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A Newman Penrose approach to Bertrand spacetime in general relativity

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A Newman Penrose approach to Bertrand spacetime in general relativity

T U Jeevitha[1], Pranav Raibagkar[2] and Sanjit Das[1*]

[1]Division of Physics, SAS, Vellore Institute of Technology, Chennai – 6000127, India
[2]Department of Physics, Indian Institute of Technology, (BHU) Varanasi UP– 221005, India
[*]Email for correspondence – sanjit.das@vit.ac.in

Abstract. Newman Penrose (NP) formalism is a new approach to general relativity was presented. In various geometrical groups are estimate on a chosen null tetrad basis for tetrad formalism. This approach widely used in analytical and numerical studies of Einstein's equations. The reasonable utilization of explicit complex straight emulsions of Ricci rotation coefficients which present, in the product, the spinor relative association. And we construct a null tetrad for general case and special attension for Bertrand's space-time type-1 metric for special case k=0.

1. Introduction

Einstein theory of general relativity gives a unique example of human ingenious, which dates back to 1915. So far, this is the best theory of gravity that explains from blackhole to gravitational wave. When solving Einstein's field equation in a background spacetime many times we do the computation in orthonormal basis for the sake of brevity [1][2]. Newman Penrose introduced one such tetrad formalism in 1962 which turns out to be a tremendously useful formalism in both analytical and numerical relativity [3]. We follow the metric signature (-, +, +, +) in this paper. A brief review of NP formalism is worthwhile [4]

2. Preparatory of Tetrad

One of the standard methods for treating the problem in GR is to consider the Einstein field conditions in a neighborhood arrange premise adjusted to the problem with which one is working. As of late, it has been demonstrated useful to choose an appropriate premise of four directly free vectors, to extend the applicable amounts on to the chosen basis and consider the conditions fulfilled by them [5][6]. It is a tetrad formalism. (i.e.) a narrowly defined intent of four uncurled self-dependent vector field called tetrad.

The decision of the tetrad basis relies upon the symmetries of the space-time with which we are working and somewhat is the piece of the problem are the utilization of the tetrad formalism. It is repeatedly used in GR to simplify many calculations. The most prominent examples are the NP and the GHP formalisms.

In this paper we are following the signature (-, +, +, +). The scalar product of any two vector

$$\vec{v}.\vec{u} = g_{ij} v^i u^j$$

$$\vec{v}.\vec{u} = v^{(a)}u_{(b)}$$

In coordinate representation

$$\vec{A} = A^{\mu}\hat{e}_{\mu}$$

At every point of spacetime. We define a set of four linearly index vectors $\{\hat{e}_{(a)}\}$ Moreover, assume that these vectors are such as,

$$\hat{e}_{(a)}.\hat{e}_{(b)} := \eta_{(a)(b)}$$

Where $\eta_{(a)(b)}$ is a constant. Matrix independent of the position in spacetime, in such a case the set of vectors $\{\hat{e}_{(a)}\}$ are called tetrad. $\eta_{(a)(b)}$ are elements of the metric tensor in the tetrad basis. When the tetrad is orthonormal, $\eta_{(a)(b)}$is nothing but Minkowski tensor [7]. Since the vectors $\{\hat{e}_{(a)}\}$ are linearly independent, it is clear that the matrix $\eta_{(a)(b)}$ can be inverted. We denote the inverse a $\eta^{(a)(b)}$ so that we have

