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Model reduction by extended minimal degree optimal Hankel norm approximation

Deepak Kumar *, S.K. Nagar

Department of Electrical Engineering, Indian Institute of Technology (BHU) Varanasi, 221005, India

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ABSTRACT

This paper presents an extended minimal degree optimal Hankel norm approximation (MDOHNA) based order reduction algorithm using a basis-free descriptor which circumvents the requirement of computing balanced realized model for order reduction. Conjunction of system decomposition algorithm (Singh and Nagar, 2004) [21], (Kumar et al., 2012) [30,31] with MDOHNA is used as extension for order reduction of unstable systems. The developed algorithm is applicable for stable/unstable, linear time invariant, minimal/non-minimal, continuous/discrete-time systems as well. Further, effectiveness of the algorithm over existing techniques is validated with the help of a numerical example.

1. Introduction

Modeling strategies often result in dynamical systems of very high dimension. It is then desirable to find systems of the same form but of lower complexity, whose input–output behavior approximates the behavior of the original system. From a mathematical and system theoretical point of view, reduction using optimal Hankel norm approximation is among the most important model reduction techniques today. It is one of the few model reduction algorithms that produce optimal approximate models.

Adamjan et al. [1] introduced a closed-form optimal solution for model reduction with respect to Hankel norm criterion for the scalar (single input–output) case. The relevance of [1] to model reduction was first mentioned by Kung [2] who later presented closed-form optimal Hankel norm solution for multivariable systems [3] and developed minimal degree approximation algorithm (MDA). The structure of linear dynamical systems with finite dimension is exploited in [4] to derive explicit algorithms and simple expressions for the Hankel norm approximation of a high dimensional discrete-time stable scalar system by a reduced model of any low order. The continuous-time scalar case is independently solved by Bettayeb et al. [5]. Kung and Genin [6] used a two-variable polynomial approach to rederive the results of [l] and described many significant properties of the MDA problems. Further Kung et al. [7] presented state space formulation of optimal Hankel norm approximation problem. Subsequently, Glover [8] investigated characterization of all optimal Hankel norm approximations that minimize the Hankel norm for multivariable linear systems and derived the frequency response error bound. Characterization of all solutions to the suboptimal Hankel problem using a different approach was also derived by Ball et al. [9,10] continuous-time and discrete-time systems. Safonov and Chiang [11] suggested an improved representation of the equations for minimal degree approximation Hankel norm model reduction which completely circumvents the need of balanced state

* Corresponding author.
E-mail addresses: deepak.kumar.rs.eee@itbhu.ac.in (D. Kumar), sknagar.eee@itbhu.ac.in (S.K. Nagar).

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space realization, permits the same simple formulas to be used for both the suboptimal and the optimal cases. Further, order reduction for different class of systems using optimal Hankel norm approximation is investigated by several authors [12–18].

As far as reduction of unstable systems is concerned, several researchers adopted different approaches. The common approach is to split the system into a stable subsystem and a unstable subsystem, and then apply an existing reduction technique to reduce the stable part while entirely retaining the unstable part. Hsu and Hou [19] considered the reduction of continuous-time systems in which the transformed system is directly used for model reduction without complete decoupling of system into stable and unstable subsystems. By using bilinear transformation, Chen [20] suggested a method for decomposition into three parts- stable, oscillatory and unstable subsystems.

In this paper extension of minimal degree optimal Hankel norm approximation (MDOHNA) technique for order reduction of unstable systems using system decomposition algorithm is proposed. The algorithm used for system decomposition [21–23] is based on real Schur transformation [24] and it is free from bilinear transformation in such a way that the original unstable system is decomposed into stable and unstable subsystems. The reduced order model is obtained by simplifying the stable subsystem and adding it to the unstable subsystem.

The organization of the paper is as follows: Section 2 describes the various steps of the MDOHNA algorithm for order reduction of stable, minimal/non-minimal, continuous/discrete-time systems. Section 3 investigates the extension of the algorithm for unstable systems using system decomposition. Finally, the comparative study of proposed method and conclusions are made in Section 4 and 5, respectively.

2. Order reduction algorithm by minimal degree optimal Hankel norm approximation

The development of the optimal Hankel norm approximation [1–3] and the balanced truncation [25] changed the perception of model reduction techniques significantly. These two techniques ensured almost perfect characteristics as their reduced order models are stable and also a priori frequency response error bounds. For the systems having uncontrollable and unobservable states, the balancing transforms are generally singular. This creates practical difficulties while applying standard Hankel norm approximation theory as balanced state space model is required to be computed first. The minimal degree optimal Hankel norm approximation (MDOHNA) algorithm [11] completely alleviates the requirement of computing balanced realized model for the system to be reduced, even when the original system is nearly uncontrollable and/or unobservable.

Consider the transfer function matrix $G(s) = C(sI - A)^{-1}B + D$ and the associated standard realization of a linear time invariant (LTI) dynamical system as,

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1a}$$

$$y(t) = Cx(t) + Du(t), \tag{1b}$$

where $A \in \Re^{n\times n},\ B \in \Re^{n\times q},\ C \in \Re^{p\times n}$ and $D \in \Re^{p\times q}$. The number of state variables n is known as the order of the system. We are interested in computing a reduced-order LTI system,

$$\dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{B}\hat{u}(t), \tag{2a}$$

$$\hat{y}(t) = \hat{C}\hat{x}(t) + \hat{D}\hat{u}(t), \tag{2b}$$

of order r, $r << n$, such that the transfer function matrix $\hat{G}(s) = \hat{C}(sI - \hat{A})^{-1}\hat{B} + \hat{D}$, approximates original system $G(s)$.

