, 35 iterations, respectively.

(ii) For $z_1 = v_1 = (1.2, .5, .8, .7, .8, .3, .6) \in \mathbb{R}^7$, Algorithms 3.1, 3.2, and 3.3 approximate the solution after 180, 17, 47 iterations, respectively.




**(a)** For Algorithm 3.1

**(b)** For Algorithm 3.3

**(c)** For Algorithm 3.2

**Fig. 2** Convergence of sequence $\{\|z_{n+1} - z_n\|\}$ for Example 5.1 for $z_1 = u_1' \in \mathbb{R}^{25}$ and $z_1 = v_1' \in \mathbb{R}^{25}$

***Remark 5.1*** (i) We observe from Example 5.1 that Algorithm 3.2 has better performance than Algorithms 3.1 and 3.3.

(ii) From Figs. 1 and 2, we observe that, when we increase the dimension of the Euclidean space, Algorithm 3.2 is stable (approximate the solution after same number of iterations), but Algorithms 3.1 and 3.3 are not stable (Tables 2, 3).

## 5.2 Test problem for split equality variational inclusion fixed point problem

***Example 5.2*** In Theorem 4.1, set $H_1 = H_2 = H_3 = \mathbb{R}^N$, $N \in \mathbb{N}$. Let $Ax = x$, $By = 4y$, where $x = (x_1, x_2, \ldots, x_N)$ and $y = (y_1, y_2, \ldots, y_N)$. Let $L_1 : H \to H$ be an operator defined by

**Table 1** Numerical values of $\|z_{n+1} - z_n\|$ for Algorithms 3.1, 3.2, 3.3 using Example 5.1

| Number of iteration $n$ | Alg 3.1 $\|z_{n+1} - z_n\|$ $z_1 = u_1$ | Alg 3.2 $\|z_{n+1} - z_n\|$ $z_0 = z_1 = u_1$ | Alg 3.3 $\|z_{n+1} - z_n\|$ for $z_1 = u_1$ | Alg 3.1 $\|z_{n+1} - z_n\|$ $z_1 = v_1$ | Alg 3.2 $\|z_{n+1} - z_n\|$ $z_0 = z_1 = v_1$ | Alg 3.3 $\|z_{n+1} - z_n\|$ for $z_1 = v_1$ |
|---|---|---|---|---|---|---|
| 1 | 1.111355459 | 1.11135545 | 1.111355459 | 0.807841464 | 0.807841464 | 0.807841464 |
| 2 | 0.560788075 | 0.525590382 | 0.546042587 | 0.469027476 | 0.438109943 | 0.458122039 |
| 3 | 0.432454832 | 0.362295749 | 0.358425276 | 0.409169418 | 0.337852137 | 0.333163686 |
| 4 | 0.514990402 | 0.245900579 | 0.241258612 | 0.397532529 | 0.200984913 | 0.19226088 |
| 5 | 0.318113797 | 0.152744832 | 0.149031721 | 0.295789681 | 0.094680962 | 0.090804417 |
| 6 | 0.606482389 | 0.063477165 | 0.063186105 | 0.24463756 | 0.057930666 | 0.0474499 |
| 7 | 0.333420241 | 0.039431031 | 0.041273598 | 0.205466418 | 0.026097965 | 0.032162785 |
| 8 | 0.301693119 | 0.02747681 | 0.017916522 | 0.200592491 | 0.013694629 | 0.012945143 |
| 9 | 0.19944876 | 0.010677138 | 0.011791397 | 0.149696064 | 0.008884985 | 0.007917026 |
| 10 | 0.293232748 | 0.005590603 | 0.004389749 | 0.176534926 | 0.004335683 | 0.005439304 |
| 11 | 0.211314695 | 0.004661433 | 0.003067762 | 0.123362506 | 0.001987702 | 0.002089563 |
| 12 | 0.207916023 | 0.002028311 | 0.00172346 | 0.175728769 | 0.000936195 | 0.001389622 |
| 13 | 0.128497974 | 0.00091327 | 0.000710803 | 0.123397524 | 0.000675208 | 0.000693258 |
| 14 | 0.221563062 | 0.000514706 | 0.000545551 | 0.141969567 | 0.00033094 | 0.000346182 |
| 15 | 0.154195036 | 0.000317561 | 0.00061265 | 0.111010728 | 0.000260525 | 0.000172959 |
| 16 | 0.221563062 | 0.000274733 | 0.00039271 | 0.1012811 | 0.000150887 | 0.000455915 |
| 17 | 0.114313918 | 0.000158071 | 0.000444428 | 0.14161302 | 8.47e-05 | 0.000532276 |