$$g^{\mu\nu}g_{\mu\sigma} = \delta^{\nu}_{\sigma}$$

$$\eta^{(a)(c)}\,\eta_{(c)(b)} = \delta^{(a)}{}_{(b)}$$

Let us now introduce a new set of vectors $\{\hat{e}_{(a)}\}$ defined as

$$\overrightarrow{e^{(a)}} = \eta^{(a)(b)}\,\overrightarrow{e_{(b)}}$$

Now it is easy to see that

$$\overrightarrow{e^{(a)}}.\overrightarrow{e_{(b)}} = \delta^{(a)}{}_{(b)}$$

3. Preliminary of BST

Many known solutions of Einstein's equation describe stable, periodic motion. One of such examples is exposed by Perlick, and he named it as ``Bertrand spacetimes''. He is the person who converted Bertrand theorem in Newtonian mechanics to general relativity. The Bertrand Spacetimes are the spherically symmetric static spacetimes which allow geodesics similar to the trajectories of objects in Kepler potential or Hooke's potential, as stated in Bertrand theorem. Perlick stated in his revolutionary paper that there could be two categories of BSTs, Type-I, and Type-II [8]. We have taken for our study Type-I to calculating Newman Penrose formalism such as,

$$ds^2 = -\frac{dt^2}{G \mp r^2\left(1 - Dr^2 \pm \sqrt{((1-Dr^2)^2 - kr^4)}\right)^{-1}} + \frac{2\left(1 - Dr^2 \pm \sqrt{((1-Dr^2)^2 - kr^4)}\right)}{\beta^2((1-Dr^2)^2 - kr^4)}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2 \quad (1)$$

For convenience in calculations, we assign k to be 0, as allowed by the parameters. Also, we will consider the upper sign in the equation to get:

$$ds^2 = -\frac{dt^2}{G - \frac{r^2}{2-2Dr^2}} + \frac{4}{\beta^2(1-Dr^2)}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2$$

The components of the metric, we have

$$g_{ab} = \begin{pmatrix} -\frac{1}{G - \frac{r^2}{2-2Dr^2}} & 0 & 0 & 0 \\ 0 & \frac{4}{\beta^2\left(1+Dr^2\right)} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \tag{2}$$

4. Newman-Penrose Formalism

It is otherwise called Spin-coefficient formalism which is a tetrad formalism with a specific choice of the basis vector. It was first proposed in 1962, in their decision of a null basis was Penrose's consistent confidence that the basic component of space-time is its light cone arrangement which makes reachable the presentation of a spinor basis [9][3].

Newman-Penrose equation has turned into a known system in the developed arrangement of the condition relating the spinor component of the Riemann curvature tensor with the components of the spinor associations [10]. Additionally, it is main acknowledged and utilized in an ongoing literature review. The significant explanation behind the popularity of NP formalism is sufficiency and inward adaptability for building the exact solution of the Einstein field equations and for different examinations.

NP equation has demonstrated productive in considering the asymptotic proprieties of the gravitational field. From the meaning of asymptotic flatness and the utilization of the NP system, it is conceivable to study the working of the fields at infinity. Alongside the possibility of conformal infinity, empower Hawking method is to examine the gravitational field of a black hole.

4.1. Complex Null tetrad and the spin coefficients

In a four-dimensional pseudo-Riemannian manifold (M,g) we choose a frame of three real and one complex null orthogonal vectors l, n, m, and $\bar{m}$ [11]. The null condition translates as follows

$$\text{l.l= n.n= m.m=} \bar{m}.\bar{m} = 0 \tag{3}$$

and the orthogonality manifests as

$$\text{l.m= l .}\bar{m}\text{= n.m= n.}\bar{m} = 0 \tag{4}$$

Apart from this the normalization obeys

$$\text{l.n= -1} \tag{5}$$
$$\text{m.}\bar{m} = 1 \tag{6}$$

$\eta_{(a)(b)}$ Produce the basic matrix, to form a constant symmetric matrix.

$$g_{\mu\nu} = -l_\mu n_\nu - l_\nu n_\mu + m_\mu \bar{m}_\nu + m_\nu \bar{m}_\mu$$

$$\eta_{(\alpha)(\beta)} = \eta^{(\alpha)(\beta)} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{7}$$

4.2. Building a Null Tetrad

Easily we can build the null tetrad by the mastery already we have, which is by the basis of one forms $\widehat{\omega^a}$.We will change the $\widehat{\omega^a}$ into a null tetrad from the matrix structure. For any given spacetime we characterized an orthonormal tetrad as timelike and spacelike vectors, where $e_{\{t\}}$ timelike vector and $e_{\{r\}}, e_{\{\theta\}}, e_{\{\phi\}}$ spacelike vectors. Now using this recipe we can build the null tetrad, the real vectors and the analogous covariant basis are given below.