The Hankel norm represents the energy transferred from past inputs to the future outputs through the system $G = \{A, B, C, D\} \in H_\infty$. If the input $u(t) = 0$ for $t \geqslant 0$, and the output is $y(t)$, then Hankel norm [8] is defined as:

$$\|G(s)\|_H = \sup_{u\in L^2(-\infty,0)} \frac{\|y\|_{L^2(0,\infty)}}{\|u\|_{L^2(-\infty,0)}}. \tag{3}$$

The associated Hankel operator [11] can be represented as,

$$H_G : L_2(-\infty, 0] \to L_2(0, \infty] : (H_G u)(t) = \int_{-\infty}^{0} G(t - \tau)u(\tau)\,d\tau. \tag{4}$$

Bettayeb et al. [5] showed that H_G has singular value decomposition which can be determined directly from state space realization of $G(s)$ with any order $n \geqslant m$, where m is the McMillan degree of $G(s)$. Further, m non-zero Hankel singular values (HSV) of the system can be obtained by finding the square root of the eigenvalues of product of P and Q.

$$HSV = \sqrt{\lambda_i(PQ)} \text{ such as } \sigma_1 \geqslant \sigma_2 \geqslant \ldots \geqslant \sigma_m > 0, \tag{5}$$

where $\lambda_i(M)$ denotes the ith eigenvalue of M and P and Q are controllability and observability Gramians which can be defined as,

$$P = \int_0^\infty e^{\tau A}BB^* e^{\tau A^*}\,d\tau, \tag{6}$$

and

$$Q = \int_0^\infty e^{\tau A^*} C^* C e^{\tau A} d\tau, \tag{7}$$

By considering the corresponding matrix differential equations it is easily verified that P and Q satisfy the following Lyapunov equations:

$$AP + PA^* + BB^* = 0, \tag{8a}$$

$$A^*Q + QA + C^*C = 0, \tag{8b}$$

where '$*$' indicates the conjugate transpose of a matrix.

For a given positive number $\alpha > 0$, the minimal degree optimal hankel norm approximation (MDOHNA) problem is to compute a state space model of minimal degree $\hat{G}(s)$ [11] such that largest Hankel singular value of the modeling error system $H_{G-\hat{G}}$, is at most α. Any rational $G(s)$, the approximant by minimal degree optimal hankel norm approximation $\hat{G}(s)$ is called minimal degree approximant $G(s)$, Minimal degree [1] is precisely equal to the number of Hankel singular values greater than α, i.e. the minimum degree is given by,

$$k := \underbrace{\max}_{i}\{i | \sigma_i > \alpha\} \tag{9}$$

Further, for the optimal case $\alpha = \sigma_{k+1}$, [8, Theorem 9.7] every k^{th} order minimal degree approximant of $G(s)$ can be defined for a constant matrix D_0 as

$$\left\| G - \hat{G} - D_0 \right\|_\infty := \underbrace{\sup}_{\omega} \bar{\sigma} = \left(G(j\omega) - \hat{G}(j\omega) - D_0 \right) \leqslant \sum_{i=k+1}^{n} \sigma_i \tag{10}$$

For computation of minimal degree optimal Hankel norm approximant of k^{th} order, a basis free descriptor system representation is used. The steps for computing MDOHNA reduced model for $G(s)$ consist of following steps:

Step 1. For $\sigma_k > \alpha \geqslant \sigma_{k+1}$, form the descriptor Γ, such that:

$$\Gamma = QP - \alpha^2 I, \tag{11}$$

and

$$\begin{bmatrix} \Gamma s - \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix} = \begin{bmatrix} \alpha^2 A^T + QAP & QB \\ CP & D \end{bmatrix} \tag{12}$$

Step 2. Take singular value decomposition of descriptor Γ and partition the result by applying the appropriate permutation such that Gramians [8] become:

$$P = Q = \begin{bmatrix} \Sigma & \\ & \sigma_{r+1} I_l \end{bmatrix}, \tag{13}$$

where σ_{r+l} has multiplicity l with,

$$\sigma_1 \geqslant \sigma_2 \geqslant .... \geqslant \sigma_k \geqslant \sigma_{k+l} \geqslant \sigma_{k+l+1} \geqslant ...\sigma_m > 0, \; l \geqslant 1 \tag{14}$$

and

$$\Gamma = [U_{E1} \quad U_{E2}] \begin{bmatrix} \Sigma_E & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_{E1}^T \\ V_{E2}^T \end{bmatrix}. \tag{15}$$

Step 3. Apply the transformation to the descriptor state-space system above we have

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} U_{E1}^T \\ U_{E2}^T \end{bmatrix} \left(\alpha^2 A^T + QAP \right) \begin{bmatrix} V_{E1}^T & V_{E2}^T \end{bmatrix}, \tag{16a}$$

