HOMOTOPY PERTURBATION METHOD FOR A STEFAN PROBLEM WITH VARIABLE LATENT HEAT

by

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In this paper, homotopy perturbation method is successfully applied to find an approximate solution of one phase Stefan problem with variable latent heat. The results thus obtained are compared graphically with a published analytical solution and are in good agreement.

Key words: *homotopy perturbation method, Taylor's series, Stefan problem, sediment transport, Shoreline problem*

Introduction

The mathematical model of the movement of the shoreline in a sedimentary ocean basin (A Shoreline Problem) is a Stefan problem with variable latent heat. Swenson *et al.* [1] utilized an analogy with one-phase melting problem and developed a mathematical model for movement of shoreline in a sedimentary basin in response to changes in sediment line flux, tectonic subsidence of Earth's crust and sea level change. Voller *et al.* [2] presented an analytical similarity solution for a Stefan problem with variable latent heat which is a limit case of the shoreline model. Later, Capart *et al.* [3] presented mathematical solutions for several sedimentary problems featuring semi-infinite alluvial channels evolving under diffusional sediment transport. Voller *et al.* [4] discussed a novel moving boundary problem related to shoreline movement in a sedimentary basin, which was solved by enthalpy method. They have shown how shoreline problem can be solved by using the same numerical tools which were already used for solving classical Stefan's melting problem. In 2009, Rajeev *et al.* [5] presented a numerical method for a moving boundary problem with variable latent heat and the comparisons were made with the results of Voller *et al*. [2].

The Stefan problem is a special non-linear problem which is difficult to get the exact solution [6, 7]. Many approximate methods have been used to solve the Stefan problem *e. g*., the perturbation method [8], combination of variable method [9]. He [10] also presented a survey of some recent developments in asymptotic techniques, which are valid not only for weakly non-linear equations, but also for strongly ones. Recently, He [11] presented some effective analytical methods to solve the problems arising in thermal science.

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In the last two decades, perturbation methods are widely used by researchers to solve non-linear problems. Perturbation methods are based on a small parameter involved in the equation. An appropriate choice of small parameters gives good results. However, unsuitable choice of small parameters leads bad results. To eliminate the limitations of the traditional perturbation techniques, He [12-14] proposed homotopy perturbation method (HPM). In this method, according to the homotopy technique, a homotopy with an imbedding parameter $p \in (0, 1)$ is constructed, and the imbedding parameter is considered as a small parameter. Thus, homotopy perturbation method is the coupling of the traditional perturbation method and the homotopy technique. He successfully applied homotopy perturbation method to solve various nonlinear problems [15, 16] and presented a comparison of this method with homotopy analysis method [17]. In 2009, Li *et al.* [18] successfully extended homotopy perturbation method to solve time-fractional diffusion equation with a moving boundary condition. He [19] presented a note on optimal homotopy asymptotic method to solve non-linear equations and shown that this method is a coupled method of the homotopy perturbation method and the method of least squares. Mohyud-Din and Yildirim [20] applied homotopy perturbation method (HPM) to solve homogeneous and inhomogeneous advection problems. Das *et al*. [21] also used homotopy perturbation method to solve a moving boundary problem involving space-time fractional derivative.

The approximate analytical approach taken in this literature is HPM. To the best of author's knowledge solution of one phase Stefan problem with variable latent heat by HPM has not yet been solved. The objective of this paper is to find an approximate analytical solution of a Stefan problem with variable latent heat by using HPM. The obtained results are compared with the existing exact solution. A brief sensitivity study is also performed.

The Shoreline problem

We consider a problem of shoreline movement in a sedimentary basin due to a sediment line flux, tectonic subsidence of the Earth's crust, and sea level change. This problem is the ocean-basin shoreline-tracking problem [1, 2]. The mathematical equations for this phenomenon are given by:

$$\frac{\partial \eta}{\partial t} = \nu \frac{\partial^2 \eta}{\partial x^2} + \frac{\partial b}{\partial t}, \quad 0 \le x \le s(t) \tag{1}$$

$$\nu \left.\frac{\partial \eta}{\partial x}\right|_{x=0} = -q(t) \quad \text{and} \quad \eta(s,t) = z(t) \tag{2}$$

where $\eta(s, t)$ is the height of the sediment above a datum, b – the height of Earth's crust above a datum, ν – the effective fluvial diffusivity, $s(t)$ – the shoreline position, q – the sediment line flux and $z(t)$ – the ocean level above the datum.

