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Boundary integral equation formulation for generalized thermoelastic diffusion – Analytical aspects

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ABSTRACT

The present work deals with the formulation of the boundary integral equations for the solution of equations under linear theory of generalized thermoelastic diffusion in a three-dimensional Euclidean space. A mixed initial-boundary value problem is considered in the present context and the fundamental solutions of the corresponding coupled differential equations are obtained in the Laplace transform domain by employing the treatment of scalar and vector potential theory. A reciprocal relation of Betti type is established. Then we formulate the boundary integral equations for generalized thermoelastic diffusion on the basis of these fundamental solutions and the reciprocal relation.

1. Introduction

In last few decades, the boundary element method (BEM) or boundary integral equation method (BIEM) has been appeared as an alternative numerical technique to finite difference method (FDM) and finite element method (FEM) and is being widely and successfully employed in several branches of applied science and technology to solve many practical engineering problems which are intractable to solve by analytical methods. Particularly, attention is being paid to engineering stress and strain analysis for elastic medium. An important feature of the BIE method is that instead of attempting to find an approximate solution for the governing differential equations throughout the relevant solution domain, the equations are converted into an integral form that often involves only integrals over the boundary of the domain. Consequently, only the boundary needs to be discretized in order to carry out the integrations and thereby the dimensionality of the unknowns for the solution of the problem reduces by one. It is also more efficient in terms of both computer time and storage than FEM for solutions of the same accuracy (see Brebbia [1], Brebbia et al. [2] and Fenner [3]). BIEM can model the elasticity problem for the infinite region without any difficulty which is difficult to model by using FEM for the infinite region. Although BIE method seems to have its origin in classical potential theory, but the first numerical treatment was formulated by Jaswon [4] and Symm [5] (see also Jaswon and Symm [6]). Rizzo & Shippy [7] outlined the first BEM procedure for steady state thermoelasticity and presented numerical results for three-dimensional linear homogeneous isotropic medium. Sladek and Sladek [8,9] obtained fundamental solutions and representation formulae for coupled and uncoupled thermoelasticity theories. Later on, Dargush and Banerjee [10,11] developed the formulations for uncoupled quasi-static thermoelasticity and for coupled thermoelasticity theories. Chen and Dargush [12] discussed the boundary element method for dynamic poroelastic and thermoelastic analyses. Application of boundary element method to 3-D problems of coupled thermoelasticity was analyzed by Tanaka et al. [13]. In thermal stress problems, some applications of BIE formulation were reported by Cruse and Rizzo [14], Rizzo and Shippy [15],

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Banerjee and Butterfield [16], Brebbia et al. [2], Ziegler and Irschik [17] and Tanaka [18]. BIE formulations for generalized thermoelasticity with relaxation times were presented by Anwar and Sherief [19,20]. Recently, a boundary element formulation for the coupled thermoelasticity for materials of arbitrary degree of anisotropy was given by Kogl and Gaul [21]. El-Karamany and Ezzat [22,23] determined the BIE formulations for generalized micro-polar thermoelasticity and for generalized thermoviscoelasticity, respectively. Prasad et al. [24] gave BIE formulations for thermoelasticity with phase-lags. El-Karamany [25] presented the BIE formulation for the generalized micro-polar thermoviscoelasticity.

The thermodiffusion in elastic solids takes into account the coupling effects of the fields of temperature, mass diffusion and that of strain in addition to heat and mass exchange with environment. The theory of thermoelastic diffusion was developed for the first time by Nowacki [26–29]. Later on, Gawinecki et al. [30] proved a theorem on existence, uniqueness and regularity of the solution for a nonlinear parabolic thermoelastic diffusion problem. A theorem about global existence of the solution for the same problem was established by Gawinecki and Szymaniec [31]. The coupled thermoelastic model of Biot [32] was used in the theory developed by Nowacki [26–29]. Recently, Sherief et al. [33] and later on Kumar and Kansal [34] introduced the generalized theories of thermoelastic diffusion by introducing thermal relaxation time parameters and diffusion relaxation parameters into the governing equations. Subsequently, several studies are carried out to investigate the nature of mutual interactions of mass diffusion with the temperature and strain fields (see Refs. [35–52]) and it has been realized that the process of thermodiffusion has a very considerable influence upon the deformation of the body.

The main objective of this work is to present a formulation of boundary integral equations in the context of the linear theory of generalized thermoelastic diffusion. Firstly, fundamental solutions of corresponding differential equations are obtained in the Laplace transform domain by considering a mixed problem and the boundary integral equations are formulated by deriving a reciprocal relation. We give special emphasis to the representation of primary fields- the temperature, displacement and chemical potential. This formulation is believed to be helpful for the solution of problems in the present theory by using boundary element method.

2. Basic governing equations

We employ a rectangular co-ordinate system x_k and usual indicial notations throughout the paper. In three-dimensional Euclidean space, let V denote a regular region of space, whose boundary B is a smooth surface, enclosing a thermoelastic material.

Following Sherief et al. [33] and Dhaliwal and Singh [53], we consider the governing equations for linear theory of generalized thermoelastic diffusion for an isotropic and homogenous elastic solid for the state variables (u_i, θ, P) as follows:

The equations of motion:

$$\mu u_{i,jj} + (\lambda + \mu - \beta_2\gamma_2)u_{j,ji} - (\beta_1 + \beta_2 d_1)\theta_{,i} - \beta_2 nP_{,i} - \rho\ddot{u}_i = -\rho F_i. \tag{1}$$

The equation of heat conduction:

$$(\rho c_E + aT_0 d_1)(\dot{\theta} + \tau_0\ddot{\theta}) + T_0(\beta_1 + a\gamma_2)(\dot{e}_{kk} + \tau_0\ddot{e}_{kk}) + T_0 d_1(\dot{P} + \tau_0\ddot{P}) - k\theta_{,ii} = \left(1 + \tau_0\frac{\partial}{\partial t}\right)Q. \tag{2}$$

The equation of mass diffusion:

$$\gamma_2(\dot{e}_{kk} + \tau_1\ddot{e}_{kk}) + d_1(\dot{\theta} + \tau_1\ddot{\theta}) + n(\dot{P} + \tau_1\ddot{P}) - dP_{,ii} = \left(1 + \tau_1\frac{\partial}{\partial t}\right)\sigma. \tag{3}$$

The constitutive equations:

$$\sigma_{ij} = 2\mu e_{ij} + (\lambda_0 e_{kk} - \gamma_1\theta - \gamma_2 P)\delta_{ij}, \tag{4}$$

$$\rho S = c\theta + \gamma_1 e_{kk} + d_1 P, \tag{5}$$

$$C = \gamma_2 e_{kk} + nP + d_1\theta. \tag{6}$$

The generalized Fourier's law:

$$q_i + \tau_0\dot{q}_i = -k\theta_{,i}. \tag{7}$$

The equation for mass flow vector:

$$\eta_i + \tau_1\dot{\eta}_i = -dP_{,i}. \tag{8}$$

The strain–displacement relation:

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}). \tag{9}$$

The energy equation:

$$q_{i,i} = Q - \rho T_0\dot{S}. \tag{10}$$

The equation of conservation of diffusive mass:

$$\eta_{i,i} = \sigma - \dot{C}, \tag{11}$$

where u_i, σ_{ij} and e_{ij} are the components of the displacement vector, stress tensor and strain tensor, respectively. F_i are the components of body force per unit mass. $\theta = Z - T_0$ where Z is the absolute temperature and T_0 is the temperature of the medium in its natural state assumed to be such that $\left|\frac{\theta}{T_0}\right| << 1$. C is the mass concentration of the diffusive material in the elastic body, P is the chemical potential per unit mass , S is the entropy per unit mass, Q is the heat source per unit volume and σ is the intensity of mass diffusing sources. q_i and η_i denote the components of heat flux vector and flow of the diffusing mass vector, respectively. λ, μ are Lamè's elastic constants, ρ is the constant mass density of the medium, and β_1 and β_2 are the material constants given by $\beta_1 = (3\lambda + 2\mu)\alpha_t$ and $\beta_2 = (3\lambda + 2\mu)\alpha_c$ where α_t is the coefficient of linear thermal expansion and α_c is the coefficient of linear diffusion expansion. δ_{ij} is the Kronecker's delta, k is the thermal conductivity, d is the diffusion coefficient and c_E is the specific heat at constant strain. τ_0 and τ_1 are the parameters of thermal relaxation time and diffusion relaxation time, respectively. 'a' and 'b' are the measures of the thermo-diffusion effect and diffusive effect, respectively. The comma notations are used to represent the partial derivatives with respect to the space variables and the superposed dots denote partial derivatives with respect to time variable t. All the functions used in Eqs. (1)–(11) are assumed to be functions of (x, t) , where $x = (x_1, x_2, x_3)$ is the position vector, and are defined on $\bar{V} = (V \cup B) \times [0, \infty)$.