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} U_{E1}^T \\ U_{E2}^T \end{bmatrix} \begin{bmatrix} QB & -C^T \end{bmatrix}, \tag{16b}$$

$$[C_1 \quad C_2] = \begin{bmatrix} CP \\ -\alpha B^T \end{bmatrix} [V_{E1}^T \quad V_{E2}^T], \tag{16c}$$

$$D_1 = D. \tag{16d}$$

Step 4. From the parameters of above transformation of Eq. (16), the equivalent state-space model can be defined as:

$$\begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} = \begin{bmatrix} \Sigma_E^{-1}(A_{11} - A_{12}A_{22}^{\perp}A_{21}) & \Sigma_E^{-1}(B_1 - A_{12}A_{22}^{\perp}B_2) \\ C_1 - C_2A_{22}^{\perp}A_{21} & D_1 - C_2A_{22}^{\perp}B_2 \end{bmatrix} \tag{17}$$

where $M^{\perp}$ denotes pseudoinverse of M. Now, additive decomposition of above state-space realization $\tilde{G} = \{\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}\}$ is obtained, such that $\tilde{G} = (\tilde{G}_- + \tilde{G}_+)$, where $\tilde{G}_-$ is stable and $\tilde{G}_+$ is antistable. Then $G_{r\_hna} = \tilde{G}_-$ is minimal degree optimal Hankel norm approximation with rth order of original system G, which can be defined as:

$$G_{r\_hna} = D_{hna} + C_{hna}(sI - A_{hna})^{-1}B_{hna}. \tag{18}$$

The reduced model by MDOHNA technique is represented by $\{A_{hna}, B_{hna}, C_{hna}, D_{hna}\}$ as,

$$G_{r\_hna} = \left[\begin{array}{c|c} A_{hna} & B_{hna} \\ \hline C_{hna} & D_{hna} \end{array}\right]. \tag{19}$$

3. Minimal degree optimal Hankel norm approximation for reduction of unstable systems

In this section extension of minimal degree optimal Hankel norm approximation technique for order reduction of unstable systems is proposed using system decomposition algorithm. System decomposition algorithm is originally developed by Nagar & Singh [21] and they computed reduced model for a higher order unstable system using balanced truncation. Later Kumar et al. [30,31] explored this algorithm using balanced singular perturbation approximation technique. MDOHNA based order reduction algorithm can be used for reduction of unstable systems after decomposition of original unstable system as below:

3.1. Unstable system decomposition

The decomposition algorithm developed in this section is inspired from [21,30,31] and contains two stages of transformations. In first stage, the block form of real Schur transformation is used whereas, in second stage of transformation, the generalized Lyapunov equation has been solved for obtaining decomposed stable and unstable subsystems. The decomposition algorithm consists of following steps:

Step 1. Consider the system represented by Eq. (2) be an unstable system. Transform this system using an unitary matrix U in block diagonal upper Schur form [26], such that the eigenvalues of the transformed system are arranged in increasing order of its real components in case of continuous-time systems (in increasing order of its absolute value in case of discrete-time systems).

If x denotes the original system states, then the first stage transformation matrix U (the unitary matrix) and the transformed system states x_t may be related as $x = Ux_t$. Thus, the first stage transformed system is,

$$G_t = \left[\begin{array}{c|c} U'AU & U'B \\ \hline CU & D \end{array}\right] = \left[\begin{array}{c|c} A_t & B_t \\ \hline C_t & D \end{array}\right] = \begin{array}{c} \begin{array}{cc} \overleftrightarrow{\;m\;} & \overleftrightarrow{\;n-m\;} \end{array} \\ \left[\begin{array}{cc|c} A_{t11} & A_{t12} & B_{t1} \\ 0 & A_{t22} & B_{t2} \\ \hline C_{t1} & C_{t2} & D \end{array}\right] \end{array} \begin{array}{l} \updownarrow m \\ \updownarrow n-m \\ \\ \end{array} \tag{20}$$

where n denotes order of the system, m denotes the number of stable eigenvalues and $n - m$ denotes the number of unstable eigenvalues.

Step 2: The transformed system of step (1), contains a coupling term A_{t12}. To bring transformed system into completely decoupled form, solve the general form of Lyapunov equation:

$$A_{t11}S - SA_{t22} + A_{t12} = 0 \tag{21}$$

Obtain the value of S and proceed for second stage of transformation using $x_t = WX$, where X is the final stage transformed state and W is the final stage transformation matrix. The second stage transformation matrix W is given as,

$$W = \begin{bmatrix} I_m & . & S \\ \dots & . & \dots \\ 0 & . & I_{n-m} \end{bmatrix}, \tag{22}$$

where I_m and I_{n-m} are identity matrix of size m and $n - m$,, respectively. The important property of W is that W^{-1} can be obtained simply by replacing S with $-S$. i.e.,

$$W^{-1} = \begin{bmatrix} I_m & . & -S \\ \dots & . & \dots \\ 0 & . & I_{n-m} \end{bmatrix}. \tag{23}$$

Thus, W^{-1} always exists and hence never found ill-conditioned. Using W, completely decoupled system (G_d) can be obtained as,

$$G_d = \left[\begin{array}{c|c} W^{-1}A_tW & W^{-1}B_t \\ \hline C_tU & D \end{array}\right] = \begin{array}{c} \overset{m}{\longleftrightarrow}\quad \overset{n-m}{\longleftrightarrow} \\ \left[\begin{array}{cc|c} A_{11} & 0 & B_1 \\ 0 & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right] \end{array} \begin{array}{l} \updownarrow m \\ \updownarrow n-m \\ \\ \end{array} \tag{24}$$

where $A_t = U'AU, B_t = U'B$ and $C_t = CU$ are obtained from step (1). This transformed model may be decomposed into stable and unstable as,

$$G_d = \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array}\right] + \left[\begin{array}{c|c} A_{22} & B_2 \\ \hline C_2 & 0 \end{array}\right] = G_s(\text{stable subsystem}) + G_u(\text{Unstable subsystem}) \tag{25}$$