Real vectors are,

$$e_{(0)} = l_a\, dx^a = \frac{1}{\sqrt{2}}(e_t + e_x)$$
$$e_{(1)} = n_a\, dx^a = \frac{1}{\sqrt{2}}(e_t - e_x)$$
$$e_{(2)} = m_a\, dx^a = \frac{1}{\sqrt{2}}(e_y + ie_z)$$
$$e_{(3)} = \overline{m}_a\, dx^a = \frac{1}{\sqrt{2}}(e_y - ie_z)$$

Corresponding covariant basis are

$e^{(0)} = e_{(1)} = n^a,\ e^{(1)} = e_{(0)} = l^a, e^{(2)} = e_{(3)} = -\overline{m}^a,$ and $e^{(3)} = e_{(2)} = -\overline{m}^a,$

The directional derivatives are obtained for basis vectors are

$$e_{(0)} = e^{(1)} = D = l^a \frac{\partial}{\partial x^a}$$
$$e_{(1)} = e^{(0)} = \Delta = n^a \frac{\partial}{\partial x^a}$$
$$e_{(2)} = -e^{(3)} = \delta = m^a \frac{\partial}{\partial x^a}$$
$$e_{(3)} = -e^{(2)} = \delta^* = \overline{m}^a \frac{\partial}{\partial x^a}$$

We can use tetrad formalism for express geometric objects. Let's see in a few definite casess how we will built the null tetrad. When we utilize metric in a coordinate basis as components $g_{\mu\nu}$, the couple of relations are useful which relate the null tetrad to the parts of the metric tensors, as

$$g_{\mu\nu} = e_{(a)\mu} e_{(b)\nu}\ \eta^{(a)(b)}$$
$$e_{(a)}{}^{\mu} = \eta_{(a)(b)} e^{(b)\mu}$$

Denoted by

$$e_{(a)}{}^{\mu} = \{l^\mu, n^\mu, m^\mu, \overline{m}^\mu\} \tag{8}$$

The normalization conditions $\vec{l}.\vec{n} = -1$ and $\vec{m}.\vec{\overline{m}} = 1$are obey by a complex null tetrad

$$e^{(a)}{}_{\mu} = \eta^{(a)(b)} e_{(b)}{}^{\mu} \tag{9}$$

$$e_{(b)} = e_{(a)}{}^{\mu} \overrightarrow{e_\mu}$$

$$e^{(a)}{}_{\mu}e_{(b)}{}^{\mu} = \delta^{(a)}{}_{(b)}$$

Now substituting a and b is 0, 1, 2, 3 we will get

$$-n_{\mu}l^{\mu} = 1,\ -l_{\mu}n^{\mu} = 1,\ \overline{m}_{\mu}m^{\mu} = 1,\ \text{and}\ m_{\mu}\overline{m^{\mu}} = 1,$$

So from the above solving equation we can see that

$$e_{\mu}^{(a)} = \{-n_{\mu}, -l_{\mu}, \overline{m}_{\mu}, m_{\mu}\}$$

$$g_{\mu\nu} = e^{(a)}{}_{\mu}\, e_{(a)\nu} \tag{10}$$

From Eqn. (3) I can change μ to ν and I pull down the ν so I will get

$$e_{(a)\nu} = \{l_{\nu}, n_{\nu}, m_{\nu}, \overline{m}_{\nu}\} \tag{11}$$

Now I will calculate Eqn. (5)

