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Reduced-order anti-synchronization of the projections of the fractional order hyperchaotic and chaotic systems

Research Article

Mayank Srivastava[1*], Saurabh K. Agrawal[1†], Subir Das[1,2‡]

1 *Department of Mathematical Sciences, Indian Institute of Technology (BHU), Varanasi 221005, India*

2 *Institute of Mathematical Sciences, University of Malaya, 50603 Kuala Lumpur, Malaysia*

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Abstract: The article aims to study the reduced-order anti-synchronization between projections of fractional order hyperchaotic and chaotic systems using active control method. The technique is successfully applied for the pair of systems viz., fractional order hyperchaotic Lorenz system and fractional order chaotic Genesio-Tesi system. The sufficient conditions for achieving anti-synchronization between these two systems are derived via the Laplace transformation theory. The fractional derivative is described in Caputo sense. Applying the fractional calculus theory and computer simulation technique, it is found that hyperchaos and chaos exists in the fractional order Lorenz system and fractional order Genesio-Tesi system with order less than 4 and 3 respectively. The lowest fractional orders of hyperchaotic Lorenz system and chaotic Genesio-Tesi system are 3.92 and 2.79 respectively. Numerical simulation results which are carried out using Adams-Bashforth-Moulton method, shows that the method is reliable and effective for reduced order anti-synchronization.

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Keywords: reduced order• anti-synchronization • fractional order derivative• hyperchaotic Lorenz system• chaotic Genesio-Tesi system• Laplace transforms method

1. Introduction

Nowadays the fractional order derivative has become a popular field of research in nonlinear dynamics since fractional order system response ultimately converges to the integer order system, also for high accuracy it can be used to describe the dynamics of systems. Fractional differen-

*E-mail: mayanksrivastava1983@gmail.com
†E-mail: saurabh.goel9944@gmail.com
‡E-mail: subir_das08@hotmail.com (Corresponding author)

tial equations have garnered a lot of attention and appreciation recently due to their ability to provide an exact description of different nonlinear phenomena.The benefit of fractional order systems is that they allow increased degrees of flexibility in the model. Other advantages of fractional order systems are that they possess memory and display much more sophisticated dynamics compared to their integral order counterpart, which is of great significance in secure communication and control process. During last few years, the applications of fractional calculus can be found in vatious fields of science and engineering [1–9]. Analysis of fractional order dynamical systems involving Riemann-Liouville as well as Caputo derivatives have been found in [2, 10–12]. The field of chaos in nonlinear dynamics has grabbed the attention of the researchers and this contributes to a significant amount of the ongoing research these days.

The applications of dynamical systems have nowadays spread to a wide spectrum of disciplines including science, engineering, biology, sociology etc. The study and analysis of nonlinear dynamics have gained immense popularity during the last few decades due to its important feature of real time dynamics system. In nonlinear systems, a small change in a parameter can lead to sudden and dramatic changes in both the qualitative and quantitative behavior of the system. Chaos is an interesting phenomenon of nonlinear systems. Chaotic systems results from deterministic systems which are aperiodic and exhibit sensitive dependence on initial conditions and parameter variations.

In a chaotic synchronization process, the aim is to force a chaotic system known as response system to synchronize another identical (or different) chaotic system called a drive system. The idea of synchronizing chaotic systems was introduced by Pecora and Carroll [13] in 1990, where they showed that it is possible to synchronize chaotic systems through a simple coupling. Synchronization of chaotic dynamical systems has been intensively studied by many researchers due to its important applications in various fields viz., ecological system, physical system, chemical system, secure communications, neuron systems and laser dynamics etc. [14–20]. Different approaches of synchronization phenomena such as active control, adaptive control, back-stepping design method, sliding mode control etc. have been successfully applied to investigate chaos synchronization [21–24]. The concept of various types of synchronization viz., complete synchronization, anti-synchronization, lag synchronization, projective synchronization, function projective synchronization etc. are used in identical and non-identical chaotic systems [25–29].

In the recent years, studying chaotic attractor in fractional order systems has become an active area of research. Chaotic synchronization between different fractional order systems has become one of the most interesting subjects in chaos theory. From the literature survey it is seen that there are many investigations that have been devoted to achieve chaos synchronization in fractional order chaotic and hyperchaotic systems. Synchronization between fractional order chaotic systems is also being widely investigated by [21, 30–32]. Srivastava et al. [21] have studied synchronization between a pair of identical chaotic Lotka-Volterra systems of fractional order for different particular cases using the Pecora-Carroll method and the active control method. Yan and Li [30] have a proposal on synchronization of fractional differential equations having chaos. Erjaee and Taghvafard [31] have designed phase and anti-phase synchronization of fractional order chaotic systems via active control. Synchronization of the fractional order hyperchaos Lorenz systems with activation feedback control is given by Wang and Song [32]. Recently Agrawal et. al. [33] performed synchronization between fractional order Ravinovich-Fabrikant and Lotka-Volterra systems using the active control method.