3.2. Reduced order model by Hankel norm approximation for decomposed stable system

The rth order minimal degree optimal Hankel norm approximated (MDOHNA) reduced order model for stable subsystem G_s can be obtained from the Section 2, using Eqs. (11)–(18) and it can be represented as,

$$G_{sr\_hna} = \left[\begin{array}{c|c} A_{shna} & B_{shna} \\ \hline C_{shna} & D_{shna} \end{array}\right]. \tag{26}$$

3.3. Overall reduced order model for original unstable system

The overall reduced model of original unstable system by proposed algorithm is obtained by adding the reduced model of decomposed stable part obtained by Eq. (26) and decomposed unstable subsystem from Eq. (25) as,

$$G_{r\_hna} = G_{sr\_hna} + G_u. \tag{27}$$

The reduced model by minimal degree optimal Hankel norm approximation (MDOHNA) technique is given by $G_{r\_mdohna} = \{A_{hna}, B_{hna}, C_{hna}, D_{hna}\}$ as,

$$G_{r\_mdohna} = \left[\begin{array}{c|c} A_{hna} & B_{hna} \\ \hline C_{hna} & D_{hna} \end{array}\right]. \tag{28}$$

4. Numerical example

In this section an 8th order linear model of the longitudinal dynamics of a forward swept wing aircraft [19,27] is considered. The wings are swept forward at angle 30° and model flight velocity is 1000 ft/s at sea level. One pair of complex poles are lightly damped phugoid mode, two real poles represent the divergent short period mode and two other sets of complex poles correspond to the two second order structural modes: wing bending mode with natural frequency 60 rad/s and the wing torsional mode with natural frequency 212 rad/s. The equations of motion [28] are developed in state variable form

Table 1
State variables of forward swept wing aircraft.

State variables	$v(t)$	Velocity	ft/s
	$\alpha(t)$	Angle of attack	rad
	$\theta(t)$	Pitch attitude	rad
	$q(t)$	Pitch rate	rad/s
	$N_1(t)$	Wing tip deflection	ft
	$\dot{N}_1(t)$	Wing tip rate	ft/s
	$N_2(t)$	Wing rotation	rad
	$\dot{N}_2(t)$	Wing rotation rate	rad/s
Input control vector	$f(t)$	Flaperon deflection	rad
	$c(t)$	Canard deflection	rad
Output vector	$\alpha(t)$	Angle of attack	rad
	$\theta(t)$	Pitch attitude	rad

with body-fixed coordinate system. The state space equations of the model are of the form given in Eq. (1). The state vector $x(t)$, input control vector $u(t)$ and output vector $y(t)$ are tabulated in Table 1.

The $\bar{A}, \bar{B}, \bar{C}$ matrices of the model are

$$A = \begin{bmatrix} 5.26\times10^{-4} & 0.0928 & -0.562 & -0.254 & -1.41\times10^{-3} & 0.00151 & 0.0479 & 3.46\times10^{-7} \\ -3.69\times10^{-3} & -2.88 & -4.67\times10^{-4} & 1.01 & 4.37 & -0.0469 & -1.49 & -1.08\times10^{-5} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1.16\times10^{-4} & 79.6 & 1.48\times10^{-5} & -0.831 & -60.5 & 1.01 & 25 & 1.46\times10^{-3} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -0.944 & -544 & -1.18\times10^{-6} & 1.16 & -3.62\times10^{4} & -20.6 & -490 & 6.73\times10^{-4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0.192 & 7.51\times10^{1} & -1.56\times10^{-5} & -0.643 & -4.37 & -0.0466 & -4.52\times10^{4} & -0.0360 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.102 & -0.463 & 0 & -19.4 & 0 & -108 & 0 & 1.34 \\ 0.0164 & -0.511 & 0 & 61.3 & 0 & 4.92 & 0 & 65 \end{bmatrix}^T, \quad C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The eigenvalues of the above system are $-10.255 \pm j189.9, -10.99, -2.84\times10^{-5} \pm j0.046, -0.018 \pm j212.6$ (stable) and 7.1881 (unstable). Following the proposed algorithm, we obtain the stable subsystem $G_s = \{A_s, B_s, C_s, D_s\}$ and unstable subsystem $G_u = \{A_u, B_u, C_u, D_u\}$ where

$$A_s = \begin{bmatrix} -10.99 & 4.944 & -678.5 & -171.5 & 73.22 & 14.84 & 66.46 \\ 6.998\times10^{-12} & -212.2 & 3.486\times10^{4} & 8793 & 587.88 & -747.8 & -3362 \\ -8.395\times10^{-13} & -2.205 & 191.7 & 98.42 & -1.105\times10^{4} & -4.843 & -20.99 \\ -2.579\times10^{-12} & 1.633\times10^{-11} & -2.218\times10^{-9} & -198.4 & 4.383\times10^{4} & 2.893 & 6.373 \\ 1.052\times10^{-12} & -3.198\times10^{-13} & -6.005\times10^{-12} & -1.929 & 198.3 & 0.01829 & 0.05224 \\ -4.413\times10^{-12} & -3.356\times10^{-13} & 1.921\times10^{-11} & 4.024\times10^{-12} & -1.05\times10^{-11} & 0.01341 & 0.5663 \\ 2.698\times10^{-13} & -2.597\times10^{-13} & 3.525\times10^{-11} & 8.539\times10^{-12} & 1.92\times10^{-11} & -0.004113 & -0.01347 \end{bmatrix}$$

$$B_s = \begin{bmatrix} 4.486 & 108.2 & 1.033 & -1.277 & -0.006218 & 0.3742 & 1.185 \\ -29.94 & -5.954 & 15.66 & -62.97 & -0.2829 & -0.9131 & -1.731 \end{bmatrix}^T,$$

