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Chaos control of fractional order Rabinovich–Fabrikant system and synchronization between chaotic and chaos controlled fractional order Rabinovich–Fabrikant system

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ABSTRACT

In this article the local stability of the Rabinovich–Fabrikant (R–F) chaotic system with fractional order time derivative is analyzed using fractional Routh–Hurwitz stability criterion. Feedback control method is used to control chaos in the considered fractional order system and after controlling the chaos the authors have introduced the synchronization between fractional order non-chaotic R–F system and the chaotic R–F system at various equilibrium points. The fractional derivative is described in the Caputo sense. Numerical simulation results which are carried out using Adams–Boshforth–Moulton method show that the method is effective and reliable for synchronizing the systems.

1. Introduction

A wide class of physical phenomena can be described by mathematical models. Simple nonlinear dynamical system and even piecewise linear system can exhibit complete unpredictable behavior known as chaos, which is an active area of research to the scientific community working in nonlinear sciences, through this has no unified and universal definitions in the scientific literature. It is described by the set of mathematical equations containing both dynamic and static variables. Chaotic system is bounded nonlinear deterministic system which has a periodic long-term nature depending on initial condition. Again due to sensitive dependence of chaotic dynamics on initial conditions [1], there is always possibility of exponential spreading of trajectories of the systems emerging from initial conditions during coupling of the systems. Dynamic of chaos has a very interesting nonlinear effect, which had been intensively studied by Lorenz [2], who found the first canonical chaotic attractor in the year 1963.

In the last few decades, fractional order modeling has been an active field of research both from the theoretical and the applied perspectives since they are naturally related to the systems with memory which prevails for most of the physical and scientific system. Fractional calculus is a field of applied mathematics that deals with derivatives and integrals of arbitrary orders and have applications in various fields of science and engineering ([3–7]). Fractional differential equations which are generalizations of classical differential equations describe the memory effect, and it is the major advantage over integer-order derivatives [8]. The chaotic dynamics of fractional order systems is an important topic of study in nonlinear dynamics. In the last few years this area of research has been growing rapidly [9].

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In the present article the authors have used a simple algorithm that allows the control laws that stabilizes the chaotic systems around the unstable equilibrium points and synchronized the chaotic system. Since the chaotic systems are very sensitive to initial conditions, so for this reason, chaotic systems are difficult to control and synchronized. Recently, controlling the chaos and synchronization of chaotic systems of complex dynamical systems has attracted researchers in the field of engineering and science. Research efforts have investigated the chaos control in many physical chaotic systems ([10,11]). Synchronization is a phenomenon occurs when two or more nonlinear systems are coupled, which is nowadays very active area of research in nonlinear dynamics. Actually synchronization is caused due to transformation of dynamical variables of two chaotic systems viz., drive (master) and response (slave) systems. The study of synchronization is extremely needed so that the trajectories of two systems will converge and they will remain in step with each other to do the structurally stable coupling.

The pioneering works of Ott et al. [12] and Pecora and Carroll [13] introduced a method about chaos control of chaotic system and synchronization between two identical or non identical systems has attracted a great deal of interest in various fields and engineering ([14–17]). In recent years, many researchers and engineers have devoted their efforts to chaos control and synchronization, including stabilization of unstable equilibrium points [18]. There are various schemes to achieve chaos control and synchronization, such as linear and nonlinear feedback control, adaptive control, active control, sliding mode control etc. ([19–22]). Among two main approaches for controlling chaos feedback control and non-feedback control, the first one is especially attractive and has been commonly applied to practical implementation due to its simplicity in configuration. In 1997, Bai and Lonngren [23] have proposed Active control method, which has received a lot of attention to the researchers working in the area of nonlinear dynamics due to its simplicity and easy to implement in applications of synchronization and anti- synchronization of coupling of a pair of chaotic for both standard order and fractional order cases ([24–29]).

The Rabinovich–Fabrikant (R–F) equations introduced by Mikhail Rabinovich and Anatoly Fabrikant [30] are a set of three coupled ordinary differential equations. For certain values of parameters the system is chaotic and for others it tends to a stable periodic orbit. Danca and Chen [31] predicted that due to the presence of quadratic and cubic terms, it is hard to analyze the system. The stochastic nature of the system arising from the modulation instability is found in non-equilibrium dissipative medium [32].

In the present article the authors have studied the dynamical behavior, chaos control and synchronization of fractional order R–F system. It is found that the chaotic attractor exists in the fractional-order R–F system. Furthermore, fractional Routh–Hurwitz conditions [33] are used to study the stability conditions in the fractional-order R–F system and the conditions for linear feedback control have been obtained for controlling chaos in the considered system. The authors have used the active control method for synchronization between fractional order chaotic and chaos controlled R–F system. Using the Adams–Boshforth–Moulton method ([34,35]), numerical simulation is carried out for different particular cases.

2. Some preliminaries and stability condition

2.1. Fractional calculus

Definition 1. A real function $f(t)$, $t > 0$, is said to be in the space C_μ, $\mu \in \Re$, if there exists a real number $p > \mu$, such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C(0,\infty)$, and it is said to be in the space C_μ^n if and only if $f^{(n)} \in C_\mu$, $n \in N$.

Definition 2. The Riemann–Liouville fractional integral operator (J_t^α) of order $\alpha > 0$, of a function $f \in C_\mu$, $\mu \geqslant -1$, is defined as

$$J_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1} f(\xi)\, d\xi, \quad \alpha > 0, \quad t > 0, \tag{1}$$

where $\Gamma(.)$ is the well-known gamma function.

Definition 3. The fractional derivative D_t^α of $f(t)$, in the Caputo sense is defined as

$$D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\xi)^{n-\alpha-1} f^{(n)}(\xi)\, d\xi, \tag{2}$$

for $n-1 < \alpha < n$, $n \in N$, $t > 0$, $f \in C_{-1}^n$.

The important reason of choosing Caputo derivatives for solving initial value fractional order differential equations is that the Caputo derivative of a constant is zero, whereas the Riemann–Liouville fractional derivative of constant is not equal to zero.

