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Derivation of operational matrix of Rabotnov fractional-exponential kernel and its application to fractional Lienard equation

Sachin Kumar [a], J.F. Gómez-Aguilar [b], J.E. Lavín-Delgado [c], D. Baleanu [d,e,f]

[a] *Department of Mathematical Sciences, Indian Institute of Technology (BHU), Varanasi 221005, India*
[b] *CONACyT-Tecnológico Nacional de México/CENIDET, Interior Internado Palmira S/N, Col. Palmira, C.P. 62490 Cuernavaca, Morelos, Mexico*
[c] *Tecnológico Nacional de México/CENIDET, Interior Internado Palmira S/N, Col. Palmira, C.P. 62490 Cuernavaca, Morelos, Mexico*
[d] *Department of Mathematics, Cankaya University, Ankara, Turkey*
[e] *Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan*
[f] *Institute of Space Sciences, Magurele–Bucharest, Romania*

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Abstract Our motive in this contribution is to find out the operational matrix of fractional derivative having non singular kernel namely Rabotnov fractional-exponential (RFE) kernel which is recently introduced and seeking numerical solution of non-linear Lienard equation which have Rabotnov fractional-exponential kernel fractional derivative. First we derive an approximation formula of the fractional order derivative of polynomial function z^k in term of RFE kernel. Using this formula and some properties of shifted Legendre polynomials, we find out the operational matrix of fractional order differentiation. In the author of knowledge this operational matrix of RFE kernel fractional derivative is derived first time. We solve a new class of fractional partial differential equation (FPDEs) by implementation of this newly derived operational matrix. We show that our newly derived operational matrix is valid by taking an fractional derivative of a polynomial. Also, we study a new model of Lienard equation with RFE kernel fractional derivative and we can easily predict the feasibility of our numerical method to this new model.

1. Introduction

In the recent years fractional differential equations have received more attention of the researchers due to its exact description of the physical phenomenon. Many physical phenomenons have been described through fractional diffusion

E-mail address: jose.ga@cenidet.tecnm.mx (J.F. Gómez-Aguilar)
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equation viz., transport in porous medium, ground water contamination problem through porous medium etc. Fractional calculus is that branch of mathematics that originates from the classical one [1]. The integral and derivative in fractional calculus is obtained from the integer order by replacing integer order exponent by real or arbitrary order.

N.H. Abel and J. Liouville first developed the theory of this fractional calculus we can found the more details on fractional calculus in [2,3]. We can extend the real order to variable order in differentiation or integration with the help of theory given in fractional calculus. There are many physical phenomena that can not be represent by classical derivative so we need the differential equation having fractional order. These fractional differential equation have a lot of application in control theory, biology, physics and medical science. In starting only fractional derivative with power law kernel was investigated. These types of fractional derivative include Caputo definition, Riemann–Liouville definition (RL), Hadamard and Grünwald-Letnikov definition. The theory of fractional differential equation boosts the application and research in many fields of science and engineering. But the main difficulty was to find out the solution of these FPDEs. By analytical method like as Laplace transform method, Homotopy analysis method and Fourier transform method it is too difficult to solve every linear and non linear FPDEs. So the researchers started to find out the method to solve these equation numerically. There are many method available in literature: eigen-vector expansion, Adomain decomposition method [4], fractional differential transform method [5], homotopy perturbation method [6], predictor–corrector method [7] and generalized block pulse operational matrix method [8], etc. A method named as operational matrix method is so popular due to its simplicity and good accuracy. Many numerical methods which are based upon operational matrices of integration and differentiation with Legendre wavelets [9], Chebyshev wavelets [10], sine wavelets, Haar wavelets [11] are given in literature to derive the numerical solution of fractional PDEs and integro differential equations. Legendre polynomial [12], Laguerre polynomial [13], Chebyshev polynomial and semi-orthogonal polynomial as Genocchi polynomial [14] are commonly used polynomial in deriving the operational matrix and then numerical solution of FPDEs.