$$C_s = \begin{bmatrix} 0.1242 & 0.002108 & 0.07722 & 0.01921 & -2.143\times10^{-5} & -0.004816 & -0.01862 \\ 0.0899 & 0.001253 & 0.09037 & 0.02263 & -4.307\times10^{-5} & 0.0206 & 0.9914 \end{bmatrix}$$

$$D_s = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, A_u = [7.188], B_u = [-2.866 \quad 7.131], C_u = [0.4466 \quad 0.623]^T \quad \text{and} \quad D_u = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

where the eigenvalues of stable subsystem (G_s) are $-10.255 \pm j189.9$, $-2.84\times10^{-5} \pm j0.046$, -10.99, $-0.018 \pm j212.6$ and the eigenvalues of unstable system (G_u) are 7.1881. Thus, the eigenvalues of the original system are retained in decomposed subsystems. The reduction of unstable system is carried out as described in Section 3. The HSVs of decomposed stable subsystem are 1.88×10^4, $0.2125, 7.236\times10^{-4}$, 0.022209, 0.022208, 1.883×10^4 and 7.975×10^{-4} and the Bar chart of HSVs is plotted in Fig. 1. From the HSVs and bar-chart it is clear that only the first two singular values are significant. Thus, the 2nd order model is suitable for reduction of 7th order decomposed stable subsystem.

Reducing the stable part to 2nd order reduced model by minimal degree optimal Hankel norm approximation technique as described in Section 2 and adding it to decomposed unstable part, the overall reduced model of the original unstable system by proposed algorithm is computed using Eqs. (25) and (27) as

$$G_{r\_mdohna} = \left[\begin{array}{ccc|cc} -2.84\times10^{-5} & 0.003822 & 0 & 6.071\times10^{-8} & -1.506\times10^{-7} \\ -0.5621 & -2.84\times10^{-5} & 0 & 2.988\times10^{-5} & -4.386\times10^{-5} \\ 0 & 0 & 7.188 & -2.866 & 7.131 \\ \hline -101.5 & -0.5816 & 0.4466 & 0 & 0 \\ 4702 & 4.028\times10^{4} & 0.623 & 0 & 0 \end{array}\right]$$

For a comparison, if we reduce stable part by balanced truncation technique [25] and adding it to unstable part, we get the reduced model of original unstable system as

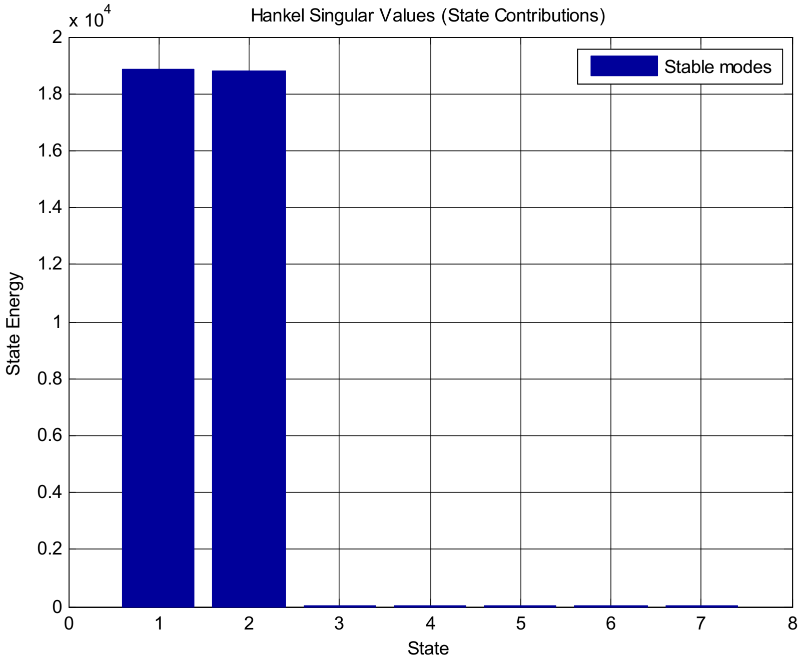

Fig. 1. Bar-chart of Hankel singular values of stable subsystem.

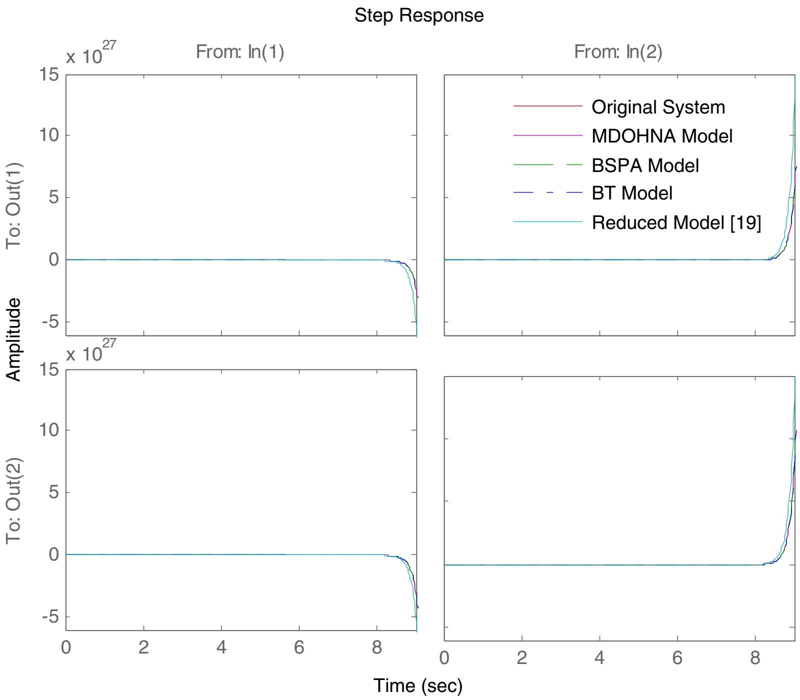

Fig. 2. Step responses of original system and 3rd order reduced model.