A boundary condition on moving interface is given by:

$$-\nu \left.\frac{\partial \eta}{\partial x}\right|_{s(t)} = (u-s)\left[\alpha \frac{\mathrm{d}s}{\mathrm{d}t} + \frac{\mathrm{d}z}{\mathrm{d}t}\right] - \int_s^u \frac{\partial b}{\partial t}\mathrm{d}x \tag{3}$$

with

$$s(0) = 0 \tag{4}$$

where α is the slope of the offshore sediment wedge and $u(t)$ – the lateral position where the toe of the submarine sediment wedge intersects the ocean basement.

Now, we assume a shoreline problem with a fixed line flux, a constant ocean level ($z = 0$), no tectonic subsidence of the Earth's crust, and a constant sloping basement $\beta < \alpha$ (as given in [2, 5]). Figure 1 represents a schematic cross-section [2] of such a basin with the variables. Under this assumption, the eqs. (1)-(4) reduce to the following problem:

$$\frac{\partial \eta}{\partial t} = v\frac{\partial^2 \eta}{\partial x^2}, \quad 0 \le x \le s(t) \tag{5}$$

with initial and boundary conditions:

$$v\frac{\partial \eta}{\partial x}\bigg|_{x=0} = -q \tag{6}$$

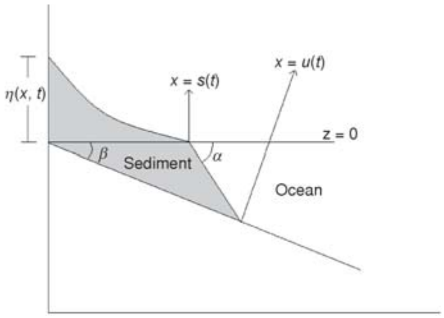

Figure 1. A schematic cross-section of limit case with no tectonic subsidence and sea level change

and
$$s(0) = 0 \tag{7}$$

The additional conditions on the moving interface are:

$$-v\frac{\partial \eta}{\partial x}\bigg|_{x=s(t)} = \gamma s\frac{\mathrm{d}s}{\mathrm{d}t} \tag{8}$$

and
$$s(0) = 0 \tag{9}$$

where $\alpha(u-s) = \alpha\beta s/(\alpha-\beta) = \gamma s$ (variable latent heat term) and γ is a constant. Equations (5)-(9) represent a Stefan problem with variable latent heat.

Solution of the problem by HPM

According to the homotopy perturbation method [17-19], we construct the following simple homotopy for eq. (5):

$$(1-p)v\frac{\partial^2 \eta}{\partial x^2} + p\left(v\frac{\partial^2 \eta}{\partial x^2} - \frac{\partial \eta}{\partial t}\right) = 0 \tag{10}$$

or

$$v\frac{\partial^2 \eta}{\partial x^2} - p\frac{\partial \eta}{\partial t} = 0 \tag{11}$$

where $p \in [0, 1]$ is an embedding parameter. When $p = 0$, eq. (11) is an ordinary differential equation, $\partial^2\eta/\partial x^2 = 0$, and if $p = 1$, eq. (11) reduce to the eq. (5).

Assuming the following power series in p as the solutions for eq. (8):

$$\eta = \sum_{n=0}^{\infty} p^n \eta_n, \quad s = \sum_{n=0}^{\infty} p^n s_n \tag{12}$$

The approximate solution of eq. (5) can be obtained by setting $p = 1$, *i. e.*

$$\eta = \sum_{n=0}^{\infty} \eta_n, \quad s = \sum_{n=0}^{\infty} s_n \tag{13}$$

Substituting eq. (12) into eq. (11), we obtain:

$$\sum_{n=0}^{\infty} p^n v\frac{\partial^2}{\partial x^2}\eta_n = \sum_{n=0}^{\infty} p^{n+1}\frac{\partial}{\partial t}\eta_n \tag{14}$$

Accordingly, the boundary condition (7) becomes:

$$\sum_{m=0}^{\infty} p^m \eta_m \left(\sum_{n=0}^{\infty} p^n s_n, t \right) = 0 \tag{15}$$