2.2. Stability of the system

Consider a three-dimensional fractional order system

$$\begin{aligned} D_t^\alpha x(t) &= f_1(x,y,z),\\ D_t^\alpha y(t) &= f_2(x,y,z),\\ D_t^\alpha z(t) &= f_3(x,y,z), \end{aligned} \tag{3}$$

where $0 < \alpha < 1$ and D_t^α is Caputo derivative. The Jacobian matrix of the system (3) at the equilibrium points is

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z}\\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z}\\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{pmatrix}. \tag{4}$$

Theorem (Matignon [36]). *The system (3) is locally asymptotically stable if all the eigenvalues of the jacobian matrix evaluated at its equilibrium point satisfy*

$$|\arg(\lambda)| > \alpha\pi/2. \tag{5}$$

The characteristic equation of the jacobian matrix (4) evaluated at their equilibrium points is given by the polynomial:

$$P(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3, \tag{6}$$

and its discriminant is given by

$$D(P) = 18a_1a_2a_3 + (a_1a_2)^2 - 4a_3a_1^3 - 4a_2^3 - 27a_3^2. \tag{7}$$

The fractional order Routh–Hurwitz conditions for the stability of the system is given as

(a) If $D(P) > 0$, then the necessary and sufficient conditions for the equilibrium point to be locally asymptotically stable, is $a_1 > 0, a_3 > 0, a_1a_2 - a_3 > 0$.
(b) If $D(P) < 0$, $a_1 \geqslant 0, a_2 \geqslant 0, a_3 > 0$, then the equilibrium point is locally asymptotically stable for $\alpha < 2/3$. However, if $D(P) < 0$, $a_1 < 0, a_2 < 0, \alpha > 2/3$, then all roots of Eq. (6) satisfy the condition $|\arg(\lambda)| < \alpha\pi/2$.
(c) If $D(P) < 0$, $a_1 > 0$, $a_2 > 0$, $a_1a_2 - a_3 = 0$, then the equilibrium point is locally asymptotically stable for all $0 < \alpha < 1$.
(d) The necessary condition for the equilibrium point, to be locally asymptotically stable is $a_3 > 0$.

3. Description of the system and its stability

3.1. The Rabinovich–Fabrikant chaotic system

The fractional order Rabinovich–Fabrikant system is given by

$$\begin{aligned} \frac{d^\alpha x}{dt^\alpha} &= y(z - 1 + x^2) + ax,\\ \frac{d^\alpha y}{dt^\alpha} &= x(3z + 1 - x^2) + ay,\\ \frac{d^\alpha z}{dt^\alpha} &= -2z(b + xy), \quad 0 < \alpha < 1. \end{aligned} \tag{8}$$

For the parameters $a = 0.87$, $b = 1.1$, the system (8) exhibits chaotic attractor for the initial conditions [−1, 0, 0.5] at $\alpha = 0.99$, which is depicted through the Figs. 1 and 2.

3.2. Stability of equilibrium points

To obtain the equilibrium points, let us consider the equations as

$$\begin{cases} y(z - 1 + x^2) + ax = 0,\\ x(3z + 1 - x^2) + ay = 0,\\ 2z(b + xy) = 0, \end{cases} \tag{9}$$

After solving Eq. (9) for $a = 0.87, b = 1.1$, we get five equilibrium points as $E_1(0,0,0)$, $E_2(-1.4797, 0.7434, 0.5422)$, $E_3(1.4797, -0.7434, 0.5422)$, $E_4(0.5182667, -2.12246, 0.94384)$ and $E_5(-0.5182667, 2.12246, 0.94384)$.

The Jacobian matrix for the system (8) at an equilibrium point $\bar{E}(\bar{x},\bar{y},\bar{z})$ is given as

$$J(\bar{E}) = \begin{pmatrix} a + 2\bar{x}\bar{y} & \bar{z} - 1 + \bar{x}^2 & \bar{y}\\ 3\bar{z} + 1 - 3\bar{x}^2 & a & 3\bar{x}\\ -2\bar{y}\bar{z} & -2\bar{z}\bar{x} & -2b - 2\bar{x}\bar{y} \end{pmatrix}. \tag{10}$$

The characteristic polynomial of the Jacobian matrix (10) for $a = 0.87$, $b = 1.1$ is given by

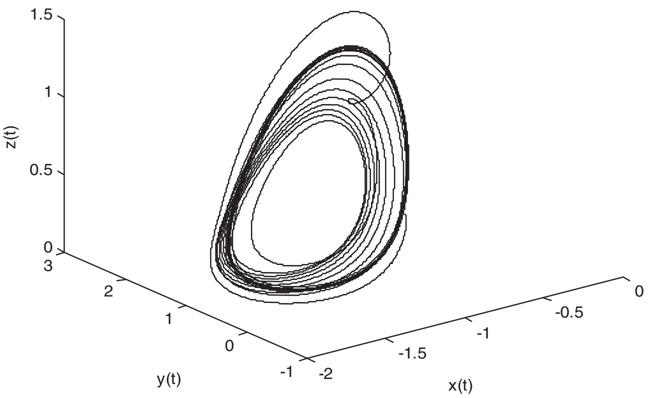

Fig. 1. Phase portrait of Rabinovich–Fabrikant system in x–y–z space for $\alpha = 0.99$.

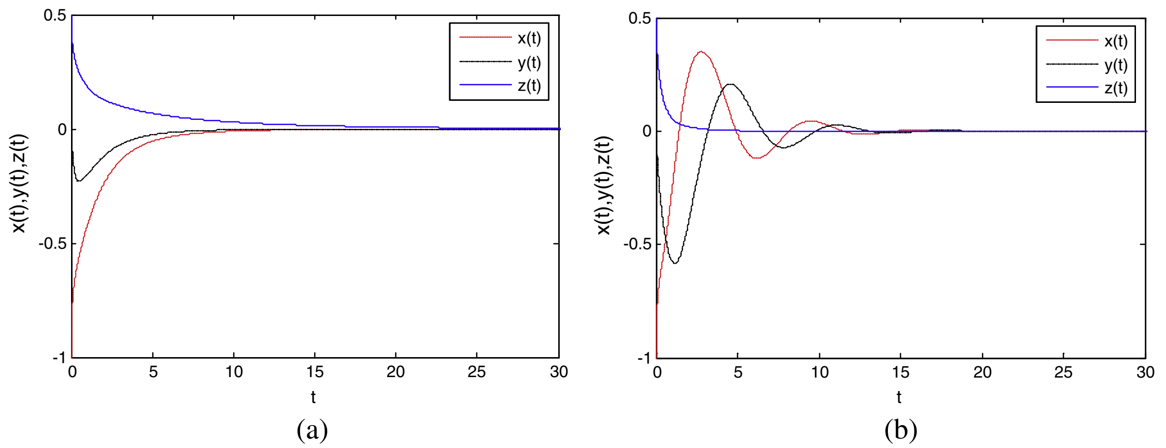

Fig. 2. Plots of $x(t), y(t)$ and $z(t)$ of the controlled system (15) stabilizing the equilibrium point E_1 for (a) $k_1 = 1, k_2 = 4, k_3 = -2$. (b) $k_1 = 1, k_2 = 10, k_3 = -1$.