**Table 1** continued

| Number of iteration $n$ | Alg 3.1 $\|z_{n+1} - z_n\|$ $z_1 = u_1$ | Alg 3.2 $\|z_{n+1} - z_n\|$ $z_0 = z_1 = u_1$ | Alg 3.3 $\|z_{n+1} - z_n\|$ for $z_1 = u_1$ | Alg 3.1 $\|z_{n+1} - z_n\|$ $z_1 = v_1$ | Alg 3.2 $\|z_{n+1} - z_n\|$ $z_0 = z_1 = v_1$ | Alg 3.3 $\|z_{n+1} - z_n\|$ for $z_1 = v_1$ |
|---|---|---|---|---|---|---|
| 18 | 0.091870286 | 0.000154502 | 0.000522484 | 0.073831215 | | 0.000406688 |
| 19 | 0.185023755 | 8.10e-05 | 0.000365838 | 0.079403073 | | 0.000268665 |
| 20 | 0.086870121 | | 0.000277389 | 0.071544591 | | 0.000472878 |
| 25 | 0.049247005 | | 0.000366993 | 0.08089255 | | 0.000596657 |
| 29 | 0.061605379 | | 0.000137194 | 0.072851194 | | 0.000548005 |
| 35 | 0.040672325 | | 9.27e-05 | 0.038717669 | | 0.000244666 |
| 45 | 0.036700107 | | | 0.025791623 | | 0.000231942 |
| 47 | 0.027580497 | | | 0.023783999 | | 9.34e-05 |
| 55 | 0.018082968 | | | 0.016136587 | | |
| 75 | 0.008881474 | | | 0.005407339 | | |
| 95 | 0.004164621 | | | 0.002636047 | | |
| 115 | 0.001327492 | | | 0.001214445 | | |
| 145 | 0.000368773 | | | 0.000425802 | | |
| 175 | 0.000142057 | | | 0.00016214 | | |
| 180 | 0.000110926 | | | 8.54e-05 | | |
| 181 | 9.64e-05 | | | | | |

**Table 2** CPU time and number of iterations for Algorithms 3.1, 3.2, 3.3 using Example 5.1 for $z_1 = u_1 \in \mathbb{R}^7$ and $z_1 = v_1 \in \mathbb{R}^7$

| | Alg 3.1 $z_1 = u_1$ | Alg 3.2 $z_0 = z_1 = u_1$ | Alg 3.3 $z_1 = u_1$ | Alg 3.1 $z_1 = v_1$ | Alg 3.2 $z_0 = z_1 = v_1$ | Alg 3.3 $z_1 = v_1$ |
|---|---|---|---|---|---|---|
| CPU time (in s) | 11.254 | 0.565 | 0.798 | 11.093 | 0.372 | 1.227 |
| Number of iterations | 181 | 19 | 35 | 180 | 17 | 47 |

**Table 3** CPU time and number of iterations for Algorithms 3.1, 3.2, 3.3 using Example 5.1 for $z_1 = u_1' = (1.2, .8, .6, .9, .7, 1, .8, .4, .8, .6, .2, .3, .4, .33, .6, 1.2, .35, .47, .8, .6, .5, .8, .4, .7, .3) \in \mathbb{R}^{25}$ and $z_1 = v_1' = (1.2, .5, .8, .7, .8, .3, .6, .2, .7, .3, .1, .2, .3, .23, .2, .1, .15, .17, .5, .4, .3, .6, .7, .1, .4) \in \mathbb{R}^{25}$