In order to compare the coefficient of various powers of p, expending $\eta_i(x, t)$ in Taylor's series form (as given in [18]) about a point (s_0, t) as:

$$\eta_i(x,t) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n \eta_i}{\partial x^n}\Bigg|_{(s_0,t)} (x - s_0)^n, \quad i = 0, 1, 2...$$

As a result, eq. (15) becomes:

$$\sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \frac{p^l}{m!} \left(\sum_{n=1}^{\infty} p^n s_n \right)^m \frac{\partial^m}{\partial x^m} \eta_l(s_0, t) = 0 \tag{16}$$

and the interface condition (8) becomes:

$$\sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \frac{p^l}{m!} \left(\sum_{n=1}^{\infty} p^n s_n \right)^m \frac{\partial^{m+1}}{\partial x^{m+1}} \eta_l(s_0, t) = -\frac{\gamma}{v} \sum_{m=0}^{\infty} p^m s_m \frac{\mathrm{d}}{\mathrm{d}t} \sum_{n=0}^{\infty} p^n s_n \quad (x = s_0) \tag{17}$$

Comparing the terms with identical powers of p in eqs. (14), (16), and (17), the following series of equations can be obtained:

$$p^0: \frac{\partial^2 \eta_0}{\partial x^2} = 0, \ \frac{\partial \eta_0(0,t)}{\partial x}\Bigg|_{x=0} = -\frac{q}{v}, \ \eta_0(s_0, t) = 0, \ \frac{\partial \eta_0(s_0,t)}{\partial x} = -\frac{\gamma}{v} s_0 \frac{\mathrm{d}s_0}{\mathrm{d}t}, \ s_0(0) = 0 \tag{18}$$

$$\begin{gathered} p^1: v\frac{\partial^2 \eta_1}{\partial x^2} = \frac{\partial \eta_0}{\partial t}, \ \frac{\partial \eta_1(0,t)}{\partial x} = 0, \ \eta_1(s_0,t) + s_1 \frac{\partial \eta_0(s_0,t)}{\partial x} = 0, \\ \frac{\partial \eta_1(s_0,t)}{\partial x} + s_1 \frac{\partial^2 \eta_0(s_0,t)}{\partial x^2} = -\frac{\gamma}{v}\left(s_0 \frac{\mathrm{d}s_1}{\mathrm{d}t} + s_1 \frac{\mathrm{d}s_0}{\mathrm{d}t} \right), \ s_1(0) = 0, \\ \vdots \end{gathered} \tag{19}$$

and so on.

Considering the first three equations of (18), we have:

$$\eta_0 = \frac{q}{v}(s_0 - x) \tag{20}$$

Substituting it into fourth equation of (18), we get:

$$s_0 = a_0 \sqrt{t} \tag{21}$$

where $a_0 = (2q/\gamma)^{1/2}$.

Substituting s_0 into eq. (20), we get:

$$\eta_0 = \frac{q}{v}(a_0 \sqrt{t} - x) \tag{22}$$

Substituting η_0 and s_0 into eq. (19) and using the above process, we can obtain the following expressions of η_1 and s_1:

$$\eta_1 = a_1 \left(a_0 \frac{x^2}{4v\sqrt{t}} + 2a_2\sqrt{t} \right) \tag{23}$$

$$s_1 = a_2 \sqrt{t} \tag{24}$$

where $a_1 = q/v$ and $a_2 = -[q^{3/2}/2^{1/2} v \gamma^{3/2}]$.

Sequentially, η_2, s_2; η_3, s_3;... can be obtained.

Substituting η_0, s_0, η_1, and s_1 into eq. (13), the first order approximate solution can be obtained as:

$$\eta = a_1\left[(a_0\sqrt{t} - x) + \left(a_0\frac{x^2}{4v\sqrt{t}} + 2a_2\sqrt{t}\right)\right] \tag{25}$$

$$s = (a_0 + a_2)\sqrt{t} \tag{26}$$

which give height of the sediment above the datum and the shoreline position at a particular time.