$$P(\lambda) = \lambda^3 + 0.46\lambda^2 + (-4\bar{x}^2 + 3\bar{x}^4 - 6.14\bar{x}\bar{y} - 4\bar{x}^2\bar{y}^2 + 2\bar{z} + 6\bar{x}^2\bar{z} + 2\bar{y}^2\bar{z} - 3\bar{z}^2 - 2.0711)\lambda + 3.86518 - 8.8\bar{x}^2 + 6.6\bar{x}^4 + 7.3418\bar{x}\bar{y} - 8\bar{x}^3\bar{y} + 6\bar{x}^5\bar{y} + 3.48\bar{x}^2\bar{y}^2 + 4.4\bar{z} - 5.22\bar{x}^2\bar{z} - 12\bar{x}^3\bar{y}\bar{z} - 1.74\bar{y}^2\bar{z} - 6.6\bar{z}^2 + 6\bar{x}\bar{y}\bar{z}^2. \tag{11}$$

At the equilibrium point $E_1(0,0,0)$, the Eq. (11), becomes

$$P(\lambda) = \lambda^3 + 0.46\lambda^2 - 2.0711\lambda + 3.86518. \tag{12}$$

The eigenvalues of the Eq. (12) are $\lambda_1 = -2.2, \lambda_{2,3} = 0.87 \pm i$.. Here λ_1 is negative real number and absolute value of $\arg(\lambda_{2,3})$ is 0.8548. So $E_1(0,0,0)$ is stable for $\alpha < 0.544186$.

At the equilibrium point $E_2(-1.4797, 0.7434, 0.5422)$, the Eq. (11) becomes

$$P(\lambda) = \lambda^3 + 0.46\lambda^2 + 13.3914\lambda + 7.45725. \tag{13}$$

The eigenvalues of the Eq. (13) are $\lambda_1 = -0.554694, \lambda_{2,3} = 0.0473469 \pm 3.66629i$. Here λ_1 is negative real number and absolute value of $\arg(\lambda_{2,3})$ is 1.55788. So $E_2(-1.4797, 0.7434, 0.5422)$ is stable for $\alpha < 0.991$. Similarly the equilibrium point $E_3(1.4797, -0.7434, 0.5422)$ is stable for $\alpha < 0.991$.

At the equilibrium point $E_4(0.5182667, -2.12246, 0.94384)$, the Eq. (11) gives

$$P(\lambda) = \lambda^3 + 0.46\lambda^2 + 8.2249\lambda - 12.9813. \tag{14}$$

The eigenvalues are $\lambda_1 = 1.25201, \lambda_{2,3} = -0.856 \pm 3.104i$. Here λ_1 is positive real number. So $E_4(0.5182667, -2.12246, 0.94384)$ is unstable for $0 < \alpha < 1$.

Similarly the equilibrium point $E_5(-0.5182667, 2.12246, 0.94384)$ is unstable for $0 < \alpha < 1$.

4. Control of Chaos

The controlled Rabinovich–Fabrikant system with fractional order time derivative is given by

$$\begin{aligned}\frac{d^\alpha x}{dt^\alpha} &= y(z-1+x^2)+ax-k_1(x-\bar{x}),\\ \frac{d^\alpha y}{dt^\alpha} &= x(3z+1-x^2)+ay-k_2(y-\bar{y}),\\ \frac{d^\alpha z}{dt^\alpha} &= -2z(b+xy)-k_3(z-\bar{z}),\end{aligned} \tag{15}$$

where k_1, k_2, k_3 are control parameters and $(\bar{x}, \bar{y}, \bar{z})$ is the equilibrium point of the system (8).

The Jacobian matrix of the system (15) at an equilibrium point $\bar{E}(\bar{x}, \bar{y}, \bar{z})$ is given as

$$S(\bar{E}) = \begin{pmatrix} a+2\bar{x}\bar{y}-k_1 & \bar{z}-1+\bar{x}^2 & \bar{y} \\ 3\bar{z}+1-3\bar{x}^2 & a-k_2 & 3\bar{x} \\ -2\bar{y}\bar{z} & -2\bar{z}\bar{x} & -2b-2\bar{x}\bar{y}-k_3 \end{pmatrix} \tag{16}$$

and the corresponding characteristic polynomial for $a = 0.87, b = 1.1$ is given by

$$\begin{aligned}P(\lambda) = {}& \lambda^3 + (0.46+k_1+k_2+k_3)\lambda^2 + (-2.071+1.33k_1+1.33k_2+k_1k_2-1.74k_3+k_1k_3+k_1k_2-4\bar{x}^2+3\bar{x}^4\\ &-6.14\bar{x}\bar{y}+2k_1\bar{x}\bar{y}-2k_3\bar{x}\bar{y}-4\bar{x}^2\bar{y}^2+2\bar{z}+6\bar{x}^2\bar{z}+2\bar{y}^2\bar{z}-3\bar{z}^2)\lambda+3.86518-1.914k_1-1.914k_2+2.2k_1k_2\\ &+1.7569k_3-0.87k_1k_3-0.87k_2k_3+k_1k_2k_3-8.8\bar{x}^2-4k_3\bar{x}^2+6.6\bar{x}^4+3k_3\bar{x}^4+7.3418\bar{x}\bar{y}-1.74k_1\bar{x}\bar{y}\\ &-6.14k_2\bar{x}\bar{y}+2k_1k_2\bar{x}\bar{y}+1.74k_3\bar{x}\bar{y}-2k_2k_3\bar{x}\bar{y}-8\bar{x}^3\bar{y}+6\bar{x}^5\bar{y}+3.48\bar{x}^2\bar{y}^2-4k_2\bar{x}^2\bar{y}^2+4.4\bar{z}+2k_3\bar{z}\\ &-5.22\bar{x}^2\bar{z}+6k_1\bar{x}^2\bar{z}-12\bar{x}^3\bar{y}\bar{z}-1.74\bar{y}^2\bar{z}-6.6\bar{z}^2-3k_3\bar{z}^2+6\bar{x}\bar{y}\bar{z}^2.\end{aligned} \tag{17}$$