Nowadays many different fractional operator which are generalization of classical ones are developed. The classical derivative as Caputo and Riemann–Liouville have power kernel. If we replace this kernel with exponential kernel and Mittag–Leffler kernel than we get a new generalized class of fractional derivative. The derivative having exponential kernel is known as Caputo-Fabrizio derivative while having Mittag–Leffler is known as Atangana-Baleanu derivative. These non-singular derivative a lot of application in chaos theory [15], groundwater flow [16], medical sciences [17] and others areas [18–33].

We have organized our article as follows. Section 2 contains the some useful definition of fractional derivative like as Caputo, R-L and RFE kernel fractional derivative. In Section 3, we derived the general formula of RFE derivative of the function t^k. Some properties of orthogonal Legendre polynomial is also included in this section. In Theorem 2 we have derive the shifted Legendre operational matrix of this RFE kernel fractional derivative. In Section 4, we described the proposed method for solving FDEs with fractional RFE kernel derivative and discussion of our Lienard model. The conclusion of all over work in given in last section.

2. Preliminary definitions

In the last few years, many definitions of fractional integration and differentiation has been come into the light. All of them have own special properties and applications. Caputo definition is more reliable as compare to Riemann–Liouville definition as application point of view. These definition are with power or singular kernel law. Nowadays many generalized definitions of fractional derivative with exponential and Mittag–Leffler kernel law have been introduced. We discussed a brief definitions and properties of R-L, Caputo and recently developed RFE derivative.

2.1. Riemann–Liouville order derivative and integration

The R-L integration of order $\varrho > 0$ of a function $h(t)$ is given by

$$I^{\vartheta}h(z) = \frac{1}{\Gamma(\varrho)}\int_0^z (z-\varpi)^{\varrho-1}h(\varpi)d\varpi, z>0, \vartheta \in R^+. \tag{1}$$

Now Riemann–Liouville fractional order differentiation of a function $h(t)$ with order $\vartheta > 0$ is defined as

$$D_l^{\vartheta}h(t) = \left(\frac{d}{dt}\right)^m (I^{m-\vartheta}h)(t), (m-1<\vartheta<m, \vartheta>0). \tag{2}$$

2.2. Definition of Caputo derivative

The Caputo derivative of a function $h(t)$ having order $\vartheta > 0$ is given as follows

$$D_c^{\vartheta}h(t) = \begin{cases} \frac{1}{\Gamma(\vartheta)}\int_0^t (-\eta+t)^{-\vartheta-1+l}h^l(\eta)d\eta & l-1<\vartheta<l, \\ \frac{d^l}{dt^l}h(t) & \vartheta = l \in N, \end{cases} \tag{3}$$

with l an integer and time interval $t > 0$.

Some important properties of Caputo differentiation are given as follows

$$D_c^{\vartheta}C = 0, \tag{4}$$

where C is a constant.

A relation between Caputo and Riemann–Liouville fractional operator can be defined as

$$(I^{\vartheta}D_c^{\vartheta}g)(t) = g(t) - \sum_{k=0}^{l-1} g^k(0^+)\frac{t^k}{k!}, l-1<\vartheta\leqslant l. \tag{5}$$

2.3. Definition of fractional derivative having RFE kernel [34]

The fractional integral with RFE kernel of function $h(x,t)$ of order θ and with respect to t is defined as follows

$$^{RFE}_{0}I_t^{\theta}h(x,t) = \int_0^t h(x,s)\times M_{\theta}\left[-\lambda(t-s)^{\theta}\right]ds, \tag{6}$$

where the parameter $\lambda \in R^+$ and Rabotnov fractional-exponential function can be defined as follows

$$M_\theta(z) = \sum_{i=0}^{\infty} \frac{(-\lambda)^i t^{(i+1)(\theta+1)-1}}{\Gamma((i+1)(\theta+1))}.$$

Now the left sided Caputo fractional order derivative on interval $[0, 1]$ with RFE kernel is defined as follows

$$ {}_0^{RFE}D_t^\theta h(x,t) = \int_0^t \frac{\partial^n h(x,s)}{\partial s^n} \times M_\theta\left[-\lambda(t-s)^\theta\right] ds, \quad n-1 < \theta \leqslant n, \tag{7}$$

with $\lambda \in R^+$.