Numerical comparison and discussion

In this section, numerical results for height of sediment and shoreline position $s(t)$ are calculated for the fixed value of slope of basement ($\beta = 1.5$) and slope of off-shore sediment ($\alpha = 1.7$). The results are carried out using MATHEMATICA software and depicted through figures. In order to show the accuracy of the proposed approximate solution, we compare it with the existing exact solution of the same problem given by Voller *et al.* [2]. It can be seen that:

$$\eta = \frac{2q}{v}\left\{\lambda\left[\frac{e^{\frac{-x^2}{4tv}} + \pi^{\frac{1}{2}}v^{\frac{-1}{2}}\dfrac{x}{2\sqrt{t}}\operatorname{erf}\left(\dfrac{x}{2\sqrt{t}\sqrt{v}}\right)}{e^{\frac{-\lambda^2}{v}} + \pi^{\frac{1}{2}}v^{\frac{-1}{2}}\lambda\operatorname{erf}\left(\lambda v^{\frac{-1}{2}}\right)}\right] - \frac{x}{2\sqrt{t}}\right\}\sqrt{t} \tag{27}$$

$$s = 2\lambda\sqrt{t} \tag{28}$$

where λ is the root of the following non-linear equation:

$$\frac{\pi^{\frac{1}{2}}v^{\frac{-1}{2}}\operatorname{erf}\left(\lambda v^{\frac{-1}{2}}\right)}{e^{\frac{-\lambda^2}{v}} + \pi^{\frac{1}{2}}v^{\frac{-1}{2}}\lambda\operatorname{erf}\left(\lambda v^{\frac{-1}{2}}\right)} - \frac{1}{\lambda} + \frac{2\gamma\lambda}{q} = 0 \tag{29}$$

are exact solutions to eqs. (5)-(9).

Figures 2 and 3 depict the dependence of height of sediment $\eta(x, t)$ on x at a fixed value of sediment line flux ($q = 0.5$) for $v = 1.0$ and $v = 2.0$, respectively. Figures 4 and 5 represent the accuracy of movement of shoreline position for a fixed value of sediment line flux ($q = 0.5$). Table 1 shows the absolute errors and the relative errors between exact and approximate solutions at $v = 1.0$ and $q = 0.5$, 1.0, 1.5. From tab. 1 and figs. 2-5, one can observe that the approximate solution by HPM is close to the exact solution given by Voller *et al.* [2].

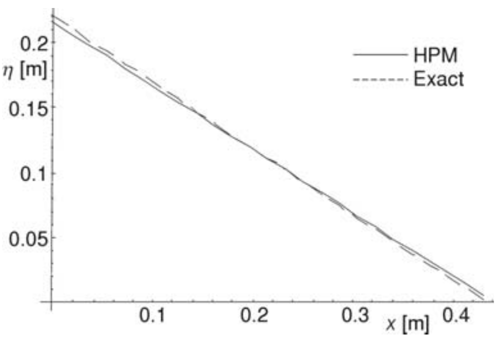

Figure 2. Plot of $\eta(x, t)$ *vs.* x for $v = 1.0$

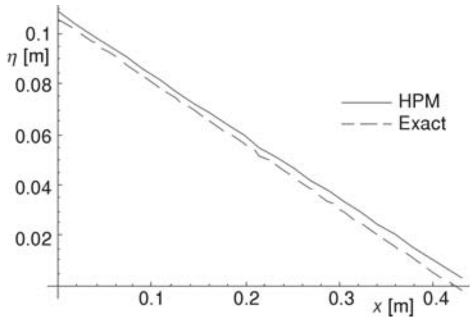

Figure 3. Plot of $\eta(x, t)$ *vs.* x for $v = 2.0$

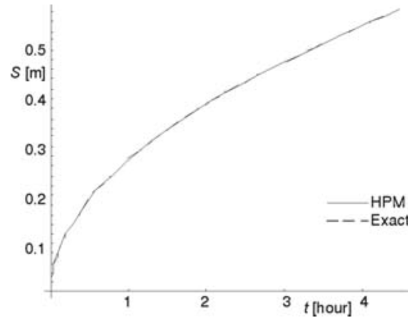

Figure 4. Plot of $s(t)$ *vs.* t for $v = 1.0$

Table 1. The exact value, approximate value, absolute error, and relative erro r at $v = 1.0$