In view of fractional order Routh–Hurwitz conditions, we have

$$\begin{cases} a_1 = 0.46+k_1+k_2+k_3,\\ a_2 = -2.071+1.33k_1+1.33k_2+k_1k_2-1.74k_3+k_1k_3+k_1k_2-4\bar{x}^2+3\bar{x}^4-6.14\bar{x}\bar{y}\\ \quad +2k_1\bar{x}\bar{y}-2k_3\bar{x}\bar{y}-4\bar{x}^2\bar{y}^2+2\bar{z}+6\bar{x}^2\bar{z}+2\bar{y}^2\bar{z}-3\bar{z}^2,\\ a_3 = 3.86518-1.914k_1-1.914k_2+2.2k_1k_2+1.7569k_3-0.87k_1k_3-0.87k_2k_3\\ \quad +k_1k_2k_3-8.8\bar{x}^2-4k_3\bar{x}^2+6.6\bar{x}^4+3k_3\bar{x}^4+7.3418\bar{x}\bar{y}-1.74k_1\bar{x}\bar{y}-6.14k_2\bar{x}\bar{y}\\ \quad +2k_1k_2\bar{x}\bar{y}+1.74k_3\bar{x}\bar{y}-2k_2k_3\bar{x}\bar{y}-8\bar{x}^3\bar{y}+6\bar{x}^5\bar{y}+3.48\bar{x}^2\bar{y}^2-4k_2\bar{x}^2\bar{y}^2+4.4\bar{z}\\ \quad +2k_3\bar{z}-5.22\bar{x}^2\bar{z}+6k_1\bar{x}^2\bar{z}-12\bar{x}^3\bar{y}\bar{z}-1.74\bar{y}^2\bar{z}-6.6\bar{z}^2-3k_3\bar{z}^2+6\bar{x}\bar{y}\bar{z}^2,\\ D(P) = 18a_1a_2a_3+(a_1a_2)^2-4a_3a_1^3-4a_2^3-27a_3^2. \end{cases} \tag{18}$$

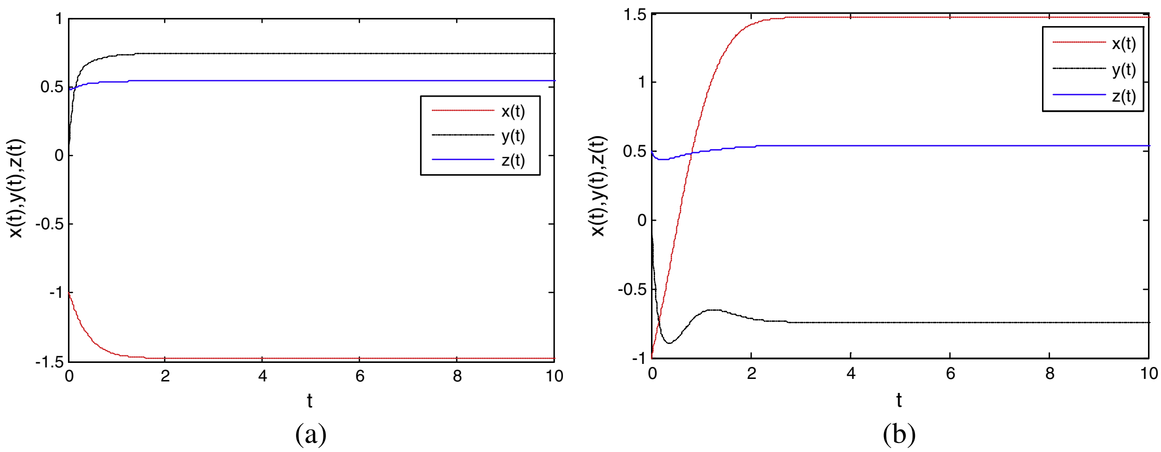

Fig. 3. Plots of $x(t), y(t)$ and $z(t)$ of the controlled system (15) stabilizing the equilibrium point E_2 and E_3 for $k_1 = 1, k_2 = 10, k_3 = 14$.

4.1. Stabilizing the point E_1

Substituting the value of E_1 in Eq. (18) and considering $k_1 = 1, k_2 = 4$ and $k_3 > -2.2$, we have $D(P) > 0, a_1 > 0, a_3 > 0, a_1 a_2 - a_3 > 0$. Here all the eigenvalues of Eq. (17) are real with negative sign. So the system (15) is locally asymptotically stable for $0 < \alpha < 1$. Also if $k_1 = 1, k_3 = -1$ and $-1 < k_2 < 1.5$, we have $D(P) < 0, a_1 > 0, a_2 > 0, a_3 > 0$. Moreover if we choose $\alpha = 0.9$, all eigenvalues satisfy Eq. (5).

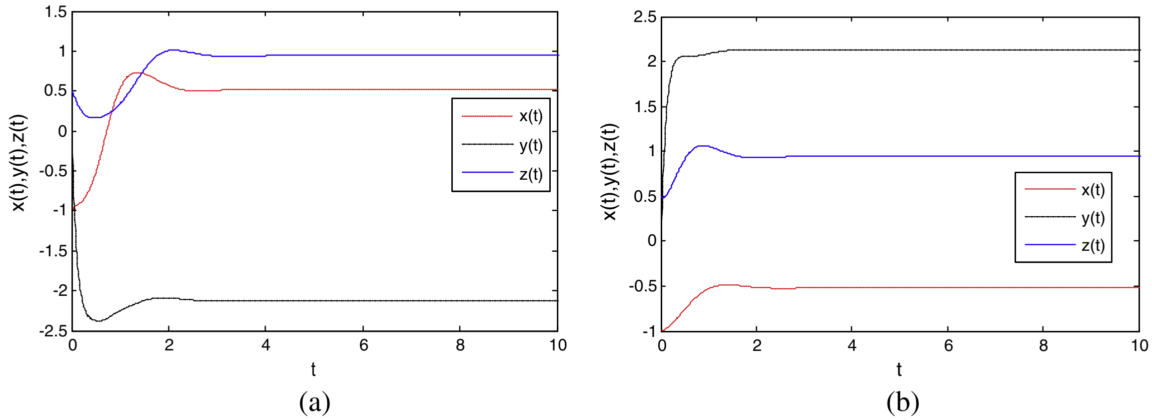

Fig. 4. Plots of $x(t), y(t)$ and $z(t)$ of the controlled system (15) stabilizing the equilibrium point E_4 and E_5 for $k_1 = 1, k_2 = 10, k_3 = 1$.