$$g_{\mu\nu} = e^{(0)}{}_{\mu}\, e_{(0)\nu} + e^{(1)}{}_{\mu}\, e_{(1)\nu} + e^{(2)}{}_{\mu}\, e_{(2)\nu} + e^{(3)}{}_{\mu}\, e_{(3)\nu}$$
$$=(-n_{\mu}l_{\nu})+(-\,l_{\mu}n_{\nu})+(\,\overline{m}_{\mu}\, m_{\nu})+(\,m_{\mu}\overline{m}_{\nu})$$

$$g_{\mu\nu} = -l_{\mu}n_{\nu} - l_{\nu}n_{\mu} + m_{\mu}\overline{m}_{\nu} + m_{\nu}\overline{m}_{\mu} \tag{12}$$

$$g^{\mu\nu} = -l^{\mu}n^{\nu} - l^{\nu}n^{\mu} + m^{\mu}, \overline{m}^{\nu} + m^{\nu}, \overline{m}^{\mu} \tag{13}$$

Now let us calculate the matrix, from the Eqn. (3), (4) and (6) so will get

$$-n_{\mu} = (0)l^{\mu} + (-1)n^{\mu} + (0)m^{\mu} + (0)\,\overline{m^{\mu}}$$

$$-l_{\mu} = (-1)l^{\mu} + (0)n^{\mu} + (0)m^{\mu} + (0)\,\overline{m^{\mu}}$$

$$\overline{m^{\mu}} = (0)l^{\mu} + (0)n^{\mu} + (0)m^{\mu} + (1)\,\overline{m^{\mu}}$$

$$m^{\mu} = (0)l^{\mu} + (0)n^{\mu} + (1)m^{\mu} + (0)\,\overline{m^{\mu}}$$

So the matrix will be

$$\begin{pmatrix} -\eta_{\mu} \\ -l_{\mu} \\ \overline{m}_{\mu} \\ m_{\mu} \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} l^{\mu} \\ n^{\mu} \\ m^{\mu} \\ \overline{m^{\mu}} \end{pmatrix} \tag{14}$$

On the off chance that we are utilizing an orthonormal tetrad, which is the favored technique for this paper, we can relate the two bases in an accompanying manner.

$$g_{\mu\nu} = -g_{tt}\, dt^2 + g_{rr}\, dr^2 + g_{\theta\theta}\, d\theta^2 + g_{\phi\phi}\, d\phi^2 \tag{15}$$

$$\omega^t = \omega^0 = \sqrt{g_{tt}}\, dt, \omega^r = \omega^1 = \sqrt{g_{rr}}\, dr, \omega^{\theta} = \omega^2 = \sqrt{g_{\theta\theta}}\, d\theta\,, \omega^{\phi} = \omega^3 = \sqrt{g_{\phi\phi}}\, d\phi \tag{16}$$

It is additionally conceivable to derive a changing relationship for the basis one forms. Once again, we review the type of the basis one form when utilizing an coordinate basis. In that case they are exact differentials:

$$\omega^a = dx^a$$

The noncoordinate basis is related to the coordinate basis in an accompanying manner:

$$\omega^{\hat{a}}=\omega^{\hat{a}}{}_b dx^b$$

Now substitute the formula will get,

$$l_a dx^a = \frac{1}{\sqrt{2}}(e_t + e_x) = \frac{1}{\sqrt{2}}(1,1,0,0)$$
$$n_a dx^a = \frac{1}{\sqrt{2}}(1,-1,0,0)$$
$$m_a dx^a = \frac{1}{\sqrt{2}}(0,0,1,i)$$
$$\bar{m}_a dx^a = \frac{1}{\sqrt{2}}(0,0,1,-i)$$

$$\begin{pmatrix} l_a \\ n_a \\ m_a \\ \bar{m}_a \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & 1 & -i \end{pmatrix} \begin{pmatrix} \omega^0 \\ \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix} \quad (17)$$

4.2.1. Ricci rotation coefficients.

As far as the complex null tetrad the twelve complex function, known as Spin- coefficients, are characterized through the Ricci rotation coefficient [12] as