In Eqs. (1)–(11), we have used the following notations

$$\gamma_1 = \beta_1 + \frac{a\beta_2}{b}, \quad \gamma_2 = \frac{\beta_2}{b}, \quad \lambda_0 = \lambda - \frac{\beta_2^2}{b}, \quad c = \frac{\rho c_E}{T_0} + \frac{a^2}{b}, \quad d_1 = \frac{a}{b}, \quad n = \frac{1}{b}$$

We suppose that

$$3\lambda + 2\mu > 0,\ \mu > 0,\ k > 0,\ d > 0,\ \tau_0 > 0,\ \tau_1 > 0,\ T_0 > 0.$$

With the Eqs. (1)–(11), we adjoin the following boundary conditions:

$$\sigma_{ij}n_j = p_{i0}(x_B, t), \quad x_B \in B_1,\ t > 0, \tag{12a}$$

$$u_i = u_{i0}(x_B, t), \quad x_B \in B_2,\ t > 0, \tag{12b}$$

$$\theta = \theta_0(x_B, t), \quad x_B \in B_3,\ t > 0, \tag{12c}$$

$$\theta_{,n} = \theta_{,i}n_i = \theta_{,n0}(x_B, t), \quad x_B \in B_4,\ t > 0, \tag{12d}$$

$$P = P_0(x_B, t), \quad x_B \in B_5,\ t > 0 \tag{12e}$$

$$P_{,n} = P_{,i}n_i = P_{,n0}(x_B, t), \quad x_B \in B_6,\ t > 0, \tag{12f}$$

where $n_i = n_i(x_B)$ are the components of outward-pointing normal vector to B at x_B and we assume that (B_1, B_2) , (B_3, B_4) and (B_5, B_6) are the three partitions of the surface B, such that $B_1 \cup B_2 = B_3 \cup B_4 = B_5 \cup B_6 = B$ and $B_1 \cap B_2 = B_3 \cap B_4 = B_5 \cap B_6 = \emptyset$.

The initial conditions are assumed to be homogeneous.

Using Eqs. (4) and (9), the components of the traction vector are obtained in the form

$$\sigma_{ij}n_j = \mu n_j(x_B)(u_{i,j} + u_{j,i}) + n_i(x_B)(\lambda_0 u_{j,j} - \gamma_1\theta - \gamma_2 P) \tag{13}$$

3. Formulation of the problem in Laplace transform domain

The Laplace transform of a function $f(x, t)$ is defined as

$$\bar{f}(x, s) = \int_0^\infty e^{-st} f(x, t)\mathrm{d}t, \quad s > 0,$$

where s is the Laplace transform parameter.

Applying Laplace transform to both sides of Eqs. (1)–(4) and Eq. (13) and using homogeneous initial conditions, we obtain

$$\mu\bar{u}_{i,jj} + (\lambda + \mu - \beta_2\gamma_2)\bar{u}_{j,ji} - (\beta_1 + \beta_2 d_1)\bar{\theta}_{,i} - \beta_2 n\bar{P}_{,i} - \rho s^2\bar{u}_i = -\rho\bar{F}_i, \tag{14}$$

$$(\rho c_E + aT_0 d_1)(s + \tau_0 s^2)\bar{\theta} + T_0(\beta_1 + a\gamma_2)(s + \tau_0 s^2)\bar{e}_{kk} + T_0 d_1(s + \tau_0 s^2)\bar{P} - k\bar{\theta}_{,ii} = (1 + \tau_0 s)\bar{Q}, \tag{15}$$

$$\gamma_2(s + \tau_1 s^2)\bar{e}_{kk} + d_1(s + \tau_1 s^2)\bar{\theta} + n(s + \tau_1 s^2)\bar{P} - d\bar{P}_{,ii} = (1 + \tau_1 s)\bar{\sigma}, \tag{16}$$

$$\bar{\sigma}_{ij} = 2\mu\bar{e}_{ij} + (\lambda_0\bar{e}_{kk} - \gamma_1\bar{\theta} - \gamma_2\bar{P})\delta_{ij}, \tag{17}$$

$$\bar{\sigma}_{ij}n_j = \mu n_j(x_B)(\bar{u}_{i,j} + \bar{u}_{j,i}) + n_i(x_B)(\lambda_0\bar{u}_{j,j} - \gamma_1\bar{\theta} - \gamma_2\bar{P}), \tag{18}$$

Now, following Helmholtz's theorem, the decomposition of the displacement and body forces can be expressed as

$$u_i = \phi_{,i} + \epsilon_{ijk}\psi_{k,j}, \quad \psi_{i,i} = 0 \tag{19}$$

$$F_i = X_{,i} + \epsilon_{ijk}Y_{k,j}, \quad Y_{i,i} = 0 \tag{20}$$

where $\boldsymbol{\epsilon}_{ijk}$ is the permutation tensor, ϕ, X are the scalar potentials and ψ_k, Y_k are the vector potentials of the vector fields u_i, F_i, respectively.

Using Eqs. (19) and (20) in Eqs. (14)–(16), we obtain

$$\mathrm{E}_1^2\bar{\phi} - m_1\bar{\theta} - m_2\bar{P} = -\frac{\bar{X}}{c_1^2}, \tag{21}$$

$$\mathrm{E}_2^2\bar{\psi}_i = -\frac{\bar{Y}_i}{c_2^2}, \tag{22}$$

$$\mathrm{D}_1\bar{\theta} - \alpha_1(s + \tau_0 s^2)\nabla^2\bar{\phi} - \alpha_2(s + \tau_0 s^2)\bar{P} = -\frac{(1 + \tau_0 s)}{k}\bar{Q}, \tag{23}$$

$$\alpha_4(s + \tau_1 s^2)\nabla^2\bar{\phi} + \alpha_5(s + \tau_1 s^2)\bar{\theta} - \mathrm{D}_2\bar{P} = \frac{(1 + \tau_1 s)}{d}\bar{\sigma}, \tag{24}$$

where we used the following notations:

$$\mathrm{E}_i^2 = \nabla^2 - \frac{s^2}{c_i^2} \text{ for } i = 1,2, \quad \mathrm{D}_1 = \nabla^2 - \frac{(\rho c_E + aT_0d_1)}{k}(s + \tau_0 s^2), \quad \mathrm{D}_2 = \nabla^2 - \frac{n}{d}(s + \tau_1 s^2), \quad c_1^2 = \frac{\lambda + 2\mu - \beta_2\gamma_2}{\rho},$$
$$c_2^2 = \frac{\mu}{\rho}, \quad \alpha_1 = \frac{T_0(\beta_1 + a\gamma_2)}{k}, \quad \alpha_2 = \frac{d_1T_0}{k}, \quad \alpha_4 = \frac{\gamma_2}{d}, \quad \alpha_5 = \frac{d_1}{d}, \quad m_1 = \frac{\beta_1 + \beta_2 d_1}{\lambda + 2\mu - \beta_2\gamma_2}, \quad m_2 = \frac{n\beta_2}{\lambda + 2\mu - \beta_2\gamma_2}.$$

4. Fundamental solutions in the Laplace- transform domain

We shall consider the following three cases to describe the action of body force, mass diffusion source and heat source of very large magnitude that act for a very short period of time.

Case I. We assume that an instantaneous source of heat located at a point y, where $y \in V \cup B$, is acting upon an elastic body in the absence of the body forces and mass diffusion sources,

i.e.

$$Q = \delta(r)\delta(t), \quad F_i = 0, \ \sigma = 0, \tag{25}$$

where $r = \sqrt{(x_i - y_i)(x_i - y_i)}$

We denote the corresponding fundamental solutions as $\bar{u}_i^{(1)}, \bar{\theta}^{(1)}, \bar{P}^{(1)}$. Therefore, the governing Eqs. (21)–(24) under the above assumptions reduce to

$$\mathrm{E}_1^2\bar{\phi}^{(1)} - m_1\bar{\theta}^{(1)} - m_2\bar{P}^{(1)} = 0 \tag{26}$$

$$\mathrm{E}_2^2\bar{\psi}_i^{(1)} = 0 \tag{27}$$

$$\mathrm{D}_1\bar{\theta}^{(1)} - \alpha_1(s + \tau_0 s^2)\nabla^2\bar{\phi}^{(1)} - \alpha_2(s + \tau_0 s^2)\bar{P}^{(1)} = -\frac{(1 + \tau_0 s)}{k}\delta(r) \tag{28}$$

$$\alpha_4(s + \tau_1 s^2)\nabla^2\bar{\phi}^{(1)} + \alpha_5(s + \tau_1 s^2)\bar{\theta}^{(1)} - \mathrm{D}_2\bar{P}^{(1)} = 0 \tag{29}$$