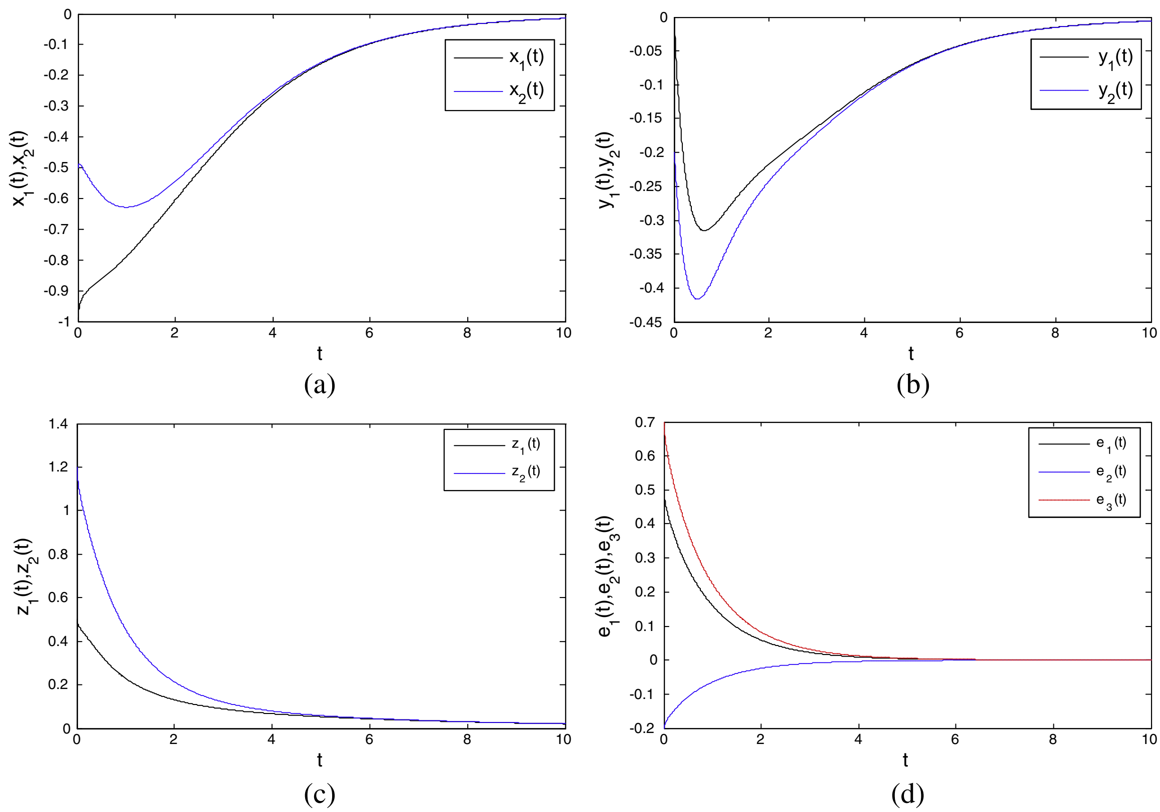

Fig. 5. Plots of state trajectories of drive system (19) and response system (20) between state vectors and between error vectors for the fractional order derivatives $q_i = 0.98 (i = 1, 2, 3)$ at $k_1 = 1, k_2 = 4$ and $k_3 = -2$.

4.2. Stabilizing the points E_2 and E_3

Substituting the value of E_2 in Eq. (18) and suppose $k_1 = 1, k_2 = 10$ and $k_3 > 13.79$, then we have $D(P) > 0, a_1 > 0, a_3 > 0, a_1a_2 - a_3 > 0$, and all eigenvalues of the system (17) are real with negative sign. So the system (15) is locally asymptotically stable for $0 \leqslant \alpha < 1$. Similarly for $k_1 = 1, k_2 = 10$ and

$k_3 > 13.79$, the system (15) is locally asymptotically stable for $0 \leqslant \alpha < 1$ at the equilibrium point E_3.

4.3. Stabilizing the points E_4 and E_5

Substituting the value of E_4 in Eq. (18) and suppose $k_1 = 1, k_2 = 10$ and $0 < k_3 < 4$, then we have $a_1 > 0, a_3 > 0, a_3 > 0$, and all the eigenvalues of the Eq. (17) are negative real number and complex numbers with negative real parts. So E_4 is locally asymptotically stable for $0 \leqslant \alpha < 1$.

Similarly for $k_1 = 1, k_2 = 10$ and $0 < k_3 < 4$, the system (17) is locally asymptotically stable for $0 < \alpha < 1$ at the equilibrium point E_5.

5. Simulation and results

In numerical simulations, the parameters of the Rabinovich–Fabrikant system are taken as $a = 0.87, b = 1.1$. Time step size is 0.01 and initial value of the system is taken as $[-1, 0, 0.5]$.

5.1. Simulation for the point E_1

At $k_1 = 1, k_2 = 4$ and $k_3 = -2$, eigenvalues are $\lambda_1 = -2.748, \lambda_2 = -0.512$ and $\lambda_3 = -0.199$. So the system (15) is asymptotically stable at $E_1(0,0,0)$. The simulation result is depicted through Fig. 2(a). When $k_1 = 1, k_3 = -1$ and $k_2 = 0.5$, then