$-\kappa = \gamma_{(2)(0)(0)} = e^i{}_{(2)}\, e_{(0)i;j} e^j{}_{(0)} = l_{i;j}\, m^i l^j$; $-\rho = \gamma_{(2)(0)(3)} = l_{i;j}\, m^i \bar{l}^j$

$-\sigma = \gamma_{(2)(0)(2)} = l_{i;j}\, m^i m^j$; $-\tau = \gamma_{(2)(0)(1)} = l_{i;j}\, m^i n^j$

$\nu = \gamma_{(1)(3)(1)} = n_{i;j}\, \bar{m}^i n^j$; $\mu = \gamma_{(1)(3)(2)} = n_{i;j}\, \bar{m}^i m^j$

$\lambda = \gamma_{(1)(3)(3)} = n_{i;j}\, \bar{m}^i \bar{m}^j$; $\pi = \gamma_{(1)(3)(0)} = n_{i;j}\, \bar{m}^i l^j$

$-\varepsilon = \frac{1}{2}(\gamma_{(1)(0)(0)} - \gamma_{(3)(2)(0)}) = \frac{1}{2}(l_{i;j}\, m^i l^j - m_{i;j}\, \bar{m}^i l^j)$

$-\beta = \frac{1}{2}(\gamma_{(1)(0)(2)} - \gamma_{(2)(3)(2)}) = \frac{1}{2}(l_{i;j}\, n^i m^j - m_{i;j}\, \bar{m}^i m^j)$

$-\gamma = \frac{1}{2}(\gamma_{(2)(3)(1)} - \gamma_{(0)(1)(1)}) = \frac{1}{2}(l_{i;j}\, n^i n^j - m_{i;j}\, \bar{m}^i n^j)$

$-\alpha = \frac{1}{2}(\gamma_{(2)(3)(3)} - \gamma_{(0)(1)(3)}) = \frac{1}{2}(l_{i;j}\, n^i \bar{m}^j - m_{i;j}\, \bar{m}^i \bar{m}^j)$

The general guidelines might be noticed that the complex conjugate of any amount can be acquired by replacing the index 3, any place it happens, by index 4, and conversely.

4.2.2. Weyl tensor.

The Ricci tensor is acquired by the contraction

$$g^{ik} R_{ijkl} = R_{jl}$$

the Weyl tensor is presently identified with the Riemann and the Ricci tensors by

$$C_{ijkl} = R_{ijkl} - \frac{1}{2}\left(g_{ik}R_{jl} + g_{jl}R_{jk} - g_{jk}R_{il} - g_{il}R_{jk}\right) + \frac{1}{6}\left(g_{ik}g_{jl} - g_{jk}g_{il}\right)R$$

The Weyl tensor is the trace-free part of the Riemann tensors, and its tetrad components are given by

$$R_{ijkl} = C_{ijkl} + \frac{1}{2}\left(g_{ik}R_{jl} - g_{jk}R_{il} - g_{il}R_{jk} + g_{jl}R_{ik}\right) - \frac{1}{6}\left(g_{ik}g_{jl} - g_{il}g_{jk}\right)R$$

In the NP formalism, the ten independent components of the Weyl tensor are spoken to by the five complex scalars [12],

$$\psi_0 = C_{0202} = C_{abcd}\, l^a m^b l^c m^d$$

$$\psi_1 = C_{0102} = C_{abcd}\, l^a n^b l^c m^d$$

$$\psi_2 = C_{0231} = C_{abcd}\, l^a m^b \bar{m}^c n^d$$

$$\psi_3 = C_{0131} = C_{abcd}\, l^a n^b \bar{m}^c n^d$$

$$\psi_4 = C_{1313} = C_{abcd}\, n^a \bar{m}^b n^c \bar{m}^d$$

while it is sure about general grounds that the Weyl tensor is totally determined by the five complex scalars $\psi_0 \ldots \ldots \psi_1$, it will be advantageous to have a general recipe which communicates the various segments of the Weyl tensor unequivocally in terms of the five scalars. [11][12].