$$G_{r\_bt} = \left[\begin{array}{ccc|cc} -1.144\times10^{-8} & 0.04635 & 0 & 0.0001675 & -0.02076 \\ -0.04635 & -5.679\times10^{-5} & 0 & 0.8231 & -1.209 \\ 0 & 0 & 7.188 & -2.866 & 7.131 \\ \hline -0.0003031 & -1.614\times10^{-5} & 0.4466 & 0 & 0 \\ -0.02076 & 1.462 & 0.623 & 0 & 0 \end{array}\right]$$

and if we reduce stable part by balanced singular perturbation approximation (BSPA) technique [29], the final reduced model is

$$G_{r\_bspa} = \left[\begin{array}{ccc|cc} -1.145\times10^{-8} & 0.04635 & 0 & 0.0001675 & -0.02076 \\ -0.04635 & -5.679\times10^{-5} & 0 & 0.8231 & -1.209 \\ 0 & 0 & 7.188 & -2.866 & 7.131 \\ \hline -0.0003035 & -4.093\times10^{-5} & 0.4466 & 0.07296 & -0.3366 \\ -0.02076 & 1.462 & 0.623 & 0.05269 & -0.2435 \end{array}\right]$$

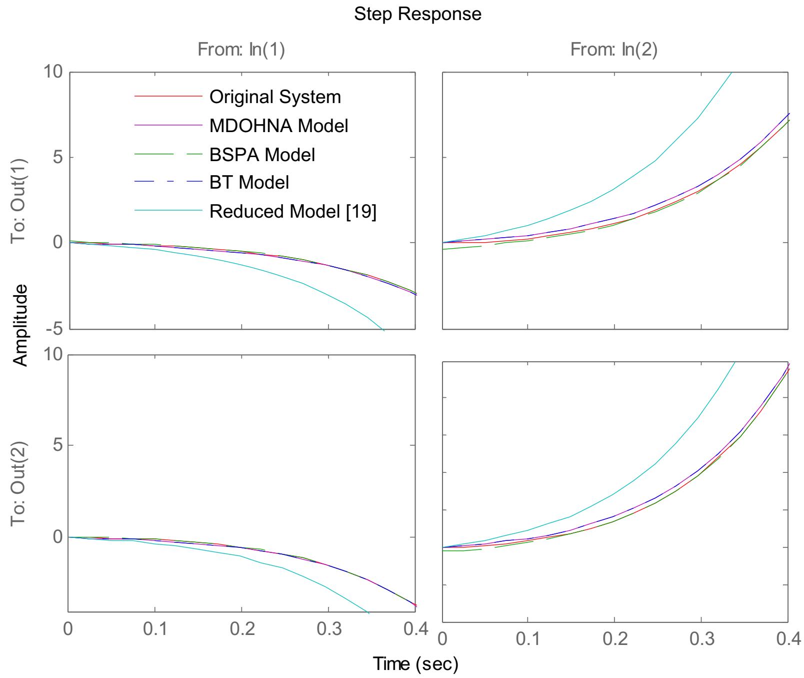

Fig. 3. Step responses of original system and 3rd order reduced model (plot shown in lower range of time scale).

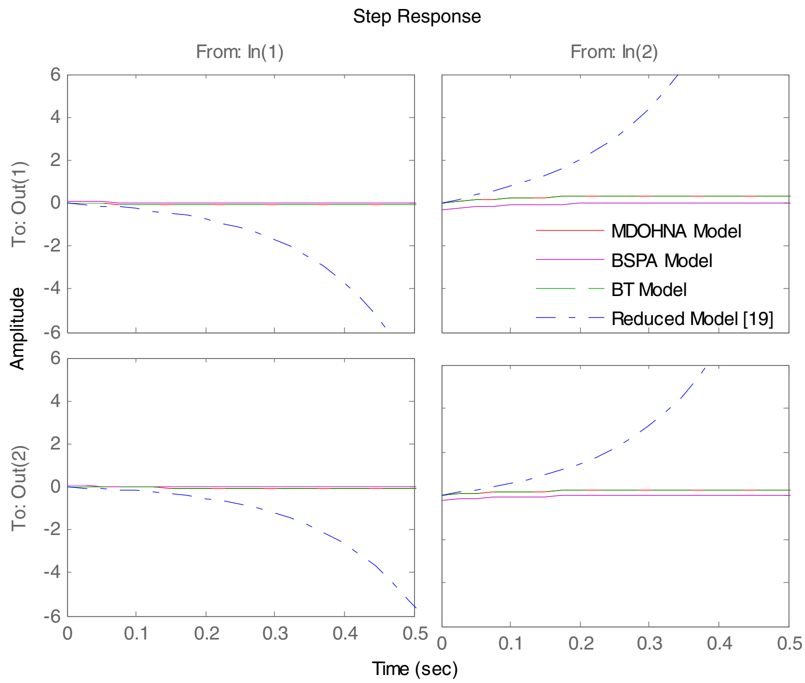

Fig. 4. Step response of modeling error transfer function models.