The problems related to the synchronization of chaotic systems with different order and with reduced order have attracted the attention of the researchers and the evidence is found in the recent articles ([34]-[38]). Femat and Perales [34] investigated synchronization of chaotic systems with different orders in 2002. Based on Lyapunov stability theory, Ho et al. [35] proposed the adaptive control and parameters modulation techniques to synchronize two chaotic systems with different order though their parameters are uncertain. Moreover, in many real systems, the synchronization is carried out even though the oscillators have different order. Bowong [36] presented a nonlinear approach for synchronizing chaotic systems with different order and its application to secure communication. Reduced order synchronization, having potential applications in various fields in engineering, is achieved through synchronization of slave system with projections of a master system, where the order of the slave system is less than the order of the master system, but during synchronization all states of the slave system will be synchronized. In 2010, Al-sawalha and Noorani [37] have investigated the reduced-order anti-synchronization of uncertain chaotic systems. Based upon the parameters modulation and the adaptive control technique, they showed that dynamical evolution of a third-order chaotic system can be anti-synchronized with the canonical projection of a fourth-order chaotic system even though their parameters are unknown. Recently, Al-sawalha and Noorani [38] discussed the adaptive reduced-order anti-synchronization of uncertain chaotic systems with different order. In 2007, Lazzouni et al. [39] showed that the synchronization of a reduced order oscillator can be done with the projections of canonical planes of a fourth-orders system. Thus it can be shown that the synchronization of two different order nonlinear systems can be achieved in reduced order.

Synchronization of non identical chaotic systems means the study of two systems which exhibit similar but chaotic behavior and analysis of the situation when the mutual system behaviors approach each other. Chaotic dynamics is sensitive to initial conditions, during coupling the trajectories of the chaotic systems emerging from two initial conditions will spread exponentially with time. Therefore the study of synchronization in coupled chaotic systems is extremely necessary. Since the synchronization is caused due to transformation between dynamical variables of drive and response systems, it is challenging to detect the existence of the transformation and also the synchronous behavior. Anti-synchronization between two chaotic systems is done where the respective states are same in magnitude but opposite in sign. Hyperchaotic systems are typically nonlinear systems having two or more positive Lyapunov exponents which are useful in networking, data encryption, chemical processes and secure communications etc. Therefore, keeping in mind the important applications of reduced-order anti-synchronization and also usefulness of hyperchaotic and chaotic systems in various fields of engineering, the authors are motivated to solve a reduced-order anti-synchronization problem and have taken an initiative to apply it to do coupling of Lorenz and Genesio-Tesi systems which have applications in lasers, electrical circuits, chemical reactions etc. Again, since fractional order models are naturally related to the systems with memory and exhibit greater flexibilities for incorporating different types of information, the authors make a sincere drive to establish a good mathematical model through coupling of fractional order chaotic systems so that it is useful to the engineers and technologists working in the aforesaid areas of research.

In the present article, the authors have described the formulation of anti-synchronization of fractional reduced-order chaotic systems and have studied the reduced-order anti-synchronization between the projections of fractional order hyperchaotic Lorenz system and the fractional order Genesio-Tesi chaotic system using Laplace Transform method. The numerical simulation results obtained for the state vectors with fractional order time derivative are displayed graphically for different particular cases. The salient feature of this article is the successful graphical presentation of the error dynamics due to memory effect during anti-synchronization of a chaotic system with various combinations of state vectors of a hyperchaotic system.

The authors hope that the article will be a useful contribution to the scientific literature on the methods of control for nonlinear dynamical system.

2. Systems description and problem formulation

2.1. System description

The hyperchaotic Lorenz system [40] is described by

$$\begin{aligned}
\frac{dx}{dt} &= a(y-x)+w \\
\frac{dy}{dt} &= cx-y-xz \\
\frac{dz}{dt} &= xy-bz \\
\frac{dw}{dt} &= -yz+dw,
\end{aligned} \tag{1}$$

where x, y, z and w are the state variables. When the real constants a, b, c, d are taken as $a = 10, b = 8/3, c = 28$ and $d = -1$, the positive Lyapunov exponents of the system (1) are $\lambda_1 = 0.3381$, $\lambda_2 = 0.1586$ and thus the system has a hyper-chaotic attractor. Here, we consider the fractional-order hyperchaotic Lorenz system [32] as

$$\begin{aligned}
\frac{d^\alpha x}{dt^\alpha} &= a(y-x)+w \\
\frac{d^\alpha y}{dt^\alpha} &= cx-y-xz \\
\frac{d^\alpha z}{dt^\alpha} &= xy-bz \\
\frac{d^\alpha w}{dt^\alpha} &= -yz+dw,
\end{aligned} \tag{2}$$

where $\alpha (0 < \alpha \leq 1)$ is the fractional order time derivative. The system (2) exhibits hyperchaotic attractors at the lowest value of α as $\alpha = 0.98$ for the same values of parameters as for the standard order system. The order of the system is sum of the orders of all the fractional derivatives in system (2) i.e., the order of the system is 3.92. Figs. 1(a)-1(d) display the projections of the hyperchaotic Lorenz system for fractional order 0.98.