| | Alg 3.1 $z_1 = u_1'$ | Alg 3.2 $z_0 = z_1 = u_1'$ | Alg 3.3 $z_1 = u_1'$ | Alg 3.1 $z_1 = v_1'$ | Alg 3.2 $z_0 = z_1 = v_1'$ | Alg 3.3 $z_1 = v_1'$ |
|---|---|---|---|---|---|---|
| CPU time (s) | 3011.928 | 4.731 | 36.224 | 2873.379 | 3.186 | 46.752 |
| Number of iterations | 1071 | 22 | 85 | 1021 | 27 | 106 |

$$L_1[x_1, \dots, x_N] = \begin{bmatrix} \frac{1}{2} & 0 & \cdots \\ \vdots & \ddots & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}$$

and $L_2 : H \to H$ be an operator defined by

$$L_2[x_1, \dots, x_N] = \begin{bmatrix} \frac{1}{3} & 0 & \cdots \\ \vdots & \ddots & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix},$$

which are nonexpansive operators. Hence, by Lemma 2.4, $T_i = (I + \frac{1}{2}L_i)$, for $i = 1, 2$ are maximal monotone operators. Let $M_i : H \to H$, for $i = 1, 2, \dots, m$ be defined by

$$M_i(x_1, x_2, \dots, x_N) = (M_{i_1}(x_1), M_{i_2}(x_2), \dots, M_{i_N}(x_N)),$$

where

$$M_{i_j}(x_j) = \begin{cases} 0, & if\ x_j = 0, \\ \frac{x_j}{i+1} \sin \frac{1}{x_j}, & if\ x_j \neq 0, \end{cases} \tag{5.2}$$

for $j = 1, 2, \dots, N$. Also, suppose that $\lambda = 2.5$, $\beta_n = \frac{n}{n+1}$ and $\delta_{i,n} = \frac{1}{m+1}, \forall i = 0, 1, 2, \dots, m$.
Let $N_i : H \to H$, for $i = 1, 2, \dots, m$ be defined by

$$N_i(x_1, x_2, \dots, x_N) = (N_{i_1}(x_1), N_{i_2}(x_2), \dots, N_{i_N}(x_N)),$$

where

$$N_{i_j}(x_j) = \begin{cases} 0, & if\ \| x_j \| \leq 1, \\ (1 - \frac{1}{(i+1)\|x_j\|})x_j, & if\ \| x_j \| > 1, \end{cases} \tag{5.3}$$

for $j = 1, 2, \dots, N$. Here, each $M_{i_j}$ and $N_{i_j}$ are quasi-nonexpansive mappings. Observe that all the assumptions of Theorem 4.1 are satisfied. So, we conclude that sequence $\{(x_n, y_n)\}$ converges strongly to $(x^*, y^*) = (0, 0) \in \Omega_1$.

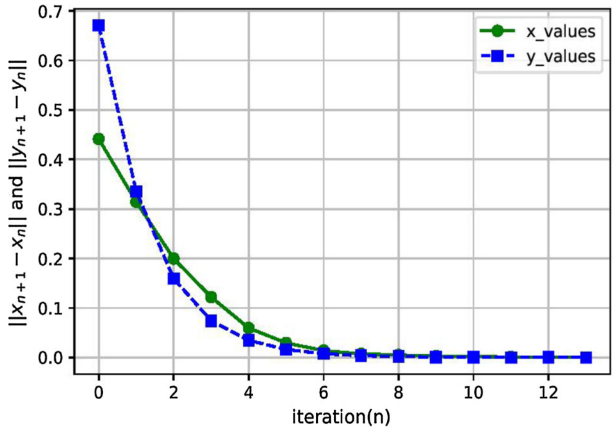



**Fig. 3** Convergence of sequences $\{\|x_{n+1} - x_n\|\}$ and $\{\|y_{n+1} - y_n\|\}$ for Example 5.2

For stopping criteria $\|x_{n+1} - x_n\| < \epsilon = 10^{-4}$ and $\|y_{n+1} - y_n\| < \epsilon = 10^{-4}$, Fig. 3 and Table 4 show the convergence of sequences $\{\|x_{n+1} - x_n\|\}$ and $\{\|y_{n+1} - y_n\|\}$ using Theorem 4.1. Table 5 and Fig. 4 show the comparison between the convergence of algorithm of Theorem 4.1 and algorithm of Theorem 1.1 (Chang et al. 2016).