The tetrad components of Weyl tensor have the following algebraic properties [9]:

1. In the event that $\psi_0 \neq 0$ and the others are zero, the gravitational field is of Petrov type N with n^i as the propagation vector.

2. In the event that $\psi_1 \neq 0$ and the others are zero, the gravitational field is of Petrov type III with n^i as the propagation vector.

3. In the event that $\psi_2 \neq 0$ and the others are zero, the gravitational field is of Petrov type D with l^i *and* n^i as the propagation vector.

4. In the event that $\psi_3 \neq 0$ and the others are zero, the gravitational field is of Petrov type III with l^i as the propagation vector.

5. In the event that $\psi_4 \neq 0$ and the others are zero, the gravitational field is of Petrov type N with l^i as the propagation vector.

4.2.3. Ricci tensor.
The ten segment of the Ricci tensor are characterized as far as the accompanying four real and three complex scalars.

$$\phi_{00} = \tfrac{1}{2}R_{ab}\, l^a l^b;\ \phi_{11} = \tfrac{1}{4}R_{ab}\,(l^a n^b + m^a \bar{m}^b);\ \phi_{22} = \tfrac{1}{2}R_{ab}\, n^a n^b$$
$$\phi_{01} = \tfrac{1}{2}R_{ab}\, l^a m^b;\ \ \phi_{02} = \tfrac{1}{2}R_{ab}\, m^a m^b;\ \ \phi_{12} = \tfrac{1}{2}R_{ab}\, m^a m^b;\ \Lambda = \frac{\mathrm{R}}{24}$$

5. Constructing a Null tetrad for general case

$$ds^2 = -e^{2v(r)}\, dt^2 + e^{2u(r)}\, dr^2 + r^2 d\Omega^2$$

The components of the metric, we have

$$g_{ab} = \begin{pmatrix} -e^{2v(r)} & 0 & 0 & 0 \\ 0 & e^{2u(r)} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \tag{18}$$

We can write down the orthonormal basis one forms. These are

$$\omega^{\hat{\theta}} = r\, d\theta$$
$$\omega^{\hat{\varphi}} = r\ \sin\theta\, d\varphi$$
$$\omega^{\hat{r}} = \left(e^{2u(r)}\right)^{1/2} dr$$
$$\omega^{\hat{t}} = \left(e^{2v(r)}\right)^{1/2} dt$$

Applying Eqn.(9) we have the following relations:

$$l = \frac{1}{\sqrt{2}}\left[\left(e^{2v(r)}\right)^{1/2} dt + \left(e^{2u(r)}\right)^{1/2} dr\right]$$
$$n = \frac{1}{\sqrt{2}}\left[\left(e^{2v(r)}\right)^{1/2} dt - \left(e^{2u(r)}\right)^{1/2} dr\right]$$
$$m = \frac{1}{\sqrt{2}}\ [rd\theta + \ ir\ \sin\theta\, d\varphi]$$
$$\bar{m} = \frac{1}{\sqrt{2}}[rd\theta - \ ir\ \sin\theta\, d\varphi]$$

In many cases, you will see the vectors written in terms of components with respect to the coordinate basis i.e., $v^a = (v^0, v^1, v^2, v^3)$ or in this case $v^a = (v^t, v^r, v^\theta, v^\varphi)$ or $v_a = (v_t, v_r, v_\theta, v_\varphi)$ using this notation, we have

$$l^\mu = \frac{1}{\sqrt{2}}\left[e^{v(r)}, e^{u(r)}, 0, 0\right]$$
$$n^\mu = \frac{1}{\sqrt{2}}\left[e^{v(r)}, -e^{u(r)}, 0, 0\right]$$
$$m^\mu = \frac{1}{\sqrt{2}}\ [0, 0, r, ir\ \sin\theta]$$
$$\bar{m}^\mu = \frac{1}{\sqrt{2}}[\ 0, 0, r, -ir\sin\theta]$$