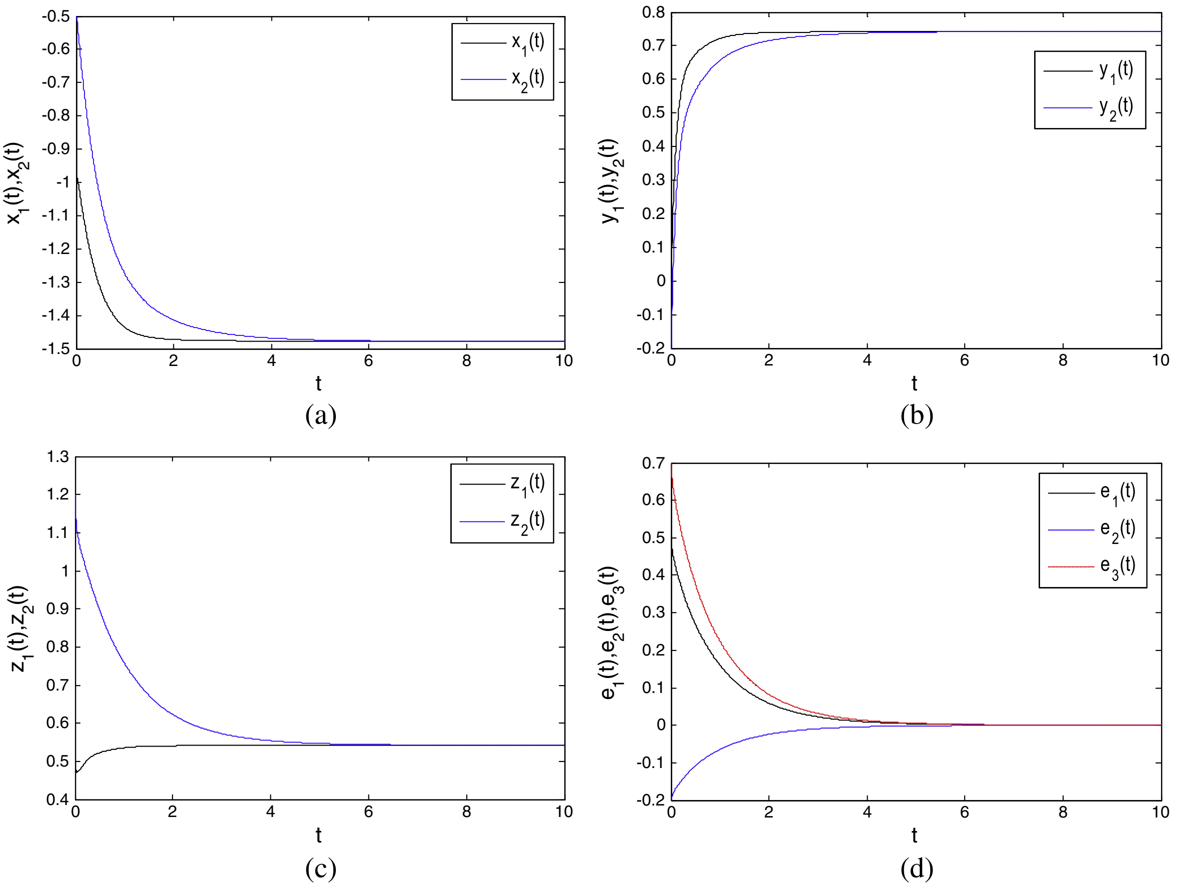

Fig. 6. Plots of state trajectories of systems (19) and (20) between state vectors and between error vectors for $q_i = 0.98 (i = 1,2,3)$ at $k_1 = 1, k_2 = 10$ and $k_3 = 14$.

eigenvalues are $\lambda_1 = -1.199$, $\lambda_{2,3} = 0.119978 \pm 0.968267i$. In this case though integer order system is unstable, but the fractional order system (15) is asymptotically stable at $E_1(0,0,0)$. The simulation result is depicted through Fig. 2(b).

5.2. Simulation for the points E_2 and E_3

At $k_1 = 1, k_2 = 10$ and $k_3 = 14$, then eigenvalues are $\lambda_1 = -11.8071, \lambda_2 = -10.2235$ and $\lambda_3 = -3.42924$. So the system (15) is asymptotically stable at $E_2(-1.4797, 0.7434, 0.5422)$. The simulation result is depicted through Fig. 3(a). Fig. 3(b) shows that system (15) is asymptotically stable at $E_3(1.4797, -0.7434, 0.5422)$

5.3. Simulation for the points E_4 and E_5:

At $k_1 = 1$, $k_2 = 10$ and $k_3 = 1$, then eigenvalues are $\lambda_1 = -8.92352$, $\lambda_{2,3} = -1.76824 \pm 2.72589i$. So the system (15) is asymptotically stable at $E_4(0.5182667, -2.12246, 0.94384)$. The simulation result is depicted through Fig. 4(a). Fig. 4(b) shows that system (15) is asymptotically stable at $E_5(-0.5182667, 2.12246, 0.94384)$.

6. Synchronization between fractional order chaotic and chaos controlled R–F systems

In this section the synchronization behavior between chaotic R–F system and unchaotic R–F system is made. We assume that chaos controlled fractional order R–F system, which is unchaotic, drives the chaotic R–F system. Therefore, unchaotic R–F system is taken as master system and chaotic R–F system as a slave system.

The master system is given by (15) as

$$\begin{aligned} \frac{d^\alpha x_1}{dt^\alpha} &= y_1(z_1 - 1 + x_1^2) + a x_1 - k_1(x_1 - \bar{x}), \\ \frac{d^\alpha y_1}{dt^\alpha} &= x_1(3z_1 + 1 - x_1^2) + a y_1 - k_2(y_1 - \bar{y}), \\ \frac{d^\alpha z_1}{dt^\alpha} &= -2z_1(b + x_1 y_1) - k_3(z_1 - \bar{z}). \end{aligned} \tag{19}$$

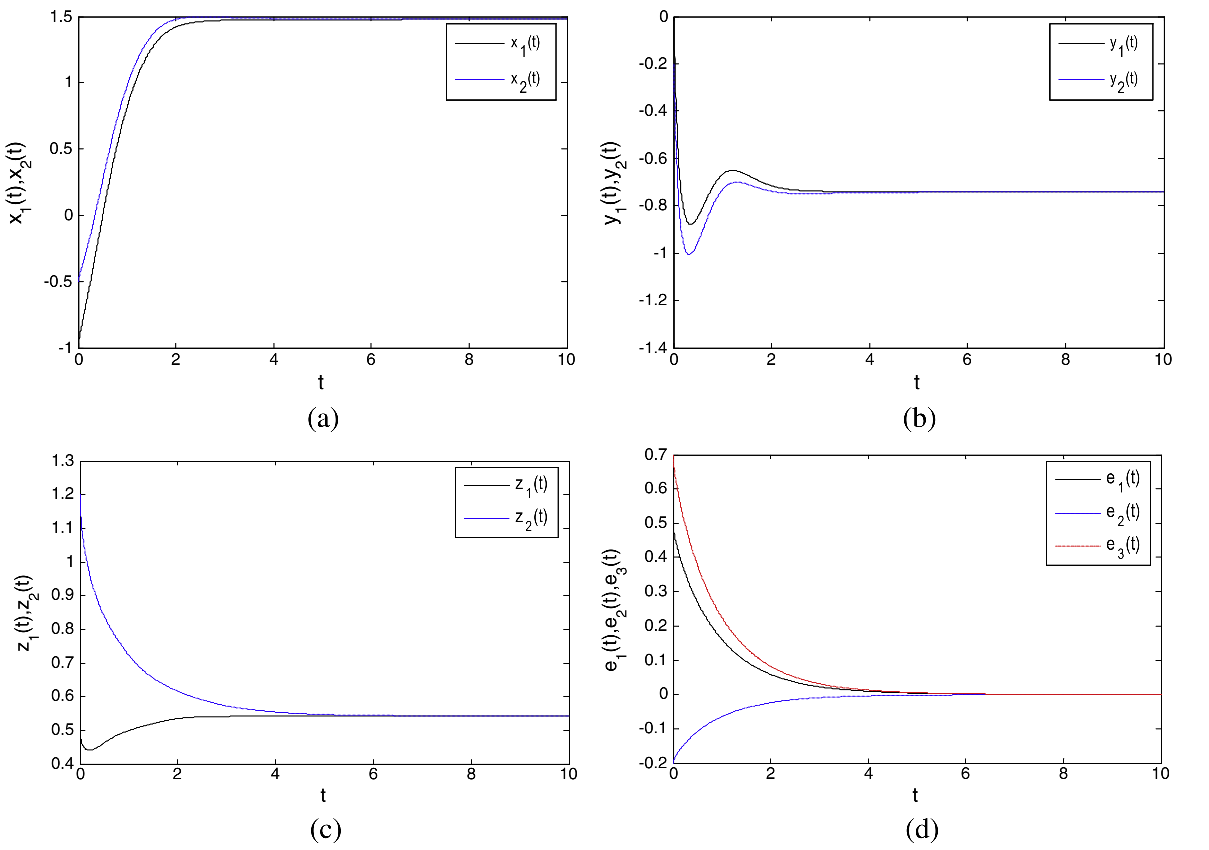

Fig. 7. Plots of state trajectories of systems (19) and (20) between state vectors and between error vectors for $q_i = 0.98 (i = 1,2,3)$ at $k_1 = 1, k_2 = 10$ and $k_3 = 14$.