For an infinite region, Eq. (27) under the consideration of homogeneous initial conditions implies that

$$\bar{\psi}_i^{(1)} = 0 \tag{30}$$

From Eqs. (26), (28), and (29), we obtain

$$(\nabla^2 - k_1^2)(\nabla^2 - k_2^2)(\nabla^2 - k_3^2)\bar{\phi}^{(1)} = -\frac{m_1(1 + \tau_0 s)}{k}(\nabla^2 - k_4^2)\delta(r) \tag{31}$$

$$(\nabla^2 - k_1^2)(\nabla^2 - k_2^2)(\nabla^2 - k_3^2)\bar{P}^{(1)} = -\frac{(m_1\alpha_4 + \alpha_5)(s + \tau_1 s^2)(1 + \tau_0 s)}{k}(\nabla^2 - k_5^2)\delta(r), \tag{32}$$

where k_1^2 , k_2^2 and k_3^2 are the solutions of the characteristic equation.

$$\nabla^6 - a_1\nabla^4 + a_2\nabla^2 - a_3 = 0, \tag{33}$$

with

$$a_1 = (s+\tau_1 s^2)\left(\frac{n}{d} + \alpha_4 m_2\right) + \frac{s^2}{c_1^2} + (s+\tau_0 s^2)\left\{m_1\alpha_1 + \frac{(\rho c_E + aT_0 d_1)}{k}\right\}$$

$$a_2 = \left[(s+\tau_1 s^2)\tfrac{s^2}{c_1^2}\tfrac{n}{d} + (s+\tau_0 s^2)\tfrac{s^2}{c_1^2}\tfrac{(\rho c_E + aT_0 d_1)}{k} + (s+\tau_1 s^2)(s+\tau_0 s^2)\left\{\begin{array}{c} -m_2\alpha_5\alpha_1 + \frac{n(\rho c_E + aT_0 d_1)}{dk} + \frac{m_1\alpha_1 n}{d} + \\ \frac{m_2\alpha_4(\rho c_E + aT_0 d_1)}{k} - \alpha_5\alpha_2 - m_1\alpha_4\alpha_2 \end{array}\right\}\right]$$

$$a_3 = \frac{(s+\tau_1 s^2)(s+\tau_0 s^2)s^2}{c_1^2}\left(\frac{n(\rho c_E + aT_0 d_1)}{dk} - \alpha_2\alpha_5\right)$$

and

$$k_4^2 = \left(\frac{n}{d} - \frac{m_2\alpha_5}{m_1}\right)(s+\tau_1 s^2), \quad k_5^2 = \frac{s^2\alpha_5}{c_1^2(\alpha_5 + m_1\alpha_4)}$$

Now using the Helmholtz equation

$$\frac{1}{(\nabla^2 - m^2)}[\delta(r)] = -\frac{1}{4\pi r}e^{-mr}. \tag{34}$$

We obtain the solution for $\bar{\phi}^{(1)}$ and $\bar{P}^{(1)}$ from Eqs. (31) and (32) as

$$\bar{\phi}^{(1)}(r,s) = -\frac{I_1}{r}\sum_{n=1}^{3} b_n(k_n^2 - k_4^2)e^{-k_n r} \tag{35}$$

$$\bar{P}^{(1)}(x,y,s) = -\frac{g_1}{r}, \tag{36}$$

where

$$I_1 = m_1 I,\ I = \frac{(1+\tau_0 s)}{4\pi k b_1 b_2 b_3},\ g_1 = I_2\sum_{n=1}^{3} b_n(k_n^2 - k_5^2)e^{-k_n r},\ I_2 = (s+\tau_1 s^2)(\alpha_5 + m_1\alpha_4)I \text{ and } b_1 = k_2^2 - k_3^2,\ b_2 = k_3^2 - k_1^2,\ b_3 = k_1^2 - k_2^2 \tag{37}$$

Therefore, it follows from Eqs. (19), (30), and (35) that

$$\bar{u}_i^{(1)}(x,y,s) = \frac{r_{,i}}{r}g_2, \tag{38}$$

where

$$r_{,i} = \frac{(x_i - y_i)}{r} \text{ and } g_2 = I_1\sum_{n=1}^{3} b_n(k_n^2 - k_4^2)\left(k_n + \frac{1}{r}\right)e^{-k_n r} \tag{39}$$

From Eqs. (26), (35), and (36), we get

$$\bar{\theta}^{(1)}(x,y,s) = \frac{g_3}{r}, \tag{40}$$

where

$$g_3 = I\sum_{n=1}^{3} b_n\left\{\frac{m_2}{m_1}(s+\tau_1 s^2)(\alpha_5 + m_1\alpha_4)(k_n^2 - k_5^2) - \left(k_n^2 - \frac{s^2}{c_1^2}\right)(k_n^2 - k_4^2)\right\}e^{-k_n r}$$

Using Eq. (38) and the relation $r_{,ij} = \frac{\delta_{ij}}{r} - \frac{r_{,i}r_{,j}}{r}$, we obtain

$$\bar{u}_{i,j}^{(1)} = \frac{g_2}{r^2}(\delta_{ij} - 3r_{,i}r_{,j}) - \frac{r_{,i}r_{,j}g_4}{r} \tag{41}$$

where $g_4 = I_1\sum_{n=1}^{3} b_n k_n^2(k_n^2 - k_4^2)e^{-k_n r}$.

By using Eqs. (36), (40), and (41) in Eq. (18), we get the solution for the components of the traction vector in Laplace transform domain as

$$\bar{\sigma}_{ij}^{(1)}n_j = \frac{1}{r}\left[2\mu n_j\left\{\frac{\delta_{ij}g_2}{r} - r_{,i}r_{,j}\left(g_4 + \frac{3g_2}{r}\right)\right\} + n_i\{-\lambda_0 g_4 - \gamma_1 g_3 + \gamma_2 g_1\}\right] \tag{42}$$

After using Eq. (40), we can find expression for $\bar{\theta}^{(1)}_{,n}$ as

$$\bar{\theta}^{(1)}_{,n} = \bar{\theta}^{(1)}_{,i} n_i = -\frac{r_{,i} n_i}{r} g_5, \tag{43}$$

where

$$g_5 = I \sum_{n=1}^{3} b_n \left(k_n + \frac{1}{r}\right) \left\{ \frac{m_2}{m_1}(s + \tau_1 s^2)(\alpha_5 + m_1 \alpha_4)(k_n^2 - k_5^2) - \left(k_n^2 - \frac{s^2}{c_1^2}\right)(k_n^2 - k_4^2) \right\} e^{-k_n r} \tag{44}$$

The expression for $\bar{P}^{(1)}_{,n}$ can be obtained from Eq. (36) as

$$\bar{P}^{(1)}_{,n} = \bar{P}^{(1)}_{,i} n_i = \frac{r_{,i} n_i}{r} g_6, \tag{45}$$

where

$$g_6 = I_2 \sum_{n=1}^{3} b_n \left(k_n + \frac{1}{r}\right)(k_n^2 - k_5^2) e^{-k_n r}$$

Case II. For this case, we assume that in absence of heat source and mass diffusion sources, an instantaneous concentrated body force is applied at point y in the direction of x_j-axis, i.e.

$$F_i = F_i^{(j)} = \delta_{ij} \delta(r) \delta(t), \quad Q = 0, \ \sigma = 0$$

where $r = \sqrt{(x_i - y_i)(x_i - y_i)}$.

Let $\bar{u}_i^{(j)}, \bar{\theta}^{(j)}, \bar{P}^{(j)}$ denote the corresponding fundamental solutions. Taking the Laplace transforms of F_i, Q and σ as mentioned above, we have

$$\bar{Q} = 0, \quad \bar{\sigma} = 0 \ \bar{F}_i = \bar{F}_i^{(j)} = \delta_{ij} \delta(r) \tag{46}$$

Now, from Eqs. (19) and (20), we get

$$\bar{u}_i = \bar{u}_i^{(j)} = \phi^{(j)}_{,i} + \in_{ilk} \bar{\psi}^{(j)}_{k,l} \tag{47}$$

$$\bar{F}_i^{(j)} = \bar{X}^{(j)}_{,i} + \in_{ilk} \bar{Y}^{(j)}_{k,l} \tag{48}$$

Taking the governing Eqs. (21)–(24), we find that the potentials in the right hand side of Eqs. (47) and (48) satisfy following equations:

$$(\nabla^2 - k_1^2)(\nabla^2 - k_2^2)(\nabla^2 - k_3^2) \bar{\phi}^{(j)} = -\frac{1}{c_1^2}(\nabla^4 - A_1 \nabla^2 + A_2) \bar{X}^{(j)} \tag{49}$$

$$\left(\nabla^2 - \frac{s^2}{c_2^2}\right) \bar{\psi}_k^{(j)} = -\frac{\bar{Y}_k^{(j)}}{c_2^2} \tag{50}$$

$$(\nabla^2 - k_1^2)(\nabla^2 - k_2^2)(\nabla^2 - k_3^2) \bar{P}^{(j)} = -\frac{\alpha_4 (s + \tau_1 s^2)}{c_1^2} (\nabla^2 - k_6^2) \nabla^2 \bar{X}^{(j)} \tag{51}$$

$$\bar{\theta}^{(j)} = -\frac{\alpha_4}{\alpha_5} \nabla^2 \bar{\phi}^{(j)} + \frac{1}{\alpha_5 (s + \tau_1 s^2)} \left(\nabla^2 - \frac{n(s + \tau_1 s^2)}{d}\right) \bar{P}^{(j)}, \tag{52}$$

where

$$A_1 = \frac{(\rho c_E + a T_0 d_1)(s + \tau_0 s^2)}{k} + \frac{n(s + \tau_1 s^2)}{d}, \quad A_2 = (s + \tau_0 s^2)(s + \tau_1 s^2)\left(\frac{(\rho c_E + a T_0 d_1) n}{kd} - \alpha_5 \alpha_2\right), \quad k_6^2 = (s + \tau_0 s^2)\left(\frac{\rho c_E + a T_0 d_1}{k} - \frac{\alpha_5 \alpha_1}{\alpha_4}\right)$$

and k_1^2 , k_2^2 and k_3^2 are the solutions of the characteristic equation given by (33).