The Genesio-Tesi chaotic system [41] is one of paradigms of chaos since it captures many features of chaotic systems. It includes a simple square part and three simple ordinary differential equations that depend on three positive real parameters. The dynamical equations of the

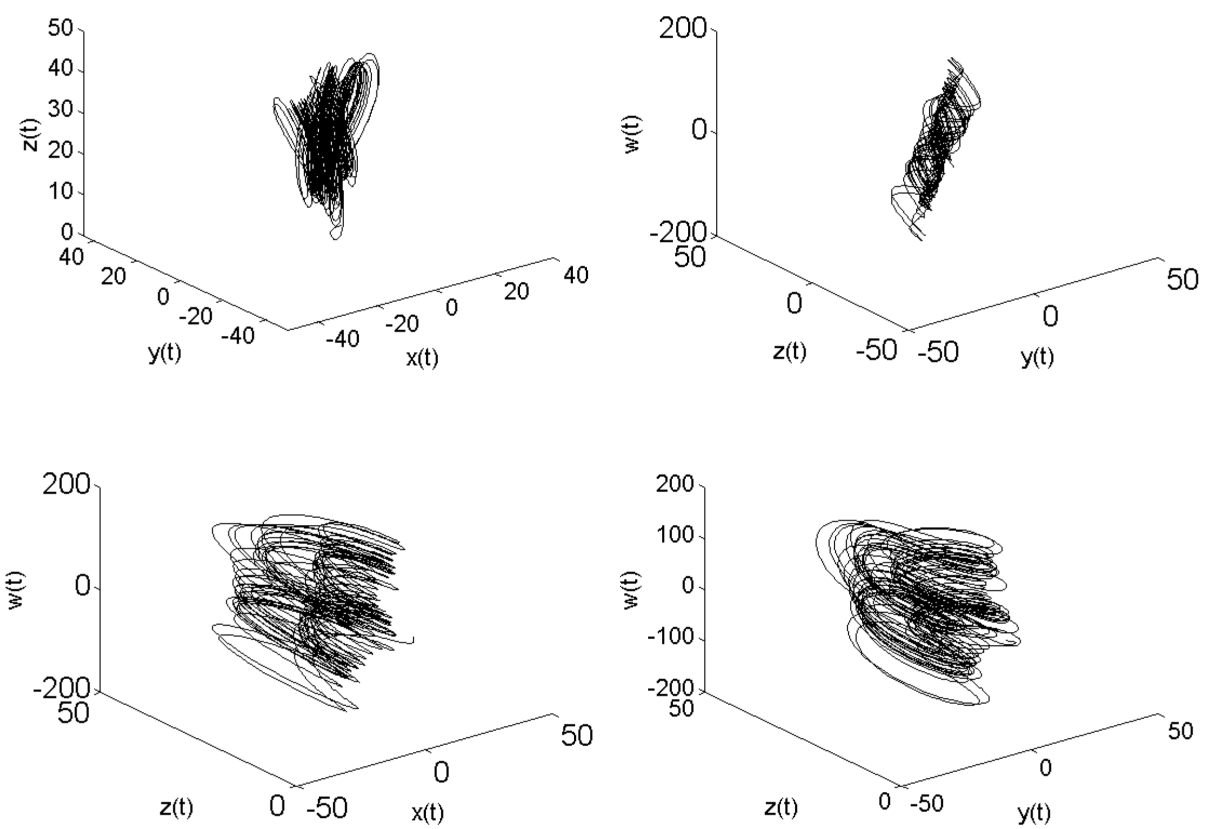

Figure 1. Phase portraits of fractional order hyperchaotic Lorenz system for $\alpha = 0.98$ in (a) the x-y-z space; (b) the x-y-w space, (c) the x-z-w space, and (d) the y-z-w space.

system is as follows

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= z \\ \frac{dz}{dt} &= -px - qy - rz + mx^2, \end{aligned} \tag{3}$$

when $p = 6, q = 2.92, r = 1.2$ and $m = 1$ there is a chaotic trajectory. Here, we consider the fractional-order Genesio-Tesi system [42] as

$$\begin{aligned} \frac{d^\alpha x}{dt^\alpha} &= y \\ \frac{d^\alpha y}{dt^\alpha} &= z \\ \frac{d^\alpha z}{dt^\alpha} &= -px - qy - rz + mx^2 \end{aligned} \tag{4}$$

The largest Lyapunov exponents of Genesio-Tesi system is 0.0022. The lowest value of α for which the system remains chaotic is 0.93. Here, the order of the system is 2.79. The chaotic attractors in $x-y-z$ space are depicted through Fig. 2 for $\alpha = 0.95$.