### 5.3 Test problem for split equality equilibrium fixed point problem

***Example 5.3*** Let $H_1 = H_2 = H_3 = \mathbb{R}$ and $C = Q = [0, \infty)$, and define the bifunctions $f : C \times C \to \mathbb{R}$ and $g : Q \times Q \to \mathbb{R}$ by

$$f(x, y) = y^2 + xy - 2x^2, \qquad g(x, y) = x(y - x).$$

We observe that the functions $f$ and $g$ satisfying the conditions $(B1) - (B4)$. Also, we have $G_\lambda^f x = \frac{x}{3\lambda+1}$ and $G_\lambda^g x = \frac{x}{\lambda+1}$. Let $Ax = x, By = 4y$. Let $M_i : H \to H$, for $i = 1, 2, \ldots, m$ be defined by

$$M_i(x) = \begin{cases} 0, & if\ x = 0, \\ \frac{x}{i+1} \sin \frac{1}{x}, & if\ x \neq 0. \end{cases} \tag{5.4}$$

Also, suppose that $\lambda = 1$, $\beta_n = \frac{n}{n+1}$ and $\delta_{i,n} = \frac{1}{m+1}$, $\forall i = 0, 1, 2, \ldots, m$. Let $N_i : H \to H$, for $i = 1, 2, \ldots, m$ be defined by

$$N_i(x) = \begin{cases} 0, & if\ |x| \leq 1, \\ (1 - \frac{1}{(i+1)|x|})x, & if\ |x| > 1. \end{cases} \tag{5.5}$$

Here, each $M_i$ and $N_i$ are quasi-nonexpansive mappings. Observe that all the assumptions of Theorem 4.2 are satisfied. So, we conclude that sequence $\{(x_n, y_n)\}$ converges strongly to $(x^*, y^*) = (0, 0) \in \Omega_1$.

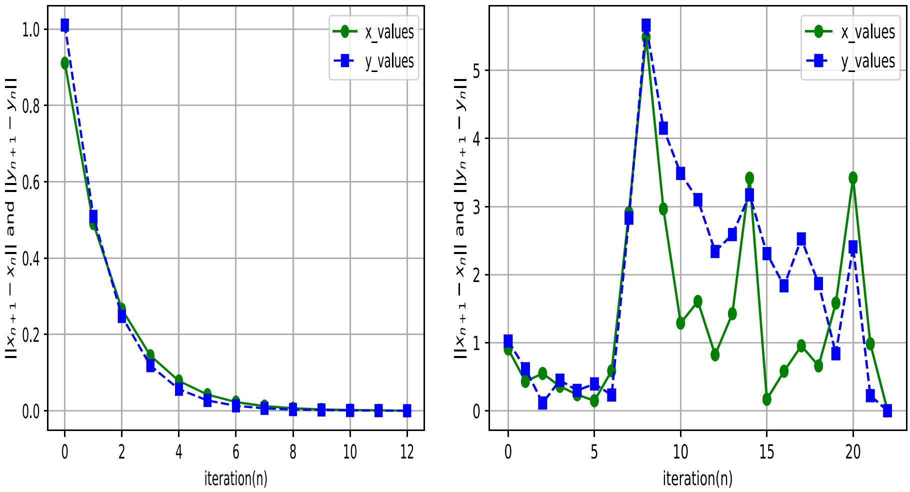
For stopping criteria $\|x_{n+1} - x_n\| < \epsilon = 10^{-4}$ and $\|y_{n+1} - y_n\| < \epsilon = 10^{-4}$, Fig. 5 and Table 6 show the convergence of sequences $\{\|x_{n+1} - x_n\|\}$ and $\{\|y_{n+1} - y_n\|\}$ using Theorem 4.2. The CPU time is 0.0920000076294.