The right sided Caputo fractional order derivative with RFE kernel on interval $[0, 1]$ can be defined as follows

$$ {}_0^{RFE}D_t^\theta h(x,t) = (-1)^n \int_t^1 \frac{\partial^n h(x,s)}{\partial s^n} \times M_\theta\left[-\lambda(t-s)^\theta\right] ds, \quad n-1 < \theta \leqslant n. \tag{8}$$

3. Operational matrix of fractional differentiation with RFE kernel derived from shifted Legendre polynomial

3.1. Approximation of fractional order derivative of z^k

In the literature there are lot of articles in which we can see the derivation of operational matrix of differentiation and integration in both Caputo and Riemann–Liouville sense. In this section we will derive the operational matrix of fractional order differentiation with respect to RFE kernel based on Legendre polynomial. In our knowledge this operational matrix is derived for the first time.

Theorem 1: The RFE derivative of order $n-1 < \theta < n$ of function $f(z) = z^k$ with $k \geqslant \lceil\theta\rceil$ is given by

$$\begin{aligned} {}_0^{RFE}D_z^\theta z^k &= \tfrac{\Gamma(k+1)}{\Gamma(k-n)} \times \tfrac{h}{3}[\Omega_{\theta,k}(s_0,z) + \Omega_{\theta,k}(s_m,z) \\ &\quad +4\{\Omega_{\theta,k}(s_1,z) + \Omega_{\theta,k}(s_3,z) + \cdots + \Omega_{\theta,k}(s_{m-1},z)\} \\ &\quad +2\{\Omega_{\theta,k}(s_2,z) + \Omega_{\theta,k}(s_4,z) + \cdots + \Omega_{\theta,k}(s_{m-2},z)\}] \end{aligned}$$

Proof: Using the definition of RFE kernel based fractional differentiation in Caputo sense we get $D^n z^k = 0, k = 0, 1, \cdots, n-1$. Now for $k \geqslant \lceil\theta\rceil$ we have

$$\begin{aligned} {}_0^{RFE}D_z^\theta z^k &= \textstyle\int_0^z D^n s^k M_\theta\left[-\lambda(z-s)^\theta\right] ds \\ &= \textstyle\int_0^z \frac{\Gamma(k+1)}{\Gamma(k-n+1)} s^{k-n} M_\theta\left[-\lambda(z-s)^\theta\right] ds \\ &= \tfrac{\Gamma(k+1)}{\Gamma(k-n+1)} \textstyle\int_0^z s^{k-n} M_\theta\left[-\lambda(z-s)^\theta\right] ds. \end{aligned}$$

The integral in above equation is a complicated integral. So we have used a numerical scheme to evaluate this. We can use any of available numerical integration scheme available in literature like as Simpson $\frac{1}{3}$

$$\begin{aligned} &= \tfrac{\Gamma(k+1)}{\Gamma(k-n+1)} \times \tfrac{h}{3}\Big[s_0^{k-n} M_\theta\left[-\lambda(z-s_0)^\theta\right] + s_m^{k-n} M_\theta\left[-\lambda(z-s_m)^\theta\right] \\ &+4\Big\{s_1^{k-n} M_\theta\left[-\lambda(z-s_1)^\theta\right] + s_3^{k-n} M_\theta\left[-\lambda(z-s_3)^\theta\right] + \cdots \\ &+s_{m-1}^{k-n} M_\theta\left[-\lambda(z-s_{m-1})^\theta\right]\Big\} \\ &+2\Big\{s_2^{k-n} M_\theta\left[-\lambda(z-s_2)^\theta\right] + s_4^{k-n} M_\theta\left[-\lambda(z-s_4)^\theta\right] + \cdots \\ &+s_{m-2}^{k-n} M_\theta\left[-\lambda(z-s_{m-2})^\theta\right]\Big\}\Big] \end{aligned}$$

$$\begin{aligned} &= \tfrac{\Gamma(k+1)}{\Gamma(k-n+1)} \times \tfrac{h}{3}[\Omega_{\theta,k}(s_0,z) + \Omega_{\theta,k}(s_m,z) \\ &+4\{\Omega_{\theta,k}(s_1,z) + \Omega_{\theta,k}(s_3,z) + \cdots + \Omega_{\theta,k}(s_{m-1},z)\} \\ &+2\{\Omega_{\theta,k}(s_2,z) + \Omega_{\theta,k}(s_4,z) + \cdots + \Omega_{\theta,k}(s_{m-2},z)\}]. \end{aligned}$$