2.2. Formulation of anti-synchronization of fractional reduced-order chaotic systems

Let us consider the fractional order chaotic system

$$\frac{d^\alpha x(t)}{dt^\alpha} = K_1 h(x) + F(x(t), t), 0 < \alpha \leq 1, \tag{5}$$

where $x(t) \in R^n$ is the state vector and $F : R^n \to R^n$ defines a nonlinear term, $h : R^n \to R^{n \times l_1}$, and $K_1 \in R^{l_1}$ is the parameter vector of the system. We consider the system (5) as the master system.
As the slave or response system, we consider the following chaotic system described by the dynamics

$$\frac{d^\alpha y(t)}{dt^\alpha} = K_2 g(y) + H(y(t), t) + \mu(t), 0 < \alpha \leq 1, \tag{6}$$

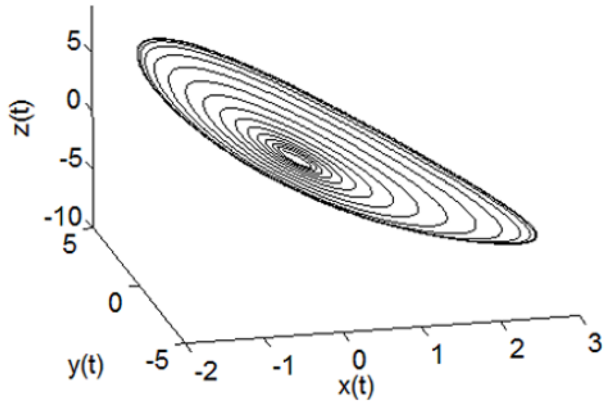

Figure 2. Phase portraits of fractional order chaotic Genesio-Tesi system for $\alpha = 0.95$.

$y(t) \in R^m$ is the state vector and $H : R^m \to R^m$ defines a nonlinear term, $g : R^m \to R^{m\times l_2}$ and $K_2 \in R^{l_2}$ is the parameter vector of system and $\mu(t) \in R^m$ is the active control parameter of the slave system.
If order $n = m$, $l_1 = l_2$ and the functions $h = g$, $F = H$, then the slave system is identical with the master system. If order $m < n$, and the functions $h \neq g$, $F \neq H$, the order of slave oscillator is lower than that of the master system. The anti-synchronization is attained only for reduced-order. The reduced-order anti-synchronization is the problem of controlling a slave system to be the projection of the master system. Therefore, we can divide the master system into two parts.
The projection is

$$\frac{d^\alpha x_i}{d\,t^\alpha} = K_1 h_i(x) + F_i(x(t), t), \tag{7}$$

where $x_i(t) \in R^m$, $F_i : R^n \to R^m$, $h_i : R^n \to R^{m\times l_1}$ and $i = 1, 2, ..., m$.
Remaining part of the systems is

$$\frac{d^\alpha x_j}{d\,t^\alpha} = K_1 h_j(x) + F_j(x(t), t), \tag{8}$$

where $x_j(t) \in R^u$, $F_j : R^n \to R^u$, $j = m + 1, ..., n$, and $h_j : R^n \to R^{u\times l_1}$ and orders m, u satisfy $m + u = n$.
The reduced-order anti-synchronization problem between two different systems is to design with a suitable choice of a controller $\mu(t)$, which anti-synchronizes the states of both the drive and response systems. i.e., $\lim_{t\to\infty} \|y + x_i\| = 0$.
We add equation (6) to equation (7) and get

$$\frac{d^\alpha e(t)}{d\,t^\alpha} = K_2 g(y) + H(y(t), t) + \mu(t) + K_1 h_i(x) + F_i(x(t), t), \tag{9}$$

where $e = y + x_i$ is the error state. Choosing the nonlinear control function as

$$\mu(t) = -K_2 g(y) - H(y(t), t) - K_1 h_i(x) - F_i(x(t), t) + V(t), \tag{10}$$

where $V(t)$ is the linear control input chosen such that the system (9) becomes stable.
Let, $V(t) = Ke(t)$, where $K \in R^{m\times m}$ is the control parameter matrix. Then equation (9) reduces to

$$\frac{d^\alpha e(t)}{d\,t^\alpha} = Ke(t), \tag{11}$$

According to the proper choice of matrix K, all the eigenvalues of the error systems should have negative real parts for achieving the anti-synchronization between master and slave systems.

Theorem 1.
If $E(s)$ is bounded then the drive and response systems (7) and (6) will be anti-synchronized under a suitable choice of control matrix K.