and if we reduce decomposed stable part by Hsu and Hou's method [19], then the final reduced order model is

$$G_{r\_hsu} = \left[\begin{array}{ccc|cc} -8.52\times10^{-10} & 0.04635 & 0.007861 & -0.005122 & 0.00713 \\ -0.04635 & -5.68\times10^{-5} & 2.393 & -0.5587 & 1.794 \\ 0 & 0 & 7.188 & -2.866 & 7.131 \\ \hline -0.0006328 & -0.0001215 & 0.9889 & 0 & 0 \\ -0.01177 & 3.043 & -2.52\times10^{-5} & 0 & 0 \end{array}\right]$$

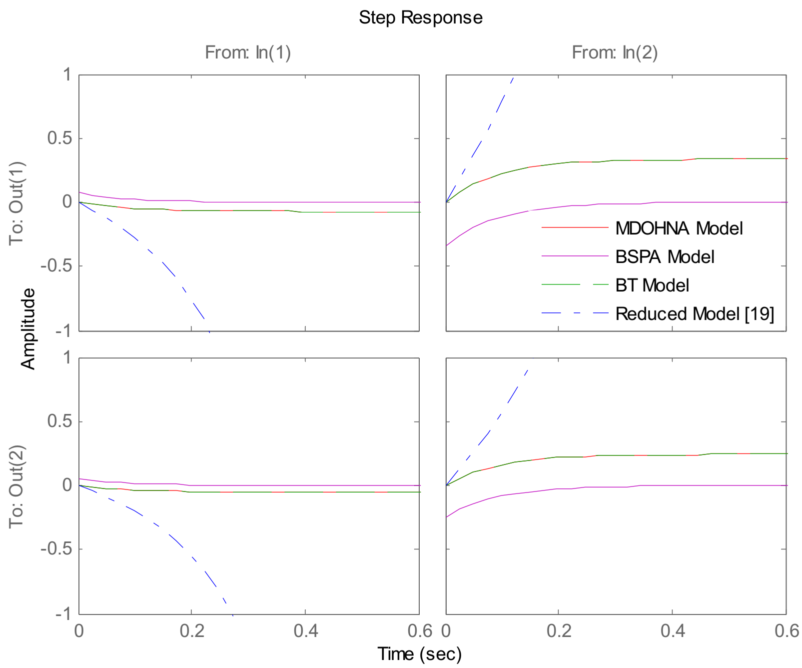

Fig. 5. Step response of modeling error transfer function models (plot shown in lower range of time scale).

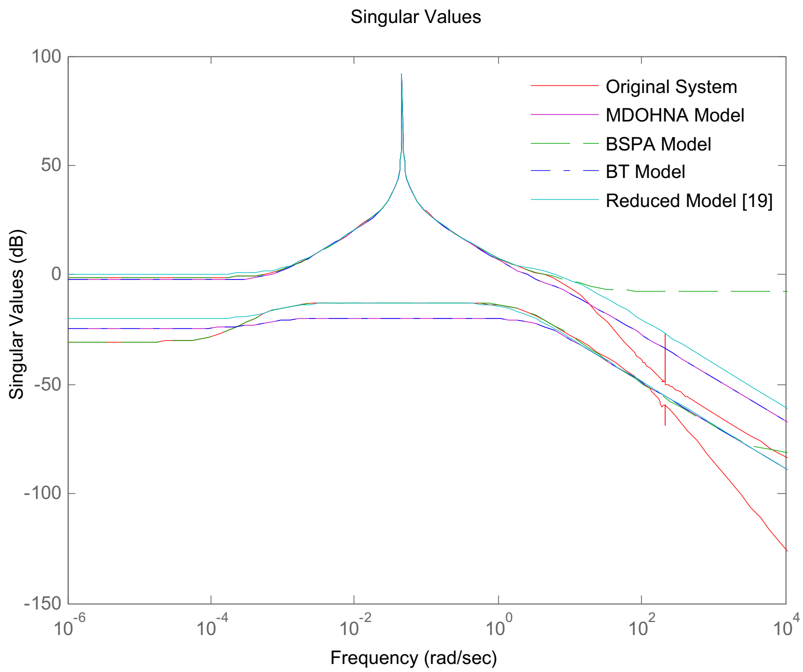

Fig. 6. Singular value plots of original system and 3rd order reduced model.

The step responses of models and the modeling errors transfer functions are plotted in Figs. 2–5, respectively. The singular value plots of models and the modeling errors transfer functions are plotted in Fig. 6–8, respectively. Frequency domain H_∞ norm bounds of models and modeling error transfer functions are tabulated in Table 2.

The computed values of the error bounds in Table 2 may be directly read and verified from the plots of Fig. 8. Reduced model by Hsu's method [19] deviates much from the plot of original system whereas reduced model by proposed algorithm preserves characteristics of original system which is observed from the plots of Fig 2–8 and Table 2. MDOHNA model possesses lesser frequency domain error in high-frequency region as in case of BT model [25], on the other hand the case is reverse in low-frequency region where BSPA model possess low-frequency error. From the Table 2 and Fig. 8, it is observed