We have divided the domain $[0, 1]$ into m equal segments and h represents the length of each segment

$$\begin{aligned} &h = \tfrac{1-0}{m}, \Omega_{\theta,k}(s,z) = s^{k-n} M_\theta\left[-\lambda(z-s)^\theta\right], \\ &s_0 = 0, s_1 = \tfrac{1}{m}, s_2 = \tfrac{2}{m} \cdots s_m = 1. \end{aligned}$$

3.2. Legendre polynomials

Now we discussed here about Legendre polynomials and their some properties. We shifted Legendre polynomials on the $[0, 1]$ from the interval $[-1, 1]$ by the transformation $z = 2x - 1$. The analytical form of these polynomials of degree i are given as follows

$$\psi_i(x) = \sum_{k=0}^{i} \frac{(-1)^{i+k}(i+k)!}{(k!)^2(l-k)!} x^k, \tag{9}$$

where $i = 0, 1, \cdots$.

These polynomials are orthogonal on the interval $[-1, 1]$ with respect to the weight function 1 and the orthogonality condition can be described as

$$\int_0^1 \psi_j(x)\psi_i(x) = \begin{cases} \frac{1}{2i+1}, & i = j, \\ 0 & i \neq j. \end{cases} \tag{10}$$

A function $\zeta(x)$ which belongs to the $L^2[0, 1]$ can be approximated by a linear combination of shifted Legendre polynomials as

$$\zeta(x) = \zeta_N(x) = \sum_{j=0}^{N} a_j \psi_i(x), \tag{11}$$

where the linear coefficients are given by

$$a_j = (2j+1) \int_0^1 \zeta(x)\psi_j(x). \tag{12}$$

Similarly, a function $\zeta(x, t)$ of two variable can be approximated as

$$\zeta(x,t) = \sum_{i=0}^{N-1}\sum_{l=0}^{N-1} a_{il}\psi_i(x)\psi_l(t), \tag{13}$$

where a_{il} are unknown coefficient. The matrix formation of above equation (14) is represented by following equation

$$\zeta(x) = \zeta_N(x) = \sum_{i=0}^{N-1}\sum_{l=0}^{N-1} a_{il}\psi_l(t)\psi_i(x) = \Theta_N^T(x)\mathbf{A}^T\Theta_N(t), \tag{14}$$

where,

$$\begin{aligned} &\mathbf{A}^T = (a_0, a_1, \cdots, a_{N-1}), \\ &\Theta_N(x) = (\psi_0(x), \psi_1(x), \cdots, \psi_{N-1}(x))^T. \end{aligned} \tag{15}$$

In view of equation (16) we will derive the Legendre operational matrix of RFE derivative in the following theorem.

Theorem 2: Let $\Theta_N(x)$ be the shifted Legendre vector with $n-1 < \theta < n$ then,

$$ {}_0^{RFE}D_t^\theta \Theta_N(t) = Q^\theta \Theta_N(t), \tag{16}$$

where Q^θ denotes the $N \times N$ RFE operational matrix of order θ. It can be defined as

$$Q^\theta = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{k=\lceil\theta\rceil}^{\lceil\theta\rceil} \varpi_{\lceil\theta\rceil,k,1} & \sum_{k=\lceil\theta\rceil}^{\lceil\theta\rceil} \varpi_{\lceil\theta\rceil,0,k} & \cdots & \sum_{k=\lceil\theta\rceil}^{\lceil\theta\rceil} \varpi_{\lceil\theta\rceil,m-1,k} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{k=\lceil\theta\rceil}^{i} \varpi_{i,0,k} & \sum_{k=\lceil\theta\rceil}^{i} \varpi_{i,1,k} & \cdots & \sum_{k=\lceil\theta\rceil}^{i} \varpi_{i,m-1,k} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{k=\lceil\theta\rceil}^{m-1} \varpi_{m-1,0,k} & \sum_{k=\lceil\theta\rceil}^{m-1} \varpi_{m-1,1,k} & \cdots & \sum_{k=\lceil\theta\rceil}^{m-1} \varpi_{m-1,m-1,k} \end{bmatrix},$$

where $\varpi_{i,j,k}$ is obtained by the following relation

$$\varpi_{i,j,k} = \frac{(-1)^{i+k}(i+k)!}{(k!)^2(l-k)!} \times \frac{\Gamma(k+1)}{\Gamma(k-n+1)} \times (2j+1)\sum_{l=0}^{j} \frac{(-1)^{j+l}(j+l)!}{(l!)^2(j-l)!} \times (\Lambda_0 + \Lambda_1 + \cdots + \Lambda_\Omega).$$