Proof. Taking Laplace transformation on both the sides of equation (11), we get
$s^\alpha E(s) - s^{\alpha-1} e(0) = KE(s)$
$E(s)(s^\alpha - K) = s^{\alpha-1} e(0)$
$E(s) = \frac{s^{\alpha-1} e(0)}{s^\alpha - K}$, where $E(s) = L(e(t))$.
Taking Mittag-Leffler function of one parameter [43] as
$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)}$,
the Laplace transform of which is given by
$L[E_\alpha(Kt^\alpha)] = \frac{s^{\alpha-1}}{s^\alpha - K}$.
Thus we have $e(t) = e(0)E_\alpha(Kt^{-\alpha})$ and hence we may conclude using the property of Mittag-Leffler function that $e(t) \to 0$ as $t \to \infty$, if $K < 0$.
Using the final-value theorem of Laplace transformation, we get
$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{s^\alpha e(0)}{s^\alpha - K} = 0$.
Since $E(s)$ is bounded, due to the attractiveness of the attractors of systems (2) and (4), there exists $l > 0$, such that $|x(t)| \leq l < \infty$ and $|y(t)| \leq l < \infty$. Therefore, $\lim_{t\to\infty} e(t) = 0$. Thus the anti-synchronization between systems (7) and (6) is implemented. □

3. Anti-synchronization between the projection of fractional order hyperchaotic Lorenz and fractional order chaotic Genesio-Tesi systems

In this section, the reduced-order anti-synchronization between three combinations of projection of fractional order Lorenz system as drive systems and fractional order Genesio-Tesi system as response system is studied. It is to be noted that during projection $x-y-z$ of the Lorenz system (2), the governing equation with other variable w will not be considered. The other cases are self explanatory.

Case-I: In this case, we consider that the projection $x-y-z$ of the hyperchaotic Lorenz system drives the Genesio-Tesi system. Therefore, we define the projection $x-y-z$ of the hyperchaotic Lorenz system as a drive system and Genesio-Tesi system as a response system as follows.
Here the drive system is described by

$$\begin{aligned}\frac{d^\alpha x_1}{d\,t^\alpha} &= a(y_1-x_1)+w_1\\ \frac{d^\alpha y_1}{d\,t^\alpha} &= cx_1-y_1-x_1z_1\\ \frac{d^\alpha z_1}{d\,t^\alpha} &= x_1y_1-bz_1\end{aligned} \tag{12}$$

and the response system is given by

$$\begin{aligned}\frac{d^\alpha x_2}{d\,t^\alpha} &= y_2+\mu_1(t)\\ \frac{d^\alpha y_2}{d\,t^\alpha} &= z_2+\mu_2(t)\\ \frac{d^\alpha z_2}{d\,t^\alpha} &= -px_2-qy_2-rz_2+mx_2^2+\mu_3(t),\end{aligned} \tag{13}$$

where $\mu(t)$=$[\mu_1(t), \mu_2(t), \mu_3(t)]^T$ is the controller to be designed. To investigate the anti-synchronization of systems (12) and (13), we define the error states as $e_1 = x_2+x_1$, $e_2 = y_2+y_1$, $e_3 = z_2+z_1$.
The corresponding error dynamics system is

$$\begin{aligned}\frac{d^\alpha e_1}{d\,t^\alpha} =& e_2+(a-1)y_1-ax_1+w_1+\mu_1(t)\\ \frac{d^\alpha e_2}{d\,t^\alpha} =& e_3-z_1+cx_1-y_1-x_1z_1+\mu_2(t)\\ \frac{d^\alpha e_3}{d\,t^\alpha} =& -pe_1-qe_2-re_3+px_1+qy_1+rz_1\\ &+mx_2^2+x_1y_1-bz_1+\mu_3(t)\end{aligned} \tag{14}$$

Choosing the control functions as

$$\begin{aligned}\mu_1(t) &= -k_1e_1-e_2-(a-1)y_1+ax_1-w_1\\ \mu_2(t) &= -k_2e_2-e_3+z_1-cx_1+y_1+x_1z_1\\ \mu_3(t) &= pe_1+qe_2-px_1-qy_1-rz_1-mx_2^2-x_1y_1+bz_1,\end{aligned} \tag{15}$$

where k_1, $k_2 > 0$, then the error system (14) is reduced to

$$\begin{aligned}\frac{d^\alpha e_1}{d\,t^\alpha} &= -k_1e_1\\ \frac{d^\alpha e_2}{d\,t^\alpha} &= -k_2e_2\\ \frac{d^\alpha e_3}{d\,t^\alpha} &= -re_3\end{aligned} \tag{16}$$

Here all the three eigenvalues of the error system (16) are negative.
Using theorem 1, we can conclude $e_i(t) \to 0$ as $t \to \infty$ for $k_1, k_2, r > 0$. Thus the anti-synchronization between systems (12) and (13) is achieved.