For the above choice of body forces, Eq. (34) with $m = 0$ and the Eqs. (47) and (48) with the considerations that $\in_{ilk} Y^{(j)}_{k,li} = 0$ and $\in_{iqp} X^{(j)}_{,iq} = 0$ lead to

$$\bar{X}^{(j)} = -\frac{1}{4\pi}\left(\frac{\delta_{ij}}{r}\right)_{,i} \tag{53}$$

$$\bar{Y}_k^{(j)} = \frac{1}{4\pi} \in_{iqk} \left(\frac{\delta_{qj}}{r}\right)_{,i} \tag{54}$$

By using Eqs. (53) and (54), we obtain the solutions of Eqs. (49)–(52) that are finite at the origin and at infinity as follows:

$$\bar{\phi}^{(j)} = \delta_{ij} \frac{1}{4\pi s^2} \frac{r_{,i}}{r^2} + \delta_{ij} r_{,i} V_1 \tag{55}$$

$$\bar{\psi}_k^{(j)} = \varepsilon_{ijk} \frac{1}{4\pi s^2} \frac{r_{,i}}{r^2} \left[\left(1 + \frac{sr}{c_2}\right) e^{-\frac{sr}{c_2}} - 1\right] \tag{56}$$

$$\bar{P}^{(j)} = \frac{\delta_{ij} r_{,i}}{r^2} V_2 \tag{57}$$

$$\bar{\theta}^{(j)} = \frac{\delta_{ij} r_{,i}}{r^2} V_3, \tag{58}$$

where

$$V_1 = R \sum_{n=1}^{3} b_n \frac{(k_n^4 - k_n^2 A_1 + A_2)}{k_n^2} \frac{(1 + k_n r)}{r^2} e^{-k_n r}$$

$$V_2 = \alpha_4 (s + \tau_1 s^2) R \sum_{n=1}^{3} b_n (k_n^2 - k_6^2)(1 + k_n r) e^{-k_n r}$$

$$V_3 = \alpha_1 (s + \tau_0 s^2) R \sum_{n=1}^{3} b_n (1 + k_n r)(k_n^2 - k_7^2) e^{-k_n r}$$

$$R = \frac{1}{4\pi c_1^2 b_1 b_2 b_3} \text{ and } k_7^2 = (s + \tau_1 s^2)\left(\frac{n}{d} - \frac{\alpha_2 \alpha_4}{\alpha_1}\right)$$

In view of Eqs. (47), (55), and (56), Eq. (47) yields

$$\bar{u}_i = \bar{u}_i^{(j)} = \frac{U_1 \delta_{ij}}{r} - \frac{U_2 r_{,i} r_{,j}}{r}, \tag{59}$$

where

$$U_1 = \frac{1}{4\pi s^2 r^2} \left(1 + \frac{sr}{c_2}\right) e^{-\frac{sr}{c_2}} + V_1$$

$$U_2 = \frac{1}{4\pi s^2} \left(\frac{3}{r^2} + \frac{3s}{c_2 r} + \frac{s^2}{c_2^2}\right) e^{-\frac{sr}{c_2}} + V_4$$

$$V_4 = R \sum_{n=1}^{3} b_n \frac{(k_n^4 - k_n^2 A_1 + A_2)}{k_n^2} \left(\frac{3}{r^2} + \frac{3k_n}{r} + k_n^2\right) e^{-k_n r}$$

From Eq. (59), we obtain

$$\bar{u}_{i,k}^{(j)} = -\frac{\delta_{ij} r_{,k}}{r^2} U_2 - \frac{\delta_{ik} r_{,j} + \delta_{jk} r_{,i}}{r^2} U_2 + \frac{r_{,i} r_{,j} r_{,k}}{r^2} U_3, \tag{60}$$

where

$$U_3 = \frac{1}{4\pi} e^{-sr/c_2} \left\{\frac{1}{c_2^2}\left(6 + \frac{sr}{c_2}\right) + \frac{15}{s^2 r^2}\left(1 + \frac{sr}{c_2}\right)\right\} + R \sum_{n=1}^{3} b_n \frac{(k_n^4 - k_n^2 A_1 + A_2)}{k_n^2} \left\{k_n^2 (6 + k_n r) + \frac{15}{r^2}(1 + k_n r)\right\} e^{-k_n r}$$

and

$$\bar{u}_{i,i}^{(j)} = \frac{\delta_{ij} r_{,i}}{r^2} V_5, \tag{61}$$

where $V_5 = \frac{1}{4\pi c_2^2} e^{-sr/c_2}\left(1 + \frac{sr}{c_2}\right) + R\sum_{n=1}^{3} b_n (k_n^4 - k_n^2 A_1 + A_2)(1 + k_n r) e^{-k_n r}$.

Now, Eqs. (57), (58), (60), and (61) yield the solution for the components of the traction vector in Laplace transform domain in this case as

$$\bar{\sigma}_{ij}^{(j)} n_j = \mu n_j \frac{(r_{,j} + r_{,i})}{r^2}(U_3 - 5U_2) + \frac{n_j r_{,i}}{r^2}[\lambda_0 V_5 - \gamma_1 V_3 - \gamma_2 V_2] \quad (62)$$

By using Eq. (58), we can find the solution for $\bar{\theta}_{,n}^{(j)}$ as

$$\bar{\theta}_{,n}^{(j)} = \bar{\theta}_{,i}^{(j)} n_i = -\frac{n_j}{r^2} V_6, \quad (63)$$

where

$$V_6 = \alpha_1 (s + \tau_0 s^2) R \sum_{n=1}^{3} b_n \left(\frac{2}{r} + 2k_n + k_n^2 r\right)(k_n^2 - k_7^2) e^{-k_n r} \quad (64)$$

Expression for $\bar{P}_{,n}^{(j)}$ can be obtained from Eq. (57) as

$$\bar{P}_{,n}^{(j)} = \bar{P}_{,i}^{(j)} n_i = -\frac{n_j}{r^2} V_7, \quad (65)$$

where

$$V_7 = \alpha_4 (s + \tau_1 s^2) R \sum_{n=1}^{3} b_n (k_n^2 - k_6^2) \left(\frac{2}{r} + 2k_n + k_n^2 r\right) e^{-k_n r}$$

Case III. Now, we assume that an instantaneous source of mass diffusion located at a point y, where $y \in V \cup B$, is acting upon an elastic body in the absence of the body forces and heat sources,

$$\text{i.e.,}\ \sigma = \delta(r)\delta(t),\ F_i = 0,\ Q = 0,\ \text{where}\ r = \sqrt{(x_i - y_i)(x_i - y_i)}. \quad (66)$$

Denoting the corresponding fundamental solutions by $\bar{u}_i^{(2)}, \bar{\theta}^{(2)}, \bar{P}^{(2)}$, the governing Eqs. (21)–(24) under the above assumptions reduce to

$$\mathrm{E}_1^2 \bar{\phi}^{(2)} - m_1 \bar{\theta}^{(2)} - m_2 \bar{P}^{(2)} = 0 \quad (67)$$

$$\mathrm{E}_2^2 \bar{\psi}_i^{(2)} = 0 \quad (68)$$

$$\mathrm{D}_1 \bar{\theta}^{(2)} - \alpha_1 (s + \tau_0 s^2) \nabla^2 \bar{\phi}^{(2)} - \alpha_2 (s + \tau_0 s^2) \bar{P}^{(2)} = 0 \quad (69)$$

$$\alpha_4 (s + \tau_1 s^2) \nabla^2 \bar{\phi}^{(2)} + \alpha_5 (s + \tau_1 s^2) \bar{\theta}^{(2)} - \mathrm{D}_2 \bar{P}^{(2)} = \frac{(1 + \tau_1 s)}{d} \delta(r) \quad (70)$$