Here range of i and j varies from $i = \lceil\theta\rceil \cdots, N-1$ and $j = 0, 1, \cdots, N-1$ respectively.

Proof: By using Theorem 1 we have

$$ {}_0^{RFE}D_t^\theta t^k = \frac{\Gamma(k+1)}{\Gamma(k-n+1)} \times \frac{h}{3}[\Omega_{\theta,k}(s_0,t) + \Omega_{\theta,k}(s_m,t) + 4\{\Omega_{\theta,k}(s_1,t) + \Omega_{\theta,k}(s_3,t) + \cdots + \Omega_{\theta,k}(s_{m-1},t)\} + 2\{\Omega_{\theta,k}(s_2,t) + \Omega_{\theta,k}(s_4,t) + \cdots + \Omega_{\theta,k}(s_{m-2},t)\}].$$

The RFE derivative of the i-th degree shifted Legendre polynomial is

$$ {}_0^{RFE}D_t^\theta \psi_i(t) = \sum_{k=0}^{i} \frac{(-1)^{i+k}(i+k)!}{(k!)^2(l-k)!} \times {}_0^{RFE}D_t^\theta t^k, \quad i = 0, 1, \cdots$$

$$= \sum_{k=\lceil\theta\rceil}^{i} \frac{(-1)^{i+k}(i+k)!}{(k!)^2(l-k)!} \times \frac{\Gamma(k+1)}{\Gamma(k-n+1)} \times \frac{h}{3}[\Omega_{\theta,k}(s_0,t) + \Omega_{\theta,k}(s_m,t) + 4\{\Omega_{\theta,k}(s_1,t) + \Omega_{\theta,k}(s_3,t) + \cdots + \Omega_{\theta,k}(s_{m-1},t)\} + 2\{\Omega_{\theta,k}(s_2,t) + \Omega_{\theta,k}(s_4,t) + \cdots + \Omega_{\theta,k}(s_{m-2},t)\}].$$

The $(i,j)^{th}$ element $\rho_{i,j}$ of operational matrix Q^θ is determined by the taking inner product with shifted Legendre polynomial $\varphi_j(x), j = 0, 1, \cdots N-1$

$$ {}_0^{RFE}D_t^\theta \varphi_i(t) = \sum_{j=0}^{m-1} \rho_{i,j}\varphi_j(t), \quad \rho_{i,j} = \langle {}_0^{RFE}D_t^\theta \varphi_i(t), \varphi_j(t)\rangle \tag{17}$$

$$= \sum_{k=\lceil\theta\rceil}^{i} \frac{(-1)^{i+k}(i+k)!}{(k!)^2(l-k)!} \times \frac{\Gamma(k+1)}{\Gamma(k-n+1)} \times \frac{h}{3}\Big[\langle\Omega_{\theta,k}(s_0,t),\varphi_j(t)\rangle + \langle\Omega_{\theta,k}(s_\Omega,t),\varphi_j(t)\rangle + 4\{\langle\Omega_{\theta,k}(s_1,t),\varphi_j(t)\rangle + \langle\Omega_{\theta,k}(s_3,t),\varphi_j(t)\rangle + \cdots + \langle\Omega_{\theta,k}(s_{\Omega-1},t),\varphi_j(t)\rangle\} + 2\{\langle\Omega_{\theta,k}(s_2,t),\varphi_j(t)\rangle + \langle\Omega_{\theta,k}(s_4,t),\varphi_j(t)\rangle + \cdots + \langle\Omega_{\theta,k}(s_{\Omega-2},t),\varphi_j(t)\rangle\}\Big]. \tag{18}$$