Case-II: Here we assume that the projection $x-y-w$ of the hyperchaotic Lorenz system drives the Genesio-Tesi system.
Here the drive system is described by

$$\begin{aligned}\frac{d^\alpha x_1}{d\,t^\alpha} =& a(y_1-x_1)+w_1\\ \frac{d^\alpha y_1}{d\,t^\alpha} =& cx_1-y_1-x_1z_1\\ \frac{d^\alpha w_1}{d\,t^\alpha} =& -y_1z_1+dw_1\end{aligned} \tag{17}$$

and the response system is given by (13).
To investigate the anti-synchronization of systems (17) and (13), we define the error states as $e_1 = x_2+x_1$, $e_2 = y_2+y_1$, $e_3 = z_2+w_1$.
Then the corresponding error dynamics system is

$$\begin{aligned}\frac{d^\alpha e_1}{d\,t^\alpha} =& e_2+(a-1)y_1-ax_1+w_1+\mu_1(t)\\ \frac{d^\alpha e_2}{d\,t^\alpha} =& e_3-w_1+cx_1-y_1-x_1z_1+\mu_2(t)\\ \frac{d^\alpha e_3}{d\,t^\alpha} =& -pe_1-qe_2-re_3+px_1+qy_1+rw_1\\ &+mx_2^2-y_1z_1+dw_1+\mu_3(t)\end{aligned} \tag{18}$$

Choosing the control functions as

$$\begin{aligned}\mu_1(t) &= -k_1e_1-e_2-(a-1)y_1+ax_1-w_1\\ \mu_2(t) &= -k_2e_2-e_3+w_1-cx_1+y_1+x_1z_1\\ \mu_3(t) &= pe_1+qe_2-px_1-qy_1-rw_1-mx_2^2+y_1z_1-dw_1,\end{aligned} \tag{19}$$

where k_1, $k_2 > 0$, the error system (18) becomes

$$\frac{d^\alpha e_1}{d\,t^\alpha} = -k_1 e_1$$
$$\frac{d^\alpha e_2}{d\,t^\alpha} = -k_2 e_2$$
$$\frac{d^\alpha e_3}{d\,t^\alpha} = -r e_3 \qquad (20)$$

As the previous case here all the three eigenvalues of the error system (20) are negative. So the error system (20) is stable and we can easily find the anti-synchronization between the systems (17) and (13).

Case-III: In this case projection $x - z - w$ of hyperchaotic Lorenz system is considered as the drive system and Genesio-Tesi system as response system.

Here the drive system is described by

$$\frac{d^\alpha x_1}{d\,t^\alpha} = a(y_1 - x_1) + w_1$$
$$\frac{d^\alpha z_1}{d\,t^\alpha} = x_1 y_1 - b z_1$$
$$\frac{d^\alpha w_1}{d\,t^\alpha} = -y_1 z_1 + d w_1 \qquad (21)$$

and the response system is given by (13).

Now for the investigation of the anti-synchronization of systems (21) and (13), defining the error states as $e_1 = x_2 + x_1$, $e_2 = y_2 + z_1$, $e_3 = z_2 + w_1$ and proceeding as the previous cases, we get the error systems as

$$\frac{d^\alpha e_1}{d\,t^\alpha} = -k_1 e_1$$
$$\frac{d^\alpha e_2}{d\,t^\alpha} = -k_2 e_2$$
$$\frac{d^\alpha e_3}{d\,t^\alpha} = -r e_3, \qquad (22)$$

where the control functions are chosen as

$$\begin{aligned} \mu_1(t) &= -\,k_1 e_1 - e_2 + z_1 - a(y_1 - x_1) - w_1 \\ \mu_2(t) &= -\,k_2 e_2 - e_3 + w_1 - x_1 y_1 + b z_1 \\ \mu_3(t) &= p e_1 + q e_2 - p x_1 - q y_1 - (r + d) w_1 - m x_2^2 + y_1 z_1 \end{aligned} \qquad (23)$$

Thus the error system (22) is stable and the anti-synchronization between the systems (21) and (13) is achieved.