For an infinite region, Eq. (68) under the consideration of homogeneous initial conditions implies that

$$\bar{\psi}_i^{(2)} = 0 \quad (71)$$

Therefore, from Eqs. (67), (69), and (70), we obtain

$$(\nabla^2 - k_1^2)(\nabla^2 - k_2^2)(\nabla^2 - k_3^2)\bar{\phi}^{(2)} = -\frac{(s + \tau_1 s^2) m_2}{d}(\nabla^2 - k_8^2)\delta(r) \quad (72)$$

$$(\nabla^2 - k_1^2)(\nabla^2 - k_2^2)(\nabla^2 - k_3^2)\bar{\theta}^{(2)} = -\frac{(s + \tau_1 s^2)(s + \tau_0 s^2)}{d}(m_2 \alpha_1 + \alpha_2)(\nabla^2 - k_9^2)\delta(r), \quad (73)$$

where$k_8^2 = \frac{(s+\tau_0 s^2)}{m_2}\left(\frac{\rho c_E + a T_0 d_1}{k} - m_1 \alpha_2\right)$, $k_9^2 = \frac{s^2 \alpha_2}{c_1^2 (m_2 \alpha_1 + \alpha_2)}$ and k_1^2 , k_2^2 and k_3^2 are the solutions of the same characteristic equation as given by (33).

By using Helmholtz equation (34), we obtain the solution for $\bar{\phi}^{(2)}$ and $\bar{\theta}^{(2)}$ from Eqs. (72) and (73) as

$$\bar{\phi}^{(2)}(r, s) = -\frac{\zeta}{r} \sum_{n=1}^{3} b_n (k_n^2 - k_8^2) e^{-k_n r} \quad (74)$$

$$\bar{\theta}^{(2)}(x, y, s) = -\frac{1}{r} h_1, \quad (75)$$

where $\zeta = \frac{(s+\tau_1 s^2) m_2}{4\pi d b_1 b_2 b_3}$ and $h_1 = \frac{\zeta (s+\tau_0 s^2)(m_2 \alpha_1 + \alpha_2)}{m_2} \sum_{n=1}^{3} b_n (k_n^2 - k_9^2) e^{-k_n r}$.

Therefore, from Eqs. (19), (71), and (74), it follows that

$$\bar{u}_i^{(2)}(x, y, s) = \frac{r_{,i}}{r} h_2, \quad (76)$$

where, $h_2 = \zeta \sum_{n=1}^{3} b_n (k_n^2 - k_8^2)(k_n + \frac{1}{r}) e^{-k_n r}$.

From Eqs. (67), (74), and (75), we obtain

$$\bar{P}^{(2)}(x, y, s) = -\frac{1}{r} h_3, \tag{77}$$

where $h_3 = \frac{\zeta}{m_2} \sum_{n=1}^{3} b_n \left[(k_n^2 - s^2/c_1^2)(k_n^2 - k_8^2) - \frac{m_1}{m_2}(s + \tau_0 s^2)(m_2\alpha_1 + \alpha_2)(k_n^2 - k_9^2) \right] e^{-k_n r}$.

From Eq. (76), we obtain

$$\bar{u}_{i,j}^{(2)} = \frac{h_2}{r^2}(\delta_{ij} - 3r_{,i}r_{,j}) - \frac{r_{,i}r_{,j}h_4}{r}, \tag{78}$$

where $h_4 = \zeta \sum_{n=1}^{3} b_n k_n^2 \left(k_n^2 - k_8^2\right) e^{-k_n r}$.

Therefore, from Eqs. (75), (77), and (78) we get the components of traction vector as

$$\bar{\sigma}_{ij}^{(2)} n_j = \frac{1}{r}\left[2\mu n_j \left\{ \frac{\delta_{ij} h_2}{r} - r_{,i}r_{,j}\left(h_4 + \frac{3h_2}{r} \right) \right\} + n_i\{-\lambda_0 h_4 + \gamma_1 h_1 + \gamma_2 h_3\} \right] \tag{79}$$

Expression for $\bar{\theta}_{,n}^{(2)}$ can be obtained from Eq. (75) as

$$\bar{\theta}_{,n}^{(2)} = \bar{\theta}_{,i}^{(2)} n_i = \frac{r_{,i} n_i}{r} h_5, \tag{80}$$

where

$$h_5 = \frac{\zeta(s + \tau_0 s^2)(m_2\alpha_1 + \alpha_2)}{m_2} \sum_{n=1}^{3} b_n (k_n^2 - k_9^2) \left(k_n + \frac{1}{r} \right) e^{-k_n r} \tag{81}$$

We can find the expression for $\bar{P}_{,n}^{(2)}$ by using Eq. (77) as

$$\bar{P}_{,n}^{(2)} = \bar{P}_{,i}^{(2)} n_i = \frac{r_{,i} n_i}{r} h_6, \tag{82}$$

where

$$h_6 = \frac{\zeta}{m_2} \sum_{n=1}^{3} b_n \left(k_n + \frac{1}{r} \right) \left[(k_n^2 - s^2/c_1^2)(k_n^2 - k_8^2) - \frac{m_1}{m_2}(s + \tau_0 s^2)(m_2\alpha_1 + \alpha_2)(k_n^2 - k_9^2) \right] e^{-k_n r}$$

Summarizing the main results, we give the Green's functions (fundamental solutions) $\bar{u}_i$, $\bar{\theta}$, and $\bar{P}$ obtained for the above three cases as follows:

$$\text{Case I}: \bar{u}_i^{(1)}(x, y, s) = \frac{r_{,i}}{r} g_2, \quad \bar{\theta}^{(1)}(x, y, s) = \frac{g_3}{r}, \quad \bar{P}^{(1)}(x, y, s) = -\frac{g_1}{r}, \tag{83}$$

$$\text{Case II}: \bar{u}_i^{(j)} = \frac{U_1 \delta_{ij}}{r} - \frac{U_2 r_{,i} r_{,j}}{r}, \quad \bar{\theta}^{(j)} = \frac{\delta_{ij} r_{,i}}{r^2} V_3, \quad \bar{P}^{(j)} = \frac{\delta_{ij} r_{,i}}{r^2} V_2, \tag{84}$$

$$\text{Case III}: \bar{u}_i^{(2)}(x, y, s) = \frac{r_{,i}}{r} h_2, \quad \bar{\theta}^{(2)}(x, y, s) = -\frac{1}{r} h_1, \quad \bar{P}^{(2)}(x, y, s) = -\frac{1}{r} h_3, \tag{85}$$

5. Reciprocity relation

In this section, we will establish a reciprocal relation for the linear theory of generalized thermoelastic diffusion which will be employed for the formulation of boundary integral equations. We consider that the body V is subjected to two different systems of thermoelastic loadings

$$\mathcal{L}^{(\beta)} = \left(F_i^{(\beta)}, Q^{(\beta)}, \sigma^{(\beta)}, p_{i0}^{(\beta)}, u_{i0}^{(\beta)}, \theta_0^{(\beta)}, P_0^{(\beta)}, \theta_{,n0}^{(\beta)}, P_{,n0}^{(\beta)} \right), \quad \beta = 1, 2$$

and the two corresponding thermoelastic configurations

$$I^{(\beta)} = (u_i^{(\beta)}, \theta^{(\beta)}, P^{(\beta)}) \quad \beta = 1, 2$$

The reciprocal relation states the relation between these two sets of thermoelastic loadings and the thermoelastic configurations.