**Table 4** Numerical values for $\|x_{n+1} - x_n\|$ and $\|y_{n+1} - y_n\|$ using Theorem 4.1 and Example 5.2

| Number of iteration n | $\|\|x_{n+1} - x_n\|\|$ for $x_1 = (0.3, 0.4, 0.1, 0.5, 0.3, 0.4, 0.8)$ | $\|\|y_{n+1} - y_n\|\|$ for $y_1 = (0.2, 0.5, 0.2, 0.6, 0.9, 0.4, 0.2)$ |
|---|---|---|
| 1 | 0.441459366509 | 0.670736200741 |
| 2 | 0.314515785474 | 0.335249956424 |
| 3 | 0.199682315695 | 0.159382646293 |
| 4 | 0.122192308631 | 0.0743614822021 |
| 5 | 0.0596315388329 | 0.0346232671892 |
| 6 | 0.0295076103278 | 0.0161793158314 |
| 7 | 0.0138869344931 | 0.00767955223807 |
| 8 | 0.00744587236442 | 0.0036488401429 |
| 9 | 0.005087552239 | 0.00215841680041 |
| 10 | 0.00299508323333 | 0.000845020212871 |
| 11 | 0.00174528964407 | 0.000498184762847 |
| 12 | 0.00116160902035 | 0.000182465498299 |
| 13 | 0.000757512055394 | 0.000225281371797 |
| 14 | 2.88242408366e-08 | 1.94312998658e-08 |
| CPU time (second) | 0.318000078201 | 0.318000078201 |

**Table 5** Numerical values for $\|x_{n+1} - x_n\|$ and $\|y_{n+1} - y_n\|$ using Theorem 4.1 (with $M_i = N_i = 0$, for each $i$) and Theorem 1.1 based on Example 5.2 for $x_1 = (.5, .4, .7, .5, .8, .4, .8, .6, .4, .9) \in \mathbb{R}^{10}$ and $y_1 = (.6, .5, .8, .6, .9, .4, 0.7, .5, .7, .4) \in \mathbb{R}^{10}$

| Number of iteration<br>n | Algorithm 3.3<br>$\|\|x_{n+1} - x_n\|\|$ | Algorithm 3.3<br>$\|\|y_{n+1} - y_n\|\|$ | Theorem 1.1<br>$\|\|x_{n+1} - x_n\|\|$ | Theorem 1.1<br>$\|\|y_{n+1} - y_n\|\|$ |
|---|---|---|---|---|
| 1 | 0.910659609256 | 1.01019111308 | 0.907617400975 | 1.02370830237 |
| 2 | 0.490095891191 | 0.509075518126 | 0.43097969185 | 0.615883156806 |
| 3 | 0.265734204135 | 0.247016410749 | 0.547013300226 | 0.113580260703 |
| 4 | 0.144181589964 | 0.118192920744 | 0.358852800103 | 0.448161929122 |
| 5 | 0.0781036029219 | 0.0563215812365 | 0.235835769276 | 0.297751273743 |
| 6 | 0.0422106783556 | 0.0268690302979 | 0.147133421686 | 0.393940861406 |
| 7 | 0.0227576965845 | 0.0128736915445 | 0.585898818053 | 0.230797587291 |
| 8 | 0.0122423135108 | 0.00620748806621 | 2.90929289433 | 2.82865918495 |
| 9 | 0.00657266781699 | 0.00301616273019 | 5.48854816984 | 5.66463114008 |
| 10 | 0.00329835389609 | 0.00201882364069 | 2.96443716184 | 4.15055400307 |
| 11 | 0.00196237478095 | 0.000633415347949 | 1.28703316785 | 3.48613989009 |
| 12 | 0.00112270196754 | 0.000296375687225 | 1.60512734864 | 3.10093855775 |
| 13 | 6.19544194522e-08 | 5.85332223818e-08 | 0.821706713234 | 2.33936005868 |
| 14 | | | 1.42552388968 | 2.58749595914 |
| 15 | | | 3.41640057063 | 3.17060261802 |
| 16 | | | 0.168131651754 | 2.30826256296 |
| 17 | | | 0.580212481133 | 1.8353028007 |
| 18 | | | 0.953683655272 | 2.52379984097 |
| 19 | | | 0.659188040845 | 1.86794039664 |
| 20 | | | 1.58175062578 | 0.837352549408 |
| 21 | | | 3.41895955827 | 2.40713441754 |
| 22 | | | 0.982561417401 | 0.220945030811 |
| 23 | | | 4.3223342555e-05 | 1.06500374407e-05 |
| CPU time (second) | 0.631999969482 | 0.631999969482 | 8.0529999733 | 8.0529999733 |