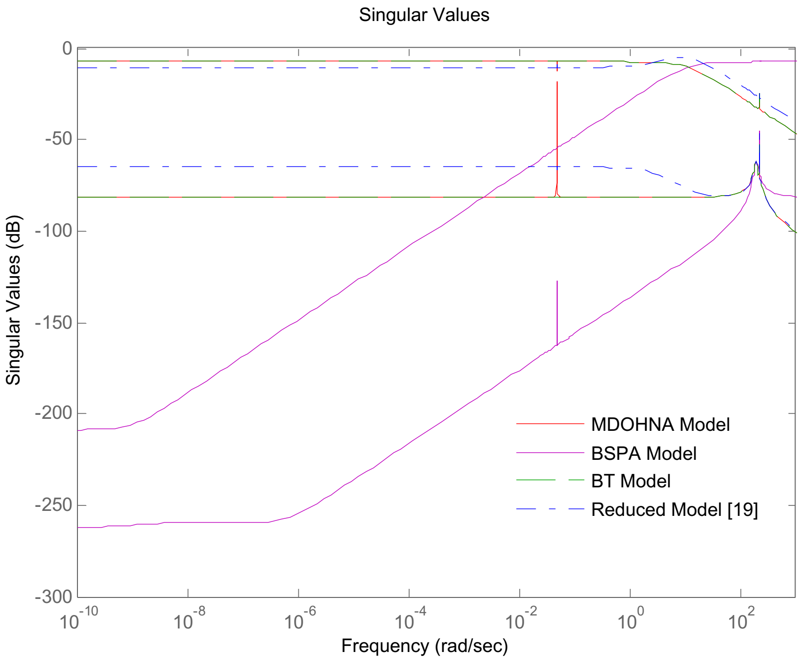

Fig. 7. Singular value plots modeling error transfer function models of original system and 3rd order reduced model.

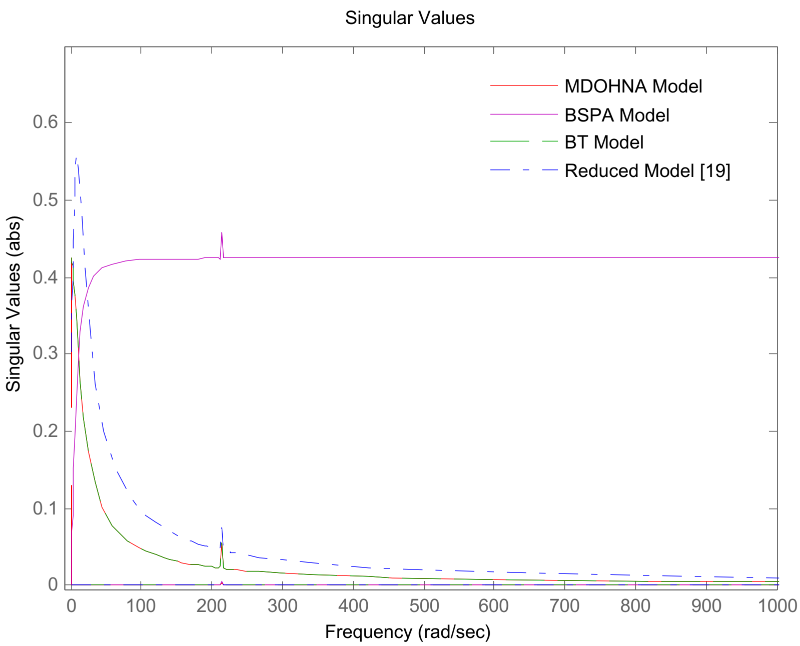

Fig. 8. Singular value plots (absolute) of original system and 3rd order reduced model.

Table 2
Frequency domain computations for models.

S.N.	Frequency domain computations $\|G(z)\|_\infty = 3.766125 \times 10^4$	Reduced model type	Reduced model order, $r = 3$
1.	H_∞ norm (peak value of bode or singular value plot of frequency response) $= \|G_r(z)\|_\infty$	(a) MDOHNA	3.76610119×10^4
		(b) BT [25]	3.76610119×10^4
		(c) BSPA [29]	3.7661251×10^4
		(d) Hsu [19]	3.7661457×10^4
2.	Actual norm error bound in modeling $= \|G_r(z) - G(z)\|_\infty$	(a) MDOHNA	0.42505
		(b) BT [25]	0.42505
		(c) BSPA [29]	0.45739
		(d) Hsu [19]	0.55927
3.	Actual relative error bound $= \frac{Actual \quad Error \quad Bound}{\|G(z)\|_\infty}$	(a) MDOHNA	1.12861×10^{-5}
		(b) BT [25]	1.12861×10^{-5}
		(c) BSPA [29]	1.21449×10^{-5}
		(d) Hsu [19]	1.48501×10^{-5}

that reduced model by MDOHNA possesses least actual H_∞ norm error bound in modeling and thus least actual relative error bound in comparison of other existing methods.

5. Conclusion

In this paper minimal degree optimal Hankel norm approximation (MDOHNA) technique is discussed for order reduction of minimal/non-minimal systems using a basis free descriptor which eliminates the requirement of computing balancing transform of original system for reduction. Further, extension of MDOHNA technique is proposed for reduction of unstable systems using system decomposition algorithm. The reduced order model is obtained by reducing the stable subsystem using MDOHNA algorithm and adding it to decomposed unstable subsystem. With the help of numerical examples, it has been observed that (1) reduced order model obtained by minimal degree optimal Hankel norm approximation technique preserves characteristics of original system almost in time and frequency domains both. (2) The proposed algorithm is free from two fold error generally occurring in bilinear transformation-based methods in discrete to continuous and then back conversions. (3) The reduced model obtained by proposed algorithm generate better results than Hsu [19], Deepak et al. [22]. (4) Proposed algorithm produce least actual H_∞ norm error bound in modeling.

Appendix A. Supplementary data

Supplementary data associated with this article can be found, in the online version, at http://dx.doi.org/10.1016/j.apm.2013.11.012.

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