Now, Eq. (17) for these two systems can be written as

$$\bar{\sigma}_{ij}^{(1)} = 2\mu \bar{e}_{ij}^{(1)} + \left[\lambda_0 \bar{e}_{kk}^{(1)} - \gamma_1 \bar{\theta}^{(1)} - \gamma_2 \bar{P}^{(1)} \right] \delta_{ij} \tag{86}$$

$$\bar{\sigma}_{ij}^{(2)} = 2\mu\bar{e}_{ij}^{(2)} + \left[\lambda_0\bar{e}_{kk}^{(2)} - \gamma_1\bar{\theta}^{(2)} - \gamma_2\bar{P}^{(2)}\right]\delta_{ij} \tag{87}$$

Multiplying Eq. (86) with $\bar{e}_{ij}^{(2)}$ and Eq. (87) with $\bar{e}_{ij}^{(1)}$ and subtracting the results, we get

$$\bar{\sigma}_{ij}^{(1)}\bar{e}_{ij}^{(2)} - \bar{\sigma}_{ij}^{(2)}\bar{e}_{ij}^{(1)} = \gamma_1(\bar{\theta}^{(2)}\bar{e}_{kk}^{(1)} - \bar{\theta}^{(1)}\bar{e}_{kk}^{(2)}) + \gamma_2(\bar{P}^{(2)}\bar{e}_{kk}^{(1)} - \bar{P}^{(1)}\bar{e}_{kk}^{(2)}) \tag{88}$$

Also applying Laplace transform to Eq. (5) for the two systems, we find

$$\rho\bar{S}^{(1)} = c\bar{\theta}^{(1)} + \gamma_1\bar{e}_{kk}^{(1)} + d_1\bar{P}^{(1)} \tag{89}$$

$$\rho\bar{S}^{(2)} = c\bar{\theta}^{(2)} + \gamma_1\bar{e}_{kk}^{(2)} + d_1\bar{P}^{(2)} \tag{90}$$

Multiplying Eq. (89) with $\bar{\theta}^{(2)}$ and Eq. (90) with $\bar{\theta}^{(1)}$ and then subtracting the results, we find

$$\rho(\bar{S}^{(1)}\bar{\theta}^{(2)} - \bar{S}^{(2)}\bar{\theta}^{(1)}) = \gamma_1(\bar{e}_{kk}^{(1)}\bar{\theta}^{(2)} - \bar{e}_{kk}^{(2)}\bar{\theta}^{(1)}) + d_1(\bar{P}^{(1)}\bar{\theta}^{(2)} - \bar{P}^{(2)}\bar{\theta}^{(1)}) \tag{91}$$

Now, Laplace transforms of Eq. (6) for the two systems yields

$$\bar{C}^{(1)} = \gamma_2\bar{e}_{kk}^{(1)} + d_1\bar{\theta}^{(1)} + n\bar{P}^{(1)} \tag{92}$$

$$\bar{C}^{(2)} = \gamma_2\bar{e}_{kk}^{(2)} + d_1\bar{\theta}^{(2)} + n\bar{P}^{(2)} \tag{93}$$

We multiply Eq. (92) with $\bar{P}^{(2)}$ and (93) with $\bar{P}^{(1)}$ and subtract the results to get

$$\bar{C}^{(1)}\bar{P}^{(2)} - \bar{C}^{(2)}\bar{P}^{(1)} = \gamma_2\left(\bar{e}_{kk}^{(1)}\bar{P}^{(2)} - \bar{e}_{kk}^{(2)}\bar{P}^{(1)}\right) + d_1\left(\bar{\theta}^{(1)}\bar{P}^{(2)} - \bar{\theta}^{(2)}\bar{P}^{(1)}\right) \tag{94}$$

From Eqs. (88), (91), and (94), we obtain

$$\bar{\sigma}_{ij}^{(1)}\bar{e}_{ij}^{(2)} - \rho\bar{S}^{(1)}\bar{\theta}^{(2)} - \bar{C}^{(1)}\bar{P}^{(2)} = \bar{\sigma}_{ij}^{(2)}\bar{e}_{ij}^{(1)} - \rho\bar{S}^{(2)}\bar{\theta}^{(1)} - \bar{C}^{(2)}\bar{P}^{(1)} \tag{95}$$

We introduce the notation

$$L_{\alpha\beta} = \int_V \left[\bar{\sigma}_{ij}^{(\alpha)}\bar{e}_{ij}^{(\beta)} - \rho\bar{S}^{(\alpha)}\bar{\theta}^{(\beta)} - \bar{C}^{(\alpha)}\bar{P}^{(\beta)}\right]dV, \quad \alpha,\beta = 1,2 \tag{96}$$

Now, by applying Laplace transform to relations (1), (4) and (10), (11) we obtain

$$\begin{aligned}\bar{\sigma}_{ij}^{(\alpha)}\bar{e}_{ij}^{(\beta)} - \rho\bar{S}^{(\alpha)}\bar{\theta}^{(\beta)} - \bar{C}^{(\alpha)}\bar{P}^{(\beta)} &= \bar{\sigma}_{ij}^{(\alpha)}\bar{u}_{i,j}^{(\beta)} - \left(-\frac{\bar{q}_{i,i}^{(\alpha)}}{T_0 s} + \frac{\bar{Q}^{(\alpha)}}{T_0 s}\right)\bar{\theta}^{(\beta)} - \left(-\frac{\bar{\eta}_{i,i}^{(\alpha)}}{s} + \frac{\bar{\sigma}^{(\alpha)}}{s}\right)\bar{P}^{(\beta)} \\ &= \left(\bar{\sigma}_{ij}^{(\alpha)}\bar{u}_i^{(\beta)}\right)_{,j} - \rho s^2\bar{u}_i^{(\alpha)}\bar{u}_i^{(\beta)} + \bar{F}_i^{(\alpha)}\bar{u}_i^{(\beta)} - \frac{1}{T_0 s}\bar{Q}^{(\alpha)}\bar{\theta}^{(\beta)} - \frac{k}{T_0 s(1+\tau_0 s)}\left(\bar{\theta}_{,i}^{(\alpha)}\bar{\theta}^{(\beta)}\right)_{,i} \\ &\quad + \frac{k}{T_0 s(1+\tau_0 s)}\bar{\theta}_{,i}^{(\alpha)}\bar{\theta}_{,i}^{(\beta)} - \frac{d}{s(1+\tau_1 s)}\left(\bar{P}_{,i}^{(\alpha)}\bar{P}^{(\beta)}\right)_{,i} + \frac{d}{s(1+\tau_1 s)}\bar{P}_{,i}^{(\alpha)}\bar{P}_{,i}^{(\beta)} - \frac{1}{s}\bar{\sigma}^{(\alpha)}\bar{P}^{(\beta)}\end{aligned} \tag{97}$$

Using Eq. (96) and (97) and employing Gauss divergence theorem, we finally obtain

$$\begin{aligned}\boldsymbol{L}_{\alpha\beta} &= \int_V \left[\bar{F}_i^{(\alpha)}\bar{u}_i^{(\beta)} - \frac{1}{T_0 s}\bar{Q}^{(\alpha)}\bar{\theta}^{(\beta)} - \frac{1}{s}\bar{\sigma}^{(\alpha)}\bar{P}^{(\beta)}\right]dV + \int_{B_1}\bar{p}_{i0}^{(\alpha)}\bar{u}_i^{(\beta)}dA + \int_{B_2}\bar{\sigma}_{ij}^{(\alpha)}\bar{u}_{i0}^{(\beta)}n_j dA \\ &\quad - \frac{k}{T_0 s(1+\tau_0 s)}\left[\int_{B_3}\bar{\theta}_{,n}^{(\alpha)}\bar{\theta}_0^{(\beta)}dA + \int_{B_4}\bar{\theta}_{,n0}^{(\alpha)}\bar{\theta}^{(\beta)}dA\right] - \frac{d}{s(1+\tau_1 s)}\left[\int_{B_5}\bar{P}_{,n}^{(\alpha)}\bar{P}_0^{(\beta)}dA + \int_{B_6}\bar{P}_{,n0}^{(\alpha)}\bar{P}^{(\beta)}dA\right] \\ &\quad - \int_V \left[\rho s^2\bar{u}_i^{(\alpha)}\bar{u}_i^{(\beta)} - \frac{k}{T_0 s(1+\tau_0 s)}\bar{\theta}_{,i}^{(\alpha)}\bar{\theta}_{,i}^{(\beta)} - \frac{d}{s(1+\tau_1 s)}\bar{P}_{,i}^{(\alpha)}\bar{P}_{,i}^{(\beta)}\right]dV\end{aligned} \tag{98}$$

Now, in view of Eqs. (95) and (98), we find that

$$\boldsymbol{L}_{12} = \boldsymbol{L}_{21} \tag{99}$$

Therefore, we find that Eqs. (98) and (99) yield the following reciprocity relation of Betti type for the generalized thermoelastic diffusion in the Laplace transform domain:

$$T_0s(1+\tau_0s)(1+\tau_1s)\int_V\left[\bar{F}_i^{(1)}\bar{u}_i^{(2)}-\bar{F}_i^{(2)}\bar{u}_i^{(1)}\right]dV-(1+\tau_0s)(1+\tau_1s)\int_V\left[\bar{Q}^{(1)}\bar{\theta}^{(2)}-\bar{Q}^{(2)}\bar{\theta}^{(1)}\right]dV-T_0(1+\tau_0s)$$
$$(1+\tau_1s)\int_V\left[\bar{\sigma}^{(1)}\bar{P}^{(2)}-\bar{\sigma}^{(2)}\bar{P}^{(1)}\right]dV+T_0s(1+\tau_0s)(1+\tau_1s)\left[\int_{B_1}[\bar{p}_{i0}^{(1)}\bar{u}_i^{(2)}-\bar{p}_{i0}^{(2)}\bar{u}_i^{(1)}]dA+\int_{B_2}[\bar{\sigma}_{ij}^{(1)}\bar{u}_{i0}^{(2)}-\bar{\sigma}_{ij}^{(2)}\bar{u}_{i0}^{(1)}]n_jdA\right]$$
$$-k(1+\tau_1s)\left[\int_{B_3}\left[\bar{\theta}_{,n}^{(1)}\bar{\theta}_0^{(2)}-\bar{\theta}_{,n}^{(2)}\bar{\theta}_0^{(1)}\right]dA+\int_{B_4}\left[\bar{\theta}_{,n0}^{(1)}\bar{\theta}^{(2)}-\bar{\theta}_{,n0}^{(2)}\bar{\theta}^{(1)}\right]dA\right]-dT_0(1+\tau_0s)\left[\int_{B_5}\left[\bar{P}_{,n}^{(1)}\bar{P}_0^{(2)}-\bar{P}_{,n}^{(2)}\bar{P}_0^{(1)}\right]dA\right.$$
$$\left.+\int_{B_6}[\bar{P}_{,n0}^{(1)}\bar{P}^{(2)}-\bar{P}_{,n0}^{(2)}\bar{P}^{(1)}]dA\right]=0 \tag{100}$$