**(a)** For Theorem 4.1 with $M_i = N_i = 0$, for each $i$ **(b)** For Theorem 1.1

**Fig. 4** Convergence of sequences $\{\|x_{n+1} - x_n\|\}$ and $\{\|y_{n+1} - y_n\|\}$ based on Example 5.2

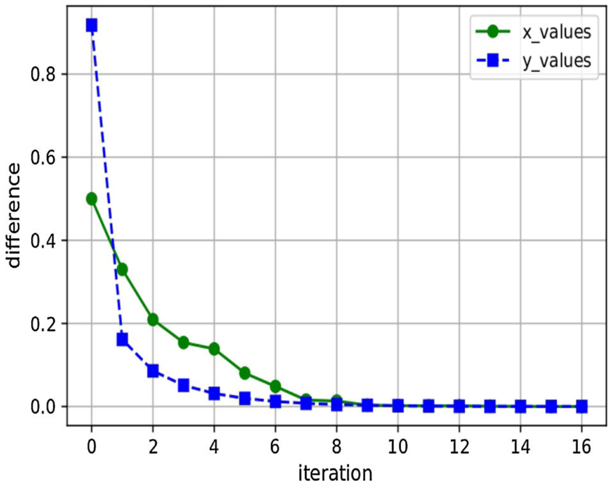



**Fig. 5** Convergence of sequences $\{\|x_{n+1} - x_n\|\}$ and $\{\|y_{n+1} - y_n\|\}$ for Example 5.3

## 6 Conclusion

In this paper, the minimization of a nonnegative lower semicontinuous function over the intersection of a finite number of fixed point sets and a zero set has been studied. The generalized version of the algorithm given by Chang et al. (2016) is obtained and new algorithms with some modifications are presented. The comparison through example is made for the three algorithms, which further suggests that the rate of convergence of the third and second algorithms are faster than that of the generalized version. Also, we have obtained a common solution of three problems so that a single solution can be used for three different

**Table 6** Numerical values for $\|x_{n+1} - x_n\|$ and $\|y_{n+1} - y_n\|$ using Theorem 4.2 and Example 5.3

| Number of iteration n | $\|\|x_{n+1} - x_n\|\|$ for $x_1 = 1.5$ | $\|\|y_{n+1} - y_n\|\|$ for $y_1 = 1.3$ |
|---|---|---|
| 1 | 0.5 | 0.918007096641 |
| 2 | 0.330035960122 | 0.161488666253 |
| 3 | 0.20922616607 | 0.08583619912 |
| 4 | 0.153955532019 | 0.0510805504004 |
| 5 | 0.138720172321 | 0.0314529235324 |
| 6 | 0.080385051789 | 0.0196247393265 |
| 7 | 0.0484058981149 | 0.0122275565211 |
| 8 | 0.0155829454539 | 0.00762143353299 |
| 9 | 0.0132753732442 | 0.00472982657387 |
| 10 | 0.00326142540624 | 0.00300107406444 |
| 11 | 0.00222545130171 | 0.00186115299614 |
| 12 | 0.00158178377996 | 0.00115021857958 |
| 13 | 0.00171855240509 | 0.000649649786976 |
| 14 | 0.00084490137982 | 0.000450952075506 |
| 15 | 0.000419623863243 | 0.000313142969688 |
| 16 | 0.000166003354156 | 0.000222866067994 |
| 17 | 5.81580335432e-05 | 0.000129818654887 |

purposes. The work to prove the convergence of these algorithms without considering the assumptions could hold the scope for future research.