In terms of the complex null tetrad the twelve complex functions, known as spin-coefficients, are defined through the Ricci rotation coefficients as

$$\pi = 0;\ \nu = 0\ ;\ \lambda = 0\ ;\ \kappa = 0\ ;\ \tau = 0\ ;\ \sigma = 0$$

$$\mu = \rho = \frac{e^{-u(r)}}{\sqrt{2}\, r}\ ;\ \varepsilon = \gamma = \frac{e^{-u(r)}\ v\prime(r)}{2\sqrt{2}}\ ;\ \alpha = -\beta = -\frac{Cot\ \theta}{2\sqrt{2}\, r}$$

In NP formalism, the ten independent components of the Wely tensor are represented by the five (complex) scalars,

$$\psi_0 = \psi_1 = \psi_3 = \psi_4 = 0$$

$$\psi_2 = \frac{e^{-2u(r)}\left(u'(r)\left(r - r^2 v'(r)\right) + r^2 v''(r)^2 - e^{2u(r)} - rv'(r) + 1\right)}{6r^2}$$

The tetrad components of Weyl tensor have $\psi_2 \neq 0$ and the others are zero ($\psi_0 = \psi_1 = \psi_3 = \psi_4 = 0$), the gravitational field is of Petrov type D with l^i and n^i as the propagation vector.

The ten components of the Ricci tensor are

$$\varphi_{00} = \frac{e^{-2u(r)}(u'(r)+v'(r))}{2r}$$

$$\varphi_{11} = \frac{e^{-2u(r)}(-r^2u'(r)v'(r)+r^2v''(r)+r^2v'(r)^2+e^{2u(r)}-1}{4r^2}$$

$$\varphi_{22} = \frac{e^{-2u(r)}(u'(r)+v'(r))}{2r}$$

$$\Lambda = \frac{e^{-2u(r)}(-r^2v''(r)+r^2(-v'(r)^2)+ru'(r)(rv'(r)+2)+e^{-2u(r)}-2rv'(r)-1)}{12r^2}$$

6. NP formalism with BST : Calculations

The orthonormal basis one-forms can be written by observing the above metric equation as:

$$\omega^{\hat{\theta}} = r\,d\theta$$
$$\omega^{\hat{\varphi}} = r\ \sin\theta\ d\varphi$$
$$\omega^{\hat{r}} = \left(\frac{4}{\beta^2(1-Dr^2)}\right)^{1/2} dr$$
$$\omega^{\hat{t}} = \left(\frac{1}{G-\dfrac{r^2}{2-2Dr^2}}\right)^{1/2} dt$$

Applying conditions on the null tetrad**,**

$$l = \frac{1}{\sqrt{2}}\left[\left(\frac{1}{G-\dfrac{r^2}{2-2Dr^2}}\right)^{1/2} dt +\ 2\left(\frac{4}{\beta^2(1-Dr^2)}\right)^{1/2} dr\right]$$
$$n = \frac{1}{\sqrt{2}}\left[\left(\frac{1}{G-\dfrac{r^2}{2-2Dr^2}}\right)^{1/2} dt -\ 2\left(\frac{4}{\beta^2(1-Dr^2)}\right)^{1/2} dr\right]$$
$$m = \frac{1}{\sqrt{2}}\ [rd\theta +\ ir\ \sin\theta\ d\varphi]$$
$$\bar{m} = \frac{1}{\sqrt{2}}[rd\theta -\ ir\ \sin\theta\ d\varphi]$$