The slave system is given by (8) as

$$\begin{aligned} \frac{d^\alpha x_2}{dt^\alpha} &= y_2(z_2 - 1 + x_2^2) + a x_2 + u_1(t), \\ \frac{d^\alpha y_2}{dt^\alpha} &= x_2(3 z_2 + 1 - x_2^2) + a y_2 + u_2(t), \\ \frac{d^\alpha z_2}{dt^\alpha} &= -2 z_2(b + x_2 y_2) + u_3(t), \quad 0 < \alpha < 1, \end{aligned} \tag{20}$$

where three active control functions $u_1(t)$, $u_2(t)$ and $u_3(t)$ are introduced in Eq. (8). Our goal is to investigate the synchronization of systems (19) and (20). We define the error states as $e_1 = x_2 - x_1, e_2 = y_2 - y_1$ and $e_3 = z_2 - z_1$. Then the corresponding error dynamics can be obtained by Eqs. (19) and (20) as

$$\begin{cases} \frac{d^{q_1} e_1}{dt^{q_1}} = a e_1 - e_2 + y_2 z_2 - y_1 z_1 + x_2^2 y_2 - x_1^2 y_1 + k_1(x_1 - \bar{x}) + u_1(t), \\ \frac{d^{q_2} e_2}{dt^{q_2}} = e_1 + a e_2 + 3 x_2 z_2 - 3 x_1 z_1 - x_2^3 + x_1^3 + k_2(y_1 - \bar{y}) + u_2(t), \\ \frac{d^{q_3} e_3}{dt^{q_3}} = -2 b e_3 - 2 x_2 y_2 z_2 + 2 x_1 y_1 z_1 + k_3(z_1 - \bar{z}) + u_3(t). \end{cases} \tag{21}$$

Then we define the active control inputs $u_1(t)$, $u_2(t)$, and $u_3(t)$ as

$$\begin{cases} u_1(t) = -y_2 z_2 + y_1 z_1 - x_2^2 y_2 + x_1^2 y_1 - k_1(x_1 - \bar{x}) + v_1(t), \\ u_2(t) = -3 x_2 z_2 + 3 x_1 z_1 + x_2^3 - x_1^3 - k_2(y_1 - \bar{y}) + v_2(t), \\ u_3(t) = 2 x_2 y_2 z_2 - 2 x_1 y_1 z_1 - k_3(z_1 - \bar{z}) + v_3(t), \end{cases} \tag{22}$$

which leads to

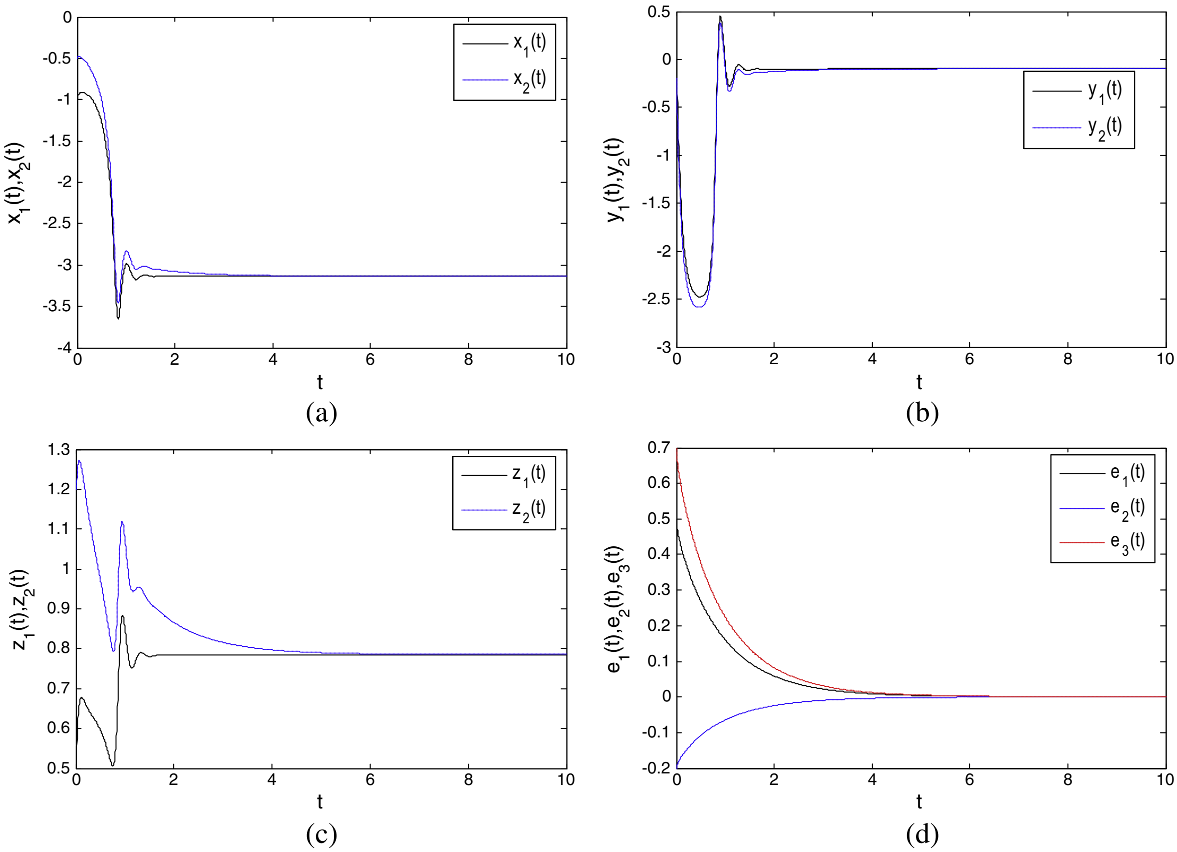

Fig. 8. Plots of state trajectories of systems (19) and (20) between state vectors and between error vectors for $q_i = 0.98 (i = 1, 2, 3)$ at $k_1 = 1, k_2 = 10$ and $k_3 = 1$.