For an infinite thermoelastic medium the surface integrals will be absent. Therefore, Eq. (100) reduces to

$$T_0s\int_V\left[\bar{F}_i^{(1)}\bar{u}_i^{(2)}-\bar{F}_i^{(2)}\bar{u}_i^{(1)}\right]dV=\int_V\left[\bar{Q}^{(1)}\bar{\theta}^{(2)}-\bar{Q}^{(2)}\bar{\theta}^{(1)}\right]dV+T_0\int_V\left[\bar{\sigma}^{(1)}\bar{P}^{(2)}-\bar{\sigma}^{(2)}\bar{P}^{(1)}\right]dV \tag{101}$$

6. Formulation of boundary integral equations

In order to obtain the integral representation of the transformed displacement, temperature and chemical potential inside the bounded region V in terms of the prescribed functions $\bar{u}_{i0},\bar{\theta}_0,\bar{P}_0,\bar{p}_{i0},\bar{\theta}_{,n0}$ and $\bar{P}_{,n0}$ on the surface B, the Green's functions $\bar{u}_i^{(1)},\bar{\theta}^{(1)},\bar{P}^{(1)},\bar{u}_i^{(j)},\bar{\theta}^{(j)},\bar{P}^{(j)},\bar{u}_i^{(2)},\bar{\theta}^{(2)},\bar{P}^{(2)}$ in infinite region and the values $\bar{u}_{i0}^{(1)}$, $\bar{\theta}_0^{(1)}$, $\bar{P}_0^{(1)}$, $\bar{p}_{i0}^{(1)}$, $\bar{\theta}_{,n0}^{(1)}$, $\bar{P}_{,n0}^{(1)}$, $\bar{u}_{i0}^{(j)},\bar{\theta}_0^{(j)},\bar{P}_0^{(j)},\bar{p}_{i0}^{(j)},\bar{\theta}_{,n0}^{(j)},\bar{P}_{,n0}^{(j)}$, $\bar{u}_{i0}^{(2)},\bar{\theta}_0^{(2)},\bar{P}_0^{(2)},\bar{p}_{i0}^{(2)},\bar{\theta}_{,n0}^{(2)},P_{,n0}^{(2)}$ on the same surface B, we consider the following cases:

First, we take $\bar{F}_i^{(1)}=0$, $\bar{\sigma}^{(1)}=0$ and $\bar{Q}^{(1)}=\delta(r)$,
where $r=\sqrt{(x_i-y_i)(x_i-y_i)}$ and $y\in V\cup B$. Thus Eq. (100) gives

$$(1+\tau_0s)(1+\tau_1s)\Delta(x)\bar{\theta}(x,s)=(1+\tau_0s)(1+\tau_1s)\int_V\bar{Q}\bar{\theta}^{(1)}dV+T_0$$
$$\times(1+\tau_0s)(1+\tau_1s)\int_V\bar{\sigma}\bar{P}^{(1)}dV-T_0s(1+\tau_0s)(1+\tau_1s)\int_V\bar{F}_i\bar{u}_i^{(1)}dV+T_0s$$
$$\times(1+\tau_0s)(1+\tau_1s)\left[\int_{B_1}\left[\bar{p}_{i0}^{(1)}\bar{u}_i-\bar{p}_{i0}\bar{u}_i^{(1)}\right]dA+\int_{B_2}\left[\bar{\sigma}_{ij}^{(1)}\bar{u}_{i0}-\bar{\sigma}_{ij}\bar{u}_{i0}^{(1)}\right]n_jdA\right]$$
$$-k(1+\tau_1s)\left[\int_{B_3}\left[\bar{\theta}_{,n}^{(1)}\bar{\theta}_0-\bar{\theta}_{,n}\bar{\theta}_0^{(1)}\right]dA+\int_{B_4}\left[\bar{\theta}_{,n0}^{(1)}\bar{\theta}-\bar{\theta}_{,n0}\bar{\theta}^{(1)}\right]dA\right]-dT_0$$
$$\times(1+\tau_0s)\left[\int_{B_5}\left[\bar{P}_{,n}^{(1)}\bar{P}_0-\bar{P}_{,n}\bar{P}_0^{(1)}\right]dA+\int_{B_6}\left[\bar{P}_{,n0}^{(1)}\bar{P}-\bar{P}_{,n0}\bar{P}^{(1)}\right]dA\right], \tag{102}$$

where

$$\int_D\delta(x-y)dV(y)=\Delta(x)=\begin{cases}1, & \text{if } x\in V\\ 0, & \text{if } x\notin V\cup B\\ \frac{1}{2}, & \text{if } x\in B\end{cases} \tag{103}$$

and $\bar{u}_i^{(1)},\bar{\theta}^{(1)},\bar{P}^{(1)}$ are the fundamental solutions as obtained in case I in Section 4.

Now, we take $\bar{F}_i^{(j)}=\delta_{ij}\delta(r)$, $\bar{Q}^{(j)}=0$, $\bar{\sigma}^{(j)}=0$ and consider the corresponding fundamental solutions $\bar{u}_i^{(j)},\bar{\theta}^{(j)},\bar{P}^{(j)}$ as found in case II to obtain

$$T_0s(1+\tau_0s)(1+\tau_1s)\Delta(x)\bar{u}_j(x,s)=T_0s(1+\tau_0s)(1+\tau_1s)\int_V\bar{F}_i\bar{u}_i^{(j)}dV-(1+\tau_0s)(1+\tau_1s)\int_V\bar{Q}\bar{\theta}^{(j)}dV$$
$$-T_0(1+\tau_0s)(1+\tau_1s)\int_V\bar{\sigma}\bar{P}^{(j)}dV-T_0s(1+\tau_0s)(1+\tau_1s)\left[\int_{B_1}\left[\bar{p}_{i0}^{(j)}\bar{u}_i-\bar{p}_{i0}\bar{u}_i^{(j)}\right]dA+\int_{B_2}\left[\bar{\sigma}_{ij}^{(j)}\bar{u}_{i0}-\bar{\sigma}_{ij}\bar{u}_{i0}^{(j)}\right]n_jdA\right]+k$$
$$\times(1+\tau_1s)\left[\int_{B_3}\left[\bar{\theta}_{,n}^{(j)}\bar{\theta}_0-\bar{\theta}_{,n}\bar{\theta}_0^{(j)}\right]dA+\int_{B_4}\left[\bar{\theta}_{,n0}^{(j)}\bar{\theta}-\bar{\theta}_{,n0}\bar{\theta}^{(j)}\right]dA\right]+dT_0$$
$$\times(1+\tau_0s)\left[\int_{B_5}\left[\bar{P}_{,n}^{(j)}\bar{P}_0-\bar{P}_{,n}\bar{P}_0^{(j)}\right]dA+\int_{B_6}\left[\bar{P}_{,n0}^{(j)}\bar{P}-\bar{P}_{,n0}\bar{P}^{(j)}\right]dA\right] \tag{104}$$