Above expression contains many inner products which are determined as follows

$$\langle\Omega_{\theta,k}(s_p,t),\varphi_j(t)\rangle = (2j+1)\int_0^1 \Omega_{\theta,k}(s_p,t)\varphi_j(t)dt, \quad p = 0, 1, \cdots \Omega \tag{19}$$

$$= (2j+1)\int_0^1 \Omega_{\theta,k}(s_p,t)\sum_{l=0}^{j} \frac{(-1)^{l+j}(l+j)!}{(j-l)!(l!)^2} t^l dt$$

$$= (2j+1)\sum_{l=0}^{j} \frac{(-1)^{j+l}(j+l)!}{(l!)^2(j-l)!}\left(\int_0^1 \Omega_{\theta,k}(s_p,t)t^l dt\right)$$

$$= (2j+1)\sum_{l=0}^{j} \frac{(-1)^{l+j}(l+j)!}{(l!)^2(-l+j)!}\frac{h}{3}\Big[\Omega_{\theta,k}(s_p,t_0)0^l + \Omega_{\theta,k}(s_p,t_\Omega)\Omega^l + 4\{\Omega_{\theta,k}(s_p,t_1)1 + \Omega_{\theta,k}(s_p,t_3)3^l + \cdots + \Omega_{\theta,k}(s_p,t_{\Omega-1})(\Omega-1)^l\} + 2\{\Omega_{\theta,k}(s_p,t_2)2^l + \Omega_{\theta,k}(s_p,t_4)4^l + \cdots + \Omega_{\theta,k}(s_p,t_{\Omega-2})(\Omega-2)^l\}\Big]$$

$$= (2j+1)\sum_{l=0}^{j} \frac{(-1)^{l+j}(l+j)!}{(l!)^2(-l+j)!} \times \Lambda_p,$$

where,

$$\Lambda_p = \frac{h}{3}\Big[\Omega_{\theta,k}(s_p,t_0)0^l + \Omega_{\theta,k}(s_p,t_\Omega)\Omega^l + 4\{\Omega_{\theta,k}(s_p,t_1)1 + \Omega_{\theta,k}(s_p,t_3)3^l + \cdots + \Omega_{\theta,k}(s_p,t_{\Omega-1})(\Omega-1)^l\} + 2\{\Omega_{\theta,k}(s_p,t_2)2^l + \Omega_{\theta,k}(s_p,t_4)4^l + \cdots + \Omega_{\theta,k}(s_p,t_{\Omega-2})(\Omega-2)^l\}\Big].$$

Putting the value of both of inner product in equation (19) we obtained the following expression of $\rho_{i,j}$

$$\rho_{i,j} = \sum_{k=\lceil\theta\rceil}^{i} \frac{(-1)^{i+k}(i+k)!}{(k!)^2(l-k)!} \times \frac{\Gamma(k+1)}{\Gamma(k-n+1)} \times (2j+1)\sum_{l=0}^{j} \frac{(-1)^{j+l}(j+l)!}{(l!)^2(j-l)!} \times (\Lambda_0 + \Lambda_1 + \cdots + \Lambda_\Omega). \tag{20}$$

Considering $\rho_{i,j} = \sum_{\lceil\theta\rceil}^{i} \varpi_{i,j,k}$ we get the final desired result

$$\varpi_{i,j,k} = \frac{(-1)^{i+k}(i+k)!}{(k!)^2(l-k)!} \times \frac{\Gamma(k+1)}{\Gamma(k-n+1)} \times (2j+1)\sum_{l=0}^{j} \frac{(-1)^{j+l}(j+l)!}{(l!)^2(j-l)!} \times (\Lambda_0 + \Lambda_1 + \cdots + \Lambda_\Omega).$$