Using the notation $v^a = (v^t, v^r, v^\theta, v^\varphi)$, we write the components of null tetrads as:

$$l^\mu = \frac{1}{\sqrt{2}}\left[\sqrt{\frac{1}{G-\dfrac{r^2}{2-2Dr^2}}}, 2\sqrt{\frac{1}{\beta^2(1-Dr^2)}}, 0, 0\right]$$
$$n^\mu = \frac{1}{\sqrt{2}}\left[\sqrt{\frac{1}{G-\dfrac{r^2}{2-2Dr^2}}}, -2\sqrt{\frac{1}{\beta^2(1-Dr^2)}}, 0, 0\right]$$
$$m^\mu = \frac{1}{\sqrt{2}}\ [0, 0, r, ir\ \sin\theta]$$

$$\bar{m}^{\mu} = \frac{1}{\sqrt{2}}[\,0, 0, r, -ir\sin\theta]$$

As far as the complex null tetrad the twelve complex functions, known as spin-coefficients, are characterized through the Ricci rotation coefficients as

$$\pi = 0,\ \nu = 0, \lambda = 0, \kappa = 0,\ \tau = 0,\ \sigma = 0$$

$$\mu = \rho = \frac{\beta\sqrt{1 - Dr^2}}{2\sqrt{2r}}$$

$$\varepsilon = \gamma = \frac{\beta r\sqrt{\frac{1}{2 - 2Dr^2}}}{8DGr^2 - 8G + 4r^2}$$

$$\alpha = -\beta = -\frac{Cot\ \theta}{2\sqrt{2r}}$$

The advantage of NP formalism is that the following five scalars give the information carried by the ten independent components of Weyl tensor,

$$\psi_0 = \psi_1 = \psi_3 = \psi_4 = 0$$

$$\psi_2 = \frac{2(\beta^2 - 4)G^2(Dr^2 - 1)^2 + Gr^2(-2\beta^2 + (5\beta^2 - 8)Dr^2 + 8) + 2(\beta^2 - 1)r^4}{12(2G(Dr^2 - 1) + r^3)^2}$$

For BST-1 with the special case k=0, ψ_2, out of all the five tetrad components of Weyl tensor, is non-zero, as can be seen. This implies that the gravitational field created by BST is of Petrov type D, which has l^i and n^i as the propagation vectors.

The non-zero components of the tetrad components of the Ricci tensor are given by:

$$\varphi_{00} = \frac{\beta^2(2DG + 1)(Dr^2 - 1)}{8(2G(Dr^2 - 1) + r^2)}$$

$$\varphi_{11} = \frac{2Gr^2(3\beta^2 + 2\beta^2D^2r^4 - 2(\beta^2 - 4)Dr^2 - 8) + 4G^2(Dr^2 - 1)^2 (\beta^2(Dr^2 - 1) + 4) + r^4(\beta^2(Dr^2 + 1) + 4)}{16(2Gr(Dr^2 - 1) + r^3)^2}$$

$$\varphi_{22} = \frac{\beta^2(2DG + 1)(Dr^2 - 1)}{8(2G(Dr^2 - 1) + r^2)}$$

$$\Lambda = \frac{\mathbf{2Gr^2\left(-\beta^2 + 6\beta^2D^2r^4 - 8(\beta^2 - 1)Dr^2 - 8\right) + 4G^2\left(Dr^2 - 1\right)^2 \left(\beta^2(3Dr^2 - 1) + 4\right) + r^4\left(\beta^2(3Dr^2 - 1) + 4\right)}}{\mathbf{48(2Gr(Dr^2 - 1) + r^3)^2}}$$

7. Conclusion

The Newman-Penrose formalism is very important in Einstein's relativity theory. In the last section, we showed that the Bertrand spacetime is Petrov type D, using the compacted spin coefficient. As fields with Petrov type D generally represent fields formed by massive astronomical objects, BST also can be considered to generate a similar type of field. We also expressed tetrad components of Ricci tensor using NP formalism for BST and found that the complex components vanish, as expected. We have presented here, a formalism which should have many applications to the type of situation in which the NP formalism has proved. Going through similar calculations done in the last section, for the general case, we can also check that all the spherically symmetric static spacetimes are of Petrov type D.

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