q	S	Exact value t_e	Approximate value t_a	Absolute error $\lvert t_e - t_a\rvert$	Relative error $\lvert t_e - t_a\rvert/t_e$
0.5	0.1	0.132377	0.132651	0.000274	0.0020698
	0.2	0.529508	0.530604	0.0010953	0.00206852
	0.3	1.19139	1.19386	0.00247	0.0020732
	0.4	2.11803	2.12242	0.00439	0.00207268
	0.5	3.30942	3.31628	0.00686	0.00207287
	0.6	4.76557	4.77544	0.00987	0.0020711
	0.7	6.48647	6.4999	0.01343	0.00207046
	0.8	8.47212	8.48966	0.0175	0.00207032
	0.9	10.7225	10.7447	0.0222	0.00207041
	1.0	13.2377	13.2651	0.0274	0.0020698
1.0	0.1	0.068517	0.0690603	0.0005433	0.0079294
	0.2	0.274068	0.27624	0.00217	0.0079250
	0.3	0.616653	0.621543	0.00489	0.0079299
	0.4	1.09627	1.10496	0.00869	0.00792688
	0.5	1.71292	1.72651	0.01359	0.0079338
	0.6	2.46661	2.48617	0.01956	0.0079299
	0.7	3.35733	3.38395	0.02662	0.0079289
	0.8	4.38509	4.41986	0.03477	0.0079291
	0.9	5.54987	5.59388	0.04401	0.00792992
	1.0	6.8517	6.90603	0.05433	0.0079294
1.5	0.1	0.0471678	0.0479785	0.0008107	0.0171876
	0.2	0.188671	0.191914	0.003243	0.0171887
	0.3	0.42451	0.431807	0.007297	0.0171892
	0.4	0.754685	0.767656	0.01297	0.0171873
	0.5	1.1792	1.19946	0.02026	0.017181
	0.6	1.69804	1.72723	0.02919	0.0171904
	0.7	2.31122	2.35095	0.03973	0.01719
	0.8	3.01874	3.07063	0.05189	0.017189
	0.9	3.82059	3.88626	0.06567	0.0171884
	1.0	4.71678	4.79785	0.08107	0.0171876

From fig. 6, it is seen that the propagation of the shoreline position increases towards sea side with the increase of sediment line flux ($q = 0.5, 1.0, 1.5$) for fixed values of $\alpha = 1.7, \beta = 1.5$, $v = 1.0$. This result is in good agreement with the result of Rajeev *et al.* [5].

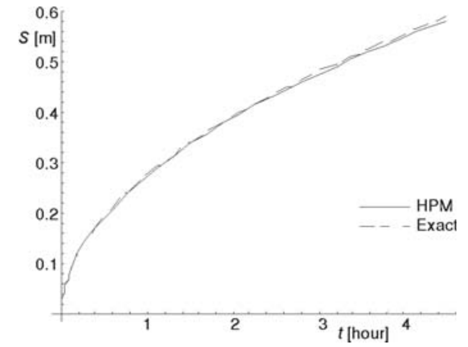

Figure 5. Plot of $s(t)$ *vs.* x for $v = 2.0$

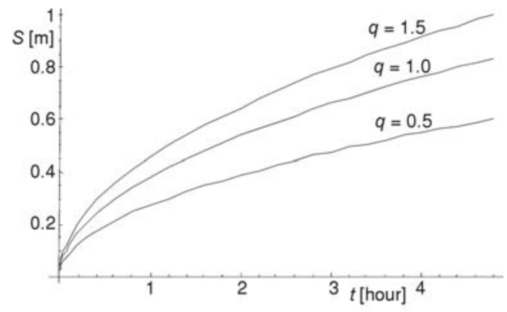

Figure 6. Plot of $s(t)$ *vs.* t for $v = 1.0$

Conclusions

The homotopy perturbation method is an efficient technique for solving various scientific and engineering problems. It is seen that HPM is a powerful and accurate method for finding the solution of Stefan problem with variable latent heat. Moreover, it is straight forward and avoids the hectic work of calculations. The author believes that the procedure as described in the present study will be applicable to linear and non-linear Stefan problems and it will considerably benefit to engineers and scientists working in this field.