The operational matrix obtained above is applicable for fractional order. For the integer case we have the following

$$\rho_{i,j} = \begin{cases} \eta_j, & j = i-k, \\ 0, & otherwise, \end{cases} \tag{21}$$

where $k = 1, 3, \cdots m$ if m is odd and $k = 1, 3, \cdots m-1$ if m is even. The function η_j is defined as

$$\eta_j = 2 \times (2j+1).$$

4. Proposed algorithms

In this section we use the previous newly derived operational matrix for RFE based fractional derivative. Now we investigate the following model of Lienard equation having RFE based fractional derivative

$$ {}_0^{RFE}D_x^\theta \zeta(x) + \kappa_1(\zeta,x){}_0^{RFE}D_x^1\zeta(x) + \kappa_2(\zeta,x) = \kappa_3(x). \tag{22}$$

with initial conditions

$$\zeta(0) = b_1, \zeta'(0) = b_2. \tag{23}$$

As we have derived the new operational matrix of fractional differentiation based on RFE kernel derivative. Now we will implement this matrix for finding the numerical solution of model (22). The unknown function $\zeta(x)$ can be written as finite linear combination of shifted Legendre polynomials

$$\zeta(x) = \sum_{i=0}^{N-1} c_i \phi_i(x), \tag{24}$$

where c_i are unknown coefficients for $i = 0, 2, \cdots$; and $j = 0, 1, 2, \cdots$.

The expression given in (24) can be written in matrix form as follows

$$\zeta(x) = C.\Theta_N(x), \tag{25}$$

where $C = [c_i]_{1\times N}$ is an $1 \times N$ matrix of unknowns and $\Theta_N(x) = (\phi_0(x), \phi_1(x), \cdots, \phi_{N-1}(x))^T$ is a column vector. With the help of derived operational matrix and approximate expression of $\zeta(x)$ given in equation (25) we have the following

$${}_0^{RFE}D_x^{\theta}\zeta(x) = Q^{\theta}\zeta(x) = C.Q^{\theta}.\Theta_N(x), \tag{26}$$

Similarly

$${}_0^{RFE}D_x^{1}\zeta(x) = Q^{1}\zeta(x) = C.Q^{1}.\Theta_N(x). \tag{27}$$

Similarly we can approximate the initial conditions in the matrix form by taking help of equation (23). we get,

$$\begin{aligned} &C.\Theta_N(0) = b_1, \\ &C.Q^1.\Theta_N(0) = b_2. \end{aligned} \tag{28}$$

With the help of Eqs. (26)–(28) and by putting the value of $\zeta(x)$ and their fractional derivative, we get the residual function as follows

$$\begin{aligned} \xi(x) = \quad & C.Q^{\theta}.\Theta_N(x) + \kappa_1(C.\Theta_N(x), x)C.Q^1.\Theta_N(x) \\ & -\kappa_2(C.\Theta_N(x), x)C.\Theta_N(x) + \kappa_3(x). \end{aligned} \tag{29}$$

Now collocating Eqs. (28) and (29) at Legendre notes, we get an non linear system of algebraic equations. By Solving that system of equations and finding C we obtained numerical solution of our proposed model.

5. Results and discussion

In this section we will show the validity of newly derived operational matrix of differentiation of RFE derivative. We use the newly derived operational matrix of shifted Legendre polynomials to finding out the numerical solution. We have used the Wolfram Mathematica version-11.3 in all numerical computations.

Example 1: Considering the function $s(x) = x^5$, we find out the RFE derivative of this function. Let us assume $s(x) = \Theta_N^T(x).C$ where coefficient vector C can be find out by using the orthogonal properties of Legendre polynomial. Now taking derivative of order 0.9 and using operational matrix

$${}_0^{RFE}D_x^{0.9}x^5 = Q^{0.9}\Theta_N^T(x).C. \tag{30}$$

We plot the graph of absolute error between exact derivative and derivative find out by the RFE operational matrix. We conclude from this plot that our newly derived operational matrix has a good accuracy. It can be easily seen by Fig. 1.

Example 2: Considering $\kappa_1(\zeta, x) = \zeta, \kappa_2(\zeta, x) = \zeta^2$ and $\theta = 0.9$ we get the following RFE kernel based fractional Lienard equation

$${}_0^{RFE}D_x^{\theta}\zeta(x) + \zeta_0^{RFE}D_x^{1}\zeta(x) + \zeta^2 = \kappa_3(x). \tag{31}$$

We can take following initial conditions

$$\begin{aligned} &\zeta(0) = 1, \\ &\zeta'(0) = 1. \end{aligned} \tag{32}$$

We are choosing the exact solution of this problem $\zeta(t,x) = e^x$and the function $\kappa_3(x)$ can be found accordingly to the exact solution. Fig. 2 represent the graphs of absolute error between approximate and existing exact solution for $N = 8$.