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## References

Agarwal RP, O'Regan D, Sahu D (2009) Fixed point theory for Lipschitzian-type mappings with applications, vol 6. Springer, Berlin

Alvarez F, Attouch H (2001) An inertial proximal method for maximal monotone operators via discretization of a nonlinear oscillator with damping. Set Value Anal 9(1–2):3–11

Bauschke HH, Combettes PL (2017) Convex analysis and monotone operator theory in Hilbert spaces. 2nd edn. CMS Books in Mathematics. Springer, Cham. https://doi.org/10.1007/978-3-319-48311-5

Bot RI, Csetnek ER, László SC (2016) An inertial forward–backward algorithm for the minimization of the sum of two nonconvex functions. EURO J Comput Optim 4(1):3–25

Byrne C (2002) Iterative oblique projection onto convex sets and the split feasibility problem. Inverse Prob 18(2):441

Censor Y, Bortfeld T, Martin B, Trofimov A (2006) A unified approach for inversion problems in intensity-modulated radiation therapy. Phys Med Biol 51(10):2353

Censor Y, Elfving T (1994) A multiprojection algorithm using Bregman projections in a product space. Numer Algorithm 8(2):221–239

Censor Y, Elfving T, Kopf N, Bortfeld T (2005) The multiple-sets split feasibility problem and its applications for inverse problems. Inverse Prob 21(6):2071

Censor Y, Gibali A, Reich S (2012) Algorithms for the split variational inequality problem. Numer Algorithm 59(2):301–323

Censor Y, Motova A, Segal A (2007) Perturbed projections and subgradient projections for the multiple-sets split feasibility problem. J Math Anal Appl 327(2):1244–1256
Chang SS, Lin W, Lijuan Q, Zhaoli M (2016) Strongly convergent iterative methods for split equality variational inclusion problems in Banach spaces. Acta Math Sci 36(6):1641–1650
Chuang CS (2017) Simultaneous subgradient algorithms for the generalized split variational inclusion problem in Hilbert spaces. Numer Funct Anal Optim 38(3):306–326
Colao V, Marino G, Muglia L (2011) Viscosity methods for common solutions for equilibrium and hierarchical fixed point problems. Optimization 60(5):553–573
Dong Q, Yuan H, Cho Y, Rassias TM (2018) Modified inertial mann algorithm and inertial CQ-algorithm for nonexpansive mappings. Optim Lett 12(1):87–102
Eslamian M (2013) Hybrid method for equilibrium problems and fixed point problems of finite families of nonexpansive semigroups. Revista de la Real Academia de Ciencias Exactas Fisicas y Naturales Serie A Matematicas 107(2):299–307
Kazmi KR, Rizvi S (2014) An iterative method for split variational inclusion problem and fixed point problem for a nonexpansive mapping. Optim Lett 8(3):1113–1124
Majee P, Nahak C (2018) A modified iterative method for split problem of variational inclusions and fixed point problems. Comput Appl Math 37(4):4710–4729
Moudafi A (2011) Split monotone variational inclusions. J Optim Theory Appl 150(2):275–283
Moudafi A (2013) A relaxed alternating CQ-algorithm for convex feasibility problems. Nonlinear Anal Theory Methods Appl 79:117–121
Moudafi A, Oliny M (2003) Convergence of a splitting inertial proximal method for monotone operators. J Comput Appl Math 155(2):447–454
Moudafi A et al (2014) Alternating CQ-algorithm for convex feasibility and split fixed-point problems. J. Nonlinear Convex Anal 15(4):809–818
Polyak BT (1964) Some methods of speeding up the convergence of iteration methods. USSR Comput Math Math Phys 4(5):1–17
Ryu EK, Boyd S (2016) Primer on monotone operator methods. Appl Comput Math 15(1):3–43
Takahashi S, Takahashi W (2007) Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces. J Math Anal Appl 331(1):506–515
Takahashi S, Takahashi W, Toyoda M (2010) Strong convergence theorems for maximal monotone operators with nonlinear mappings in Hilbert spaces. J Optim Theory Appl 147(1):27–41
Xu HK (2002) Iterative algorithms for nonlinear operators. J Lond Math Soc 66(1):240–256
Xu HK (2006) A variable Krasnosel'skii–Mann algorithm and the multiple-set split feasibility problem. Inverse Prob 22(6):2021
Zegeye H, Shahzad N (2011) Convergence of mann's type iteration method for generalized asymptotically nonexpansive mappings. Comput Math Appl 62(11):4007–4014
Zhao J (2015) Solving split equality fixed-point problem of quasi-nonexpansive mappings without prior knowledge of operators norms. Optimization 64(12):2619–2630

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