4. Numerical simulations and results

The parameters of fractional order hyperchaotic Lorenz system and fractional order chaotic Genesio-Tesi system are taken as $(a, b, c, d) = (10, 8/3, 28, -1)$ and $(p, q, r, m) = (6, 2.92, 1.2, 1)$ such that the systems exhibit hyperchaotic and chaotic behaviours respectively. During the anti-synchronization between Lorenz system in $x - y - z$ projected plane as drive system and Genesio-Tesi as response system, the initial values are taken as $(x_1(0), y_1(0), z_1(0), w_1(0)) = (2, -3, 4, 3)$ and $(x_2(0), y_2(0), z_2(0)) = (-7, -4, 5)$ such that the systems exhibit hyperchaotic and chaotic behaviour respectively. Thus, the initial errors will be $(e_1(0), e_2(0), e_3(0)) = (-5, -7, 9)$. During the anti-synchronization between Lorenz system in $x - y - w$ projected plane and Genesio-Tesi system, the initial conditions are taken as $(x_1(0), y_1(0), w_1(0), z_1(0)) = (2, -3, 3, 4)$ and $(x_2(0), y_2(0), z_2(0)) = (2, -3, 3)$. Thus the initial errors are $(e_1(0), e_2(0), e_3(0)) = (4, -6, 6)$. Again at the time of anti-synchronization between Lorenz system in $x - z - w$ projected plane and Genesio-Tesi system, the initial conditions are taken as $(x_1(0), z_1(0), w_1(0), y_1(0)) = (2, 4, 3, -3)$ and $(x_2(0), y_2(0), z_2(0)) = (-7, -13, 6)$ with initial errors $(e_1(0), e_2(0), e_3(0)) = (-5, -9, 9)$. In all the three cases the Adams-Bashforth-Moulton method ([44],[45]) is used to solve the systems with time step size is taken as 0.005. Figs. 3(a)-(c), 4(a)-(c) and 5 (a)-(c) show the state responses of the systems (12), (17) and (21) with the system (13) for $k_1 = k_2 = 1$. Figs. 3(d), 4(d) and 5(d) respectively show the anti-synchronization error systems (16), (20) and (22) converge to zero as time becomes large for the order of the derivative $\alpha = 0.98$.

5. Conclusion

The present research work investigates the anti-synchronization between a fractional order chaotic Genesio-Tesi system and the projections of the fractional order hyperchaotic Lorenz systems for different combinations of state variables. The conditions for the anti-synchronization of considered pairs of projection of fractional order hyperchaotic and chaotic systems are derived using Laplace transforms. The simulation results show that the states of the projections of the fractional order systems are anti-synchronized asymptotically. Numerical simulations clearly exhibit the reliability and effectiveness of the proposed anti-synchronization scheme even for the fractional order derivative and generalizes previous theoretical results.

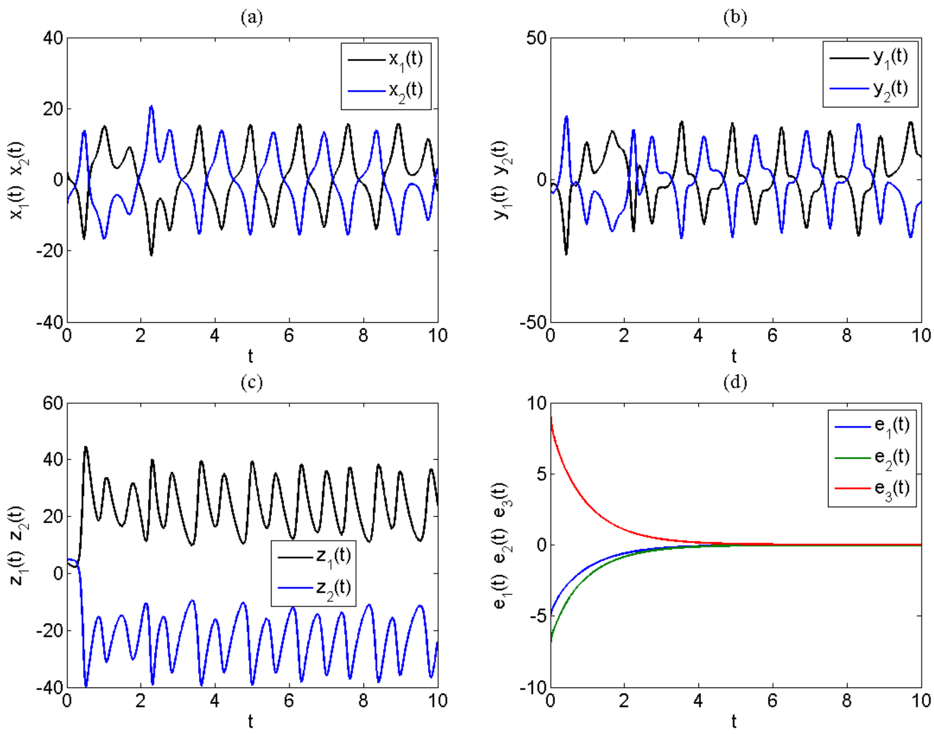

Figure 3. Anti-synchronization between the projection $x-y-z$ of the fractional order hyperchaotic Lorenz and chaotic Genesio-Tesi systems for signals (a) between x_1 and x_2 (b) between y_1 and y_2 (c) between z_1 and z_2 (d) the error functions of three state variables versus time t for fractional order time derivative $\alpha = 0.98$.