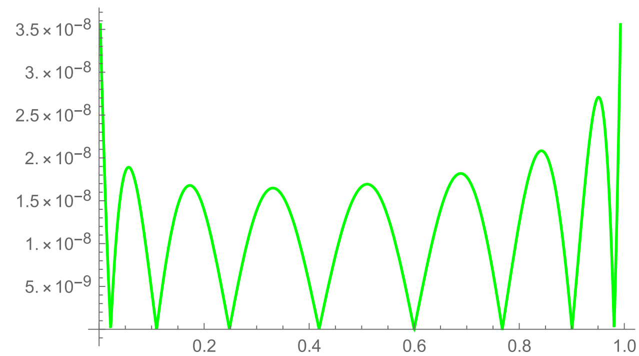

Fig. 1 Plots of absolute error for $N = 7$.

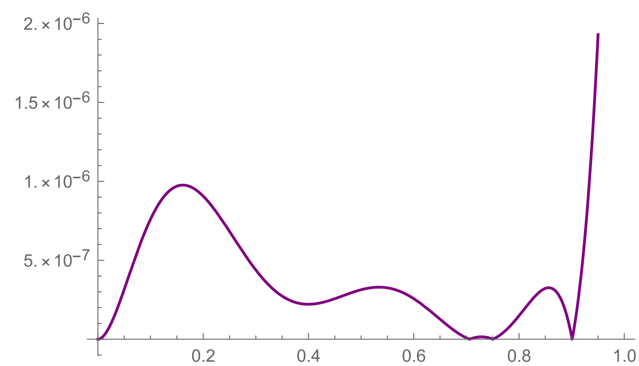

Fig. 2 Plots of absolute error for $N = 7$.

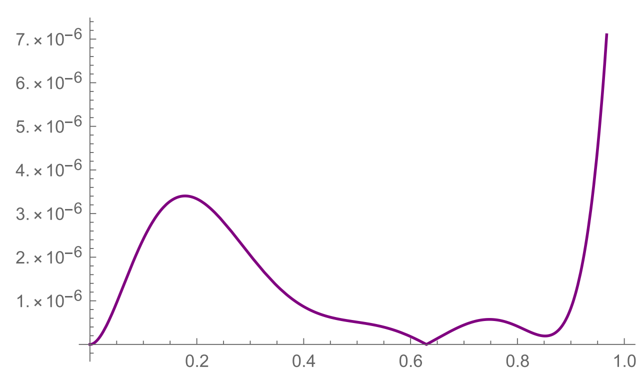

Fig. 3 Plots of absolute error for $N = 7$.

Example 3: Considering $\kappa_1 = \zeta(1-\zeta), \kappa_2 = e^{\zeta}$ and $\theta = 0.9$ our model reduces to the following equation

$$ {}_0^{RFE}D_x^{\theta}\zeta(x) + \zeta(1-\zeta){}_0^{RFE}D_x^{1}\zeta(x) + e^{\zeta} = \kappa_3(x). \tag{33} $$

with following initial conditions

$$ \begin{aligned} &\zeta(0) = 0, \\ &\zeta'(0) = 0. \end{aligned} \tag{34} $$

We take the exact solution $\zeta(x) = x^2$ with suitable force function. Fig. 3 represent the variations of absolute error drawn between approximate and exact solution.

6. Conclusion

In this paper, we have solved numerically the RFE kernel based fractional Lienard equation. We derived an accurate formula of RFE derivative of z^k for the first time. For the first time the RFE operational matrix of shifted Legendre polynomials is derived. The accuracy and validity of this operational matrix is also shown by operating it on function $s(t) = x^5$. We show the successful implementation of this operational matrix to solve the RFE kernel based fractional ordinary differential equation. We easily conclude that our proposed method is valid for such type of FDEs which have RFE fractional derivative. The graphical and exhibitions is presented to validate the effectiveness of the proposed method used for solving various cases of Lienard equation having RFE fractional derivative.

Declaration of Competing Interest

The authors declare no conflict of interest.

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