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Nomenclature

- b – height of the Earth's crust (basement) above datum, [m]
- q – prescribed sediment line flux, [$m^3m^{-1}t^{-1}$]
- $s(t)$ – shoreline position, [m]
- t – time
- $u(t)$ – position of intersection between offshore sediment wedge and basement, [m]
- v – diffusion coefficient, [m^2t^{-1}]
- x – space variable, [m]
- $z(t)$ – ocean level above datum, [m]

Greek symbols

- α – slope of off-shore sediment wedge
- β – slope of basement
- γ – constant
- η – height of sediment above datum, [m]

References

[1] Swenson, J. B., *et al*., Fluvio-Deltaic Sedimentation a Generalized Stefan Problem, *Euro. J. Appl. Maths., 11* (2000), 5, pp. 433-452
[2] Voller, V. R., *et al*., An Analytic Solution for a Stefan Problem with Variable Latent Heat, *Int. J. Heat Mass Transfer, 47* (2004), 24, pp. 5387-5390
[3] Capart , H., *et al*., Self-Similar Evolution of Semi-Infinite Alluvial Channels with Moving Boundaries, *J. of Sedimentary Research, 77* (2007), 1, pp. 13-22
[4] Voller, V. R., *et al*., An Enthalpy Method for Moving Boundary Problems on the Earth's Surface, *Int. J. Num. Heat and Fluid Flow, 16* (2006), 5, pp. 641-654
[5] Rajeev, *et al*., Numerical Solution of a Moving-Boundary Problem with Variable Latent Heat, *Int. J. Heat Mass Transfer, 52* (2009), 7-8, pp. 1913-1917
[6] Crank, J., *Free and Moving Boundary Problem*, Clarendon Press, Oxford, UK, 1987
[7] Carslaw, H. S., Jaeger, J. C., *Conduction of Heat in Solids*, Oxford University Press, Oxford, UK, 1987

[8] Lin, J. S., Peng, Y. L., Swelling Controlled Release of Drug in Spherical Polymer-Penetrant Systems, *Int. J. Heat Mass Transfer, 48* (2005), 6, pp.1186-1194
[9] Abdekhodaie, M. J., Cheng, Y. L., Diffusional Release of a Dispersed Solute from a Spherical Polymer Matrix, *J. Memb. Sci., 115* (1996), 2, pp. 171-178
[10] He, J. H., Some Asymptotic Methods for Strongly Nonlinear Equations, *Int. J. Modern Physics B, 20* (2006), 10, pp.1141-1199
[11] He, J. H., Analytical Methods for Thermal Science – an Elementary Introduction, *Thermal Science, 15* (2011), Suppl. 1, pp. S1-S3
[12] He, J. H. , Homotopy Perturbation Technique, *Comput. Methods Appl. Mech. Engg., 178* (1999), 3-4, pp. 257-262
[13] He, J. H. , A Coupling Method of a Homotopy Technique and a Perturbation Technique for Non-Linear Problems, *Int. J. Non-linear Mech., 35* (2000), 1, pp. 37-43
[14] He, J. H., Homotopy Perturbation Method: A New Nonlinear Analytical Technique, *Appl. Math. Comput., 135* (2003), 1, pp. 73-79
[15] He, J. H., The Homotopy Perturbation Method for Nonlinear Oscillators with Discontinuities, *Appl. Math. Comput., 151* (2004),1, pp. 287-292
[16] He, J. H. , Application of Homotopy Perturbation Method to Nonlinear Wave Equations, *Chaos, Soliton, Fract., 26* (2005), 3, pp. 695-700
[17] He, J. H., Comparison of Homotopy Perturbation Method and Homotopy Analysis Method, *Appl. Math. Comput., 156* (2004), 2, pp. 527-539
[18] Li, X., *et al.*, Homotopy Perturbation Method to Time-Fractional Diffusion Equation with a Moving Boundary Condition, *Applied Mathematics and Computation, 208* (2009), 2, pp. 434-439
[19] He, J. H., A Note on the Homotopy Perturbation Method, *Thermal Science, 14* (2010), 2, pp. 565-568
[20] Mohyud-Din, S.T., Yildirim, A., Homotopy Perturbation Method for Advection Problems, *Nonlinear Science Letters A, 1* (2010), 3, 307-312
[21] Das, S., *et al.*, Analytical Approximate Solution of Space-Time Fractional Diffusion Equation with a Moving Boundary Condition, *Zeitschrift fur Naturforschung A, 66a* (2011), 1-2, pp. 281-288

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