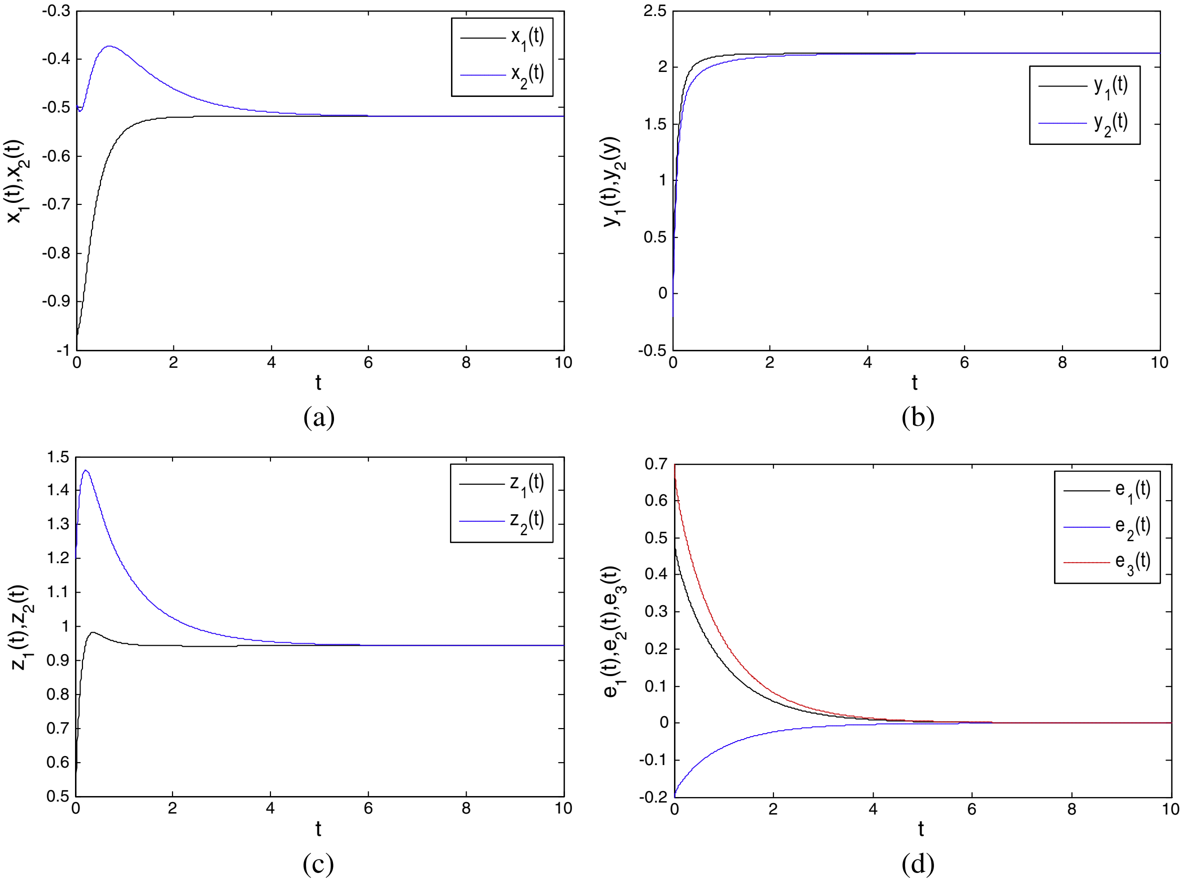

Fig. 9. Plots of state trajectories of systems (19) and (20) between state vectors and between error vectors for $q_i = 0.98 (i = 1, 2, 3)$ at $k_1 = 1, k_2 = 10$ and $k_3 = 1$.

$$\begin{cases} \frac{d^{q_1} e_1}{dt^{q_1}} = ae_1 - e_2 + v_1(t), \\ \frac{d^{q_2} e_2}{dt^{q_2}} = e_1 + ae_2 + v_2(t), \\ \frac{d^{q_3} e_3}{dt^{q_3}} = -2be_3 + v_3(t), \end{cases} \tag{23}$$

The synchronization error system (23) is a linear system with active control inputs v_1, $v_2(t)$ and $v_3(t)$. Next we design an appropriate feedback control which stabilizes the system so that e_1, e_2, e_3 converge to zero as time t tends to infinity, which implies that the system (19) and (20) are synchronized with feedback control. There are many possible choices for the control inputs v_1, $v_2(t)$ and $v_3(t)$. We choose

$$\begin{bmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \end{bmatrix} = A \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \tag{24}$$

where A is a 3×3 constant matrix. In order to make the closed loop system stable, the matrix A should be selected in such a way that the feedback system has eigenvalues λ_i which satisfies the condition $|\arg(\lambda_i)| > 0.5\pi q$, $i = 1, 2, 3$. There is not a unique choice for such matrix A, a good choice can taken be as follows:

$$A = \begin{pmatrix} -(a+1) & 1 & 0 \\ -1 & -(a+1) & 0 \\ 0 & 0 & 2b-1 \end{pmatrix} \tag{25}$$

Then the error system is changed to

$$D_t^{q_1} e_1 = -e_1, \quad D_t^{q_2} e_2 = -e_2, \quad D_t^{q_3} e_3 = -e_3. \tag{26}$$

Here all eigenvalues λ_i of matrix A are -1, which satisfy the condition $|\arg(\lambda_i)| > q\pi/2$, for $0 < q \leqslant 1$. Therefore, the linear system (26) is stable and thus we get the required synchronization.

7. Simulation and results

For the purpose of numerical simulations, the parameters of the R–F system are taken as $a = 0.87$, $b = 1.1$. The initial values of the systems (19) and (20) are considered as [−1, 0, 0.5] and [−0.5, −0.2, 1.2], respectively. Thus, the initial conditions of error system become [0.5, −0.2, 0.7].

7.1. Synchronization at the point E_1

See Fig. 5.

7.2. Synchronization at the point E_2

See Fig. 6.

7.3. Synchronization at the point E_3

See Fig. 7.

7.4. Synchronization at the point E_4

See Fig. 8.

7.5. Synchronization at the point E_5

See Fig. 9.

8. Conclusion

There are three important goals that the authors have achieved in the present article. First one is the local stability of the R–F chaotic system with fractional order time derivative is analyzed. Second one is employing the control function of fractional order R–F chaotic system with different equilibrium points. The stability of the equilibrium points using the fractional Routh–Hurwitz criterion and the sufficient conditions for control of the fractional order R–F system by linear feedback control have been studied. It is observed that the R–F system can be controlled to its equilibrium points. The stability theorems of fractional-order systems guarantee that the chaos control occurs if the necessary conditions are satisfied. Simulation results show that the feedback control is easy to implement even for controlling the fractional order chaotic systems. Thirdly, authors have applied the powerful active control method which provides us a simple way to synchronize a pair of chaotic systems and finally investigate the synchronization between fractional order chaotic and non-chaotic R–F system. Numerical simulations are used to verify the efficiency, effectiveness and validity of the proposed method. The chaos control of chaotic systems and synchronization between pair of fractional order chaotic systems assume considerable significance in the study of nonlinear dynamics. The outcome of this research work would be appreciated and could be utilized by those researchers involved in the field of mathematical modeling of fractional order dynamical systems.

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