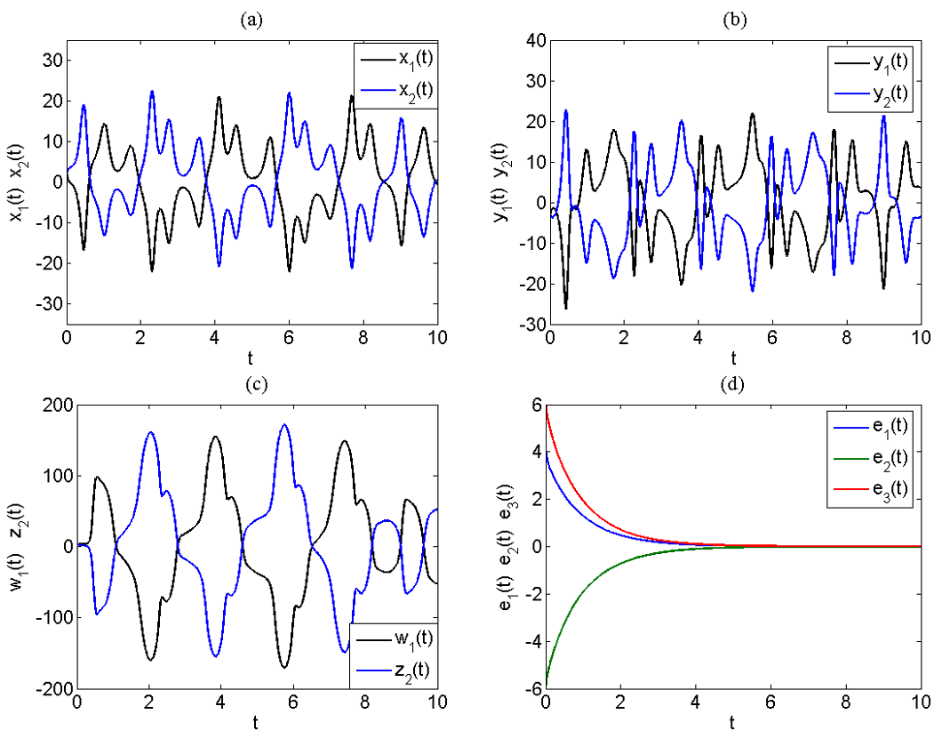

Figure 4. Anti-synchronization between the projection $x-y-w$ of the fractional order hyperchaotic Lorenz and chaotic Genesio-Tesi systems for signals (a) between x_1 and x_2 (b) between y_1 and y_2 (c) between w_1 and z_2 (d) the error functions of three state variables versus time t for fractional order time derivative $\alpha = 0.98$.

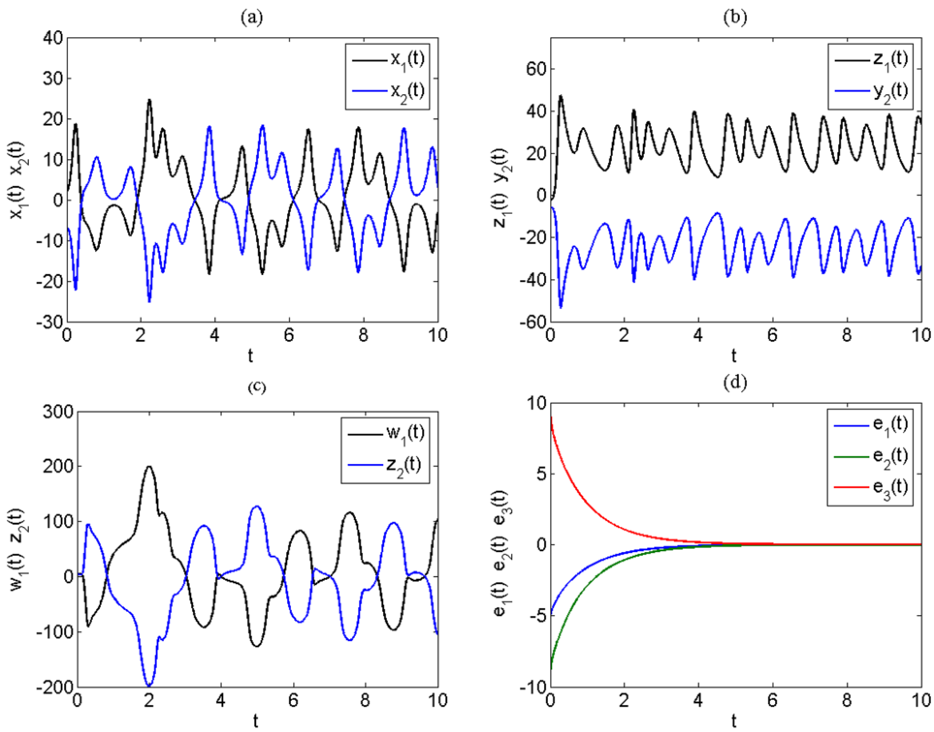

Figure 5. The anti-synchronization between the projection $x-z-w$ of the fractional order hyperchaotic Lorenz and chaotic Genesio-Tesi systems for signals (a) between x_1 and x_2 (b) between z_1 and y_2 (c) between w_1 and z_2 (d) the error functions of three state variables versus time t for fractional order $\alpha = 0.98$.

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