Next, by assuming $\bar{F}_i^{(2)}=0$, $\bar{Q}^{(2)}=0$ and $\bar{\sigma}^{(2)}=\delta(r)$, and considering the corresponding fundamental solutions $\bar{u}_i^{(2)},\bar{\theta}^{(2)},\bar{P}^{(2)}$ as described in case III, we obtain

$$T_0(1+\tau_0 s)(1+\tau_1 s)\Delta(x)\bar{P}(x,s) = (1+\tau_0 s)(1+\tau_1 s)\int_V \bar{Q}\bar{\theta}^{(2)}dV + T_0(1+\tau_0 s)$$
$$\times (1+\tau_1 s)\int_V \bar{\sigma}\bar{P}^{(2)}dV - T_0 s(1+\tau_0 s)(1+\tau_1 s)\int_V \bar{F}_i\bar{u}_i^{(2)}dV + T_0 s(1+\tau_0 s)$$
$$\times (1+\tau_1 s)\left[\int_{B_1}\left[\bar{p}_{i0}^{(2)}\bar{u}_i - \bar{p}_{i0}\bar{u}_i^{(2)}\right]dA + \int_{B_2}\left[\bar{\sigma}_{ij}^{(2)}\bar{u}_{i0} - \bar{\sigma}_{ij}\bar{u}_{i0}^{(2)}\right]dA\right] - k$$
$$\times (1+\tau_1 s)\left[\int_{B_3}\left[\bar{\theta}_{,n}^{(2)}\bar{\theta}_0 - \bar{\theta}_{,n}\bar{\theta}_0^{(2)}\right]dA + \int_{B_4}\left[\bar{\theta}_{,n0}^{(2)}\bar{\theta} - \bar{\theta}_{,n0}\bar{\theta}^{(2)}\right]dA\right] - dT_0$$
$$\times (1+\tau_0 s)\left[\int_{B_5}\left[\bar{P}_{,n}^{(2)}\bar{P}_0 - \bar{P}_{,n}\bar{P}_0^{(2)}\right]dA + \int_{B_6}\left[\bar{P}_{,n0}^{(2)}\bar{P} - \bar{P}_{,n0}\bar{P}^{(2)}\right]dA\right] \tag{105}$$

Inverting the Laplace transforms of Eqs. (102), (104), and (105) by using the convolution theorem of Laplace transform, we get

$$\Delta(x)L_1\theta(x,t) = T_0 W_1(x,t) \tag{106}$$

$$\Delta(x)L_1 u_j(x,t) = W_2(x,t) \tag{107}$$

$$\Delta(x)L_1 P(x,t) = W_3(x,t) \tag{108}$$

where

$$W_1(x,t) = \frac{1}{T_0}\int_0^t\int_V Q(y,t-\tau)L_1\left[\theta^{(1)}(y,x,\tau)\right]dV(y)\,d\tau - \int_0^t\int_V F_i(y,t-\tau)\frac{\partial\left[L_1(u_i^{(1)}(y,x,\tau))\right]}{\partial\tau}dV(y)\,d\tau$$
$$+\int_0^t\int_V \sigma(y,t-\tau)L_1\left[P^{(1)}(y,x,\tau)\right]dV(y)\,d\tau + \int_0^t$$
$$\times\int_{B_1}\left\{u_i(y,t-\tau)\frac{\partial\left[L_1(p_{i0}^{(1)}(y,x,\tau))\right]}{\partial\tau} - p_{i0}(y,t-\tau)\frac{\partial\left[L_1(u_i^{(1)}(y,x,\tau))\right]}{\partial\tau}\right\}dA(y)\,d\tau + \int_0^t$$
$$\times\int_{B_2}\left\{u_{i0}(y,t-\tau)\frac{\partial\left[L_1(\sigma_{ij}^{(1)}(y,x,\tau))\right]}{\partial\tau} - \sigma_{ij}(y,t-\tau)\frac{\partial\left[L_1(u_{i0}^{(1)}(y,x,\tau))\right]}{\partial\tau}\right\}n_j dA(y)d\tau$$
$$-\frac{k}{T_0}\int_0^t\int_{B_3}\left\{\theta_0(y,t-\tau)L_2\left[\theta_{,n}^{(1)}(y,x,\tau)\right] - \theta_{,n}(y,t-\tau)L_2\left[\theta_0^{(1)}(y,x,\tau)\right]\right\}dA(y)\,d\tau$$
$$-\frac{k}{T_0}\int_0^t\int_{B_4}\left\{\theta(y,t-\tau)L_2\left[\theta_{,n0}^{(1)}(y,x,\tau)\right] - \theta_{,n0}(y,t-\tau)L_2\left[\theta^{(1)}(y,x,\tau)\right]\right\}dA(y)\,d\tau - d\int_0^t$$
$$\times\int_{B_5}\left\{P_0(y,t-\tau)L_3\left[P_{,n}^{(1)}(y,x,\tau)\right] - P_{,n}(y,t-\tau)L_3\left[P_0^{(1)}(y,x,\tau)\right]\right\}dA(y)\,d\tau - d\int_0^t$$
$$\times\int_{B_6}\left\{P(y,t-\tau)L_3\left[P_{,n0}^{(1)}(y,x,\tau)\right] - P_{,n0}(y,t-\tau)L_3\left[P^{(1)}(y,x,\tau)\right]\right\}dA(y)\,d\tau$$

$$W_2(x,t) = -\frac{1}{T_0}\int_0^t\int_V Q(y,t-\tau)L_1^*\left[\theta^{(j)}(y,x,\tau)\right]dV(y)\,d\tau + \int_0^t\int_V F_i(y,t-\tau)L_1\left[u_i^{(j)}(y,x,\tau)\right]dV(y)\,d\tau$$
$$-\int_0^t\int_V \sigma(y,t-\tau)L_1^*\left[P^{(j)}(y,x,\tau)\right]dV(y)\,d\tau - \int_0^t$$
$$\times\int_{B_1}\left\{u_i(y,t-\tau)L_1\left[p_{i0}^{(j)}(y,x,\tau)\right] - p_{i0}(y,t-\tau)L_1\left[u_i^{(j)}(y,x,\tau)\right]\right\}dA(y)\,d\tau - \int_0^t$$
$$\times\int_{B_2}\left\{u_{i0}(y,t-\tau)L_1\left[\sigma_{ij}^{(j)}(y,x,\tau)\right]n_j - \sigma_{ij}(y,t-\tau)n_j L_1\left[u_{i0}^{(j)}(y,x,\tau)\right]\right\}dA(y)\,d\tau + \frac{k}{T_0}\int_0^t$$
$$\times\int_{B_3}\left\{\theta_0(y,t-\tau)L_2^*\left[\theta_{,n}^{(j)}(y,x,\tau)\right] - \theta_{,n}(y,t-\tau)L_2^*\left[\theta_0^{(j)}(y,x,\tau)\right]\right\}dA(y)\,d\tau + \frac{k}{T_0}\int_0^t$$
$$\times\int_{B_4}\left\{\theta(y,t-\tau)L_2^*\left[\theta_{,n0}^{(j)}(y,x,\tau)\right] - \theta_{,n0}(y,t-\tau)L_2^*\left[\theta^{(j)}(y,x,\tau)\right]\right\}dA(y)\,d\tau + d\int_0^t$$
$$\times\int_{B_5}\left\{P_0(y,t-\tau)L_3^*\left[P_{,n}^{(j)}(y,x,\tau)\right] - P_{,n}(y,t-\tau)L_3^*\left[P_0^{(j)}(y,x,\tau)\right]\right\}dA(y)\,d\tau + d\int_0^t$$
$$\times\int_{B_6}\left\{P(y,t-\tau)L_3^*\left[P_{,n0}^{(j)}(y,x,\tau)\right] - P_{,n0}(y,t-\tau)L_3^*\left[P^{(j)}(y,x,\tau)\right]\right\}dA(y)\,d\tau$$

$$W_3(x,t) = \frac{1}{T_0}\int_0^t \int_V Q(y,t-\tau)L_1\left[\theta^{(2)}(y,x,\tau)\right]dV(y)\,d\tau - \int_0^t \int_V F_i(y,t-\tau)\frac{\partial\left[L_1(u_i^{(2)}(y,x,\tau))\right]}{\partial\tau}dV(y)\,d\tau$$
$$+\int_0^t \int_V \sigma(y,t-\tau)L_1\left[P^{(2)}(y,x,\tau)\right]dV(y)\,d\tau + \int_0^t$$
$$\times\int_{B_1}\left\{u_i(y,t-\tau)\frac{\partial\left[L_1(p_{i0}^{(2)}(y,x,\tau))\right]}{\partial\tau} - p_{i0}(y,t-\tau)\frac{\partial\left[L_1(u_i^{(2)}(y,x,\tau))\right]}{\partial\tau}\right\}dA(y)\,d\tau + \int_0^t$$
$$\times\int_{B_2}\left\{u_{i0}(y,t-\tau)\frac{\partial\left[L_1(\sigma_{ij}^{(2)}(y,x,\tau))\right]}{\partial\tau}n_j - \sigma_{ij}(y,t-\tau)n_j\frac{\partial\left[L_1(u_{i0}^{(2)}(y,x,\tau))\right]}{\partial\tau}\right\}dA(y)\,d\tau - \frac{k}{T_0}\int_0^t$$
$$\times\int_{B_3}\left\{\theta_0(y,t-\tau)L_2\left[\theta_{,n}^{(2)}(y,x,\tau)\right] - \theta_{,n}(y,t-\tau)L_2\left[\theta_0^{(2)}(y,x,\tau)\right]\right\}dA(y)\,d\tau - \frac{k}{T_0}\int_0^t$$
$$\times\int_{B_4}\left\{\theta(y,t-\tau)L_2\left[\theta_{,n0}^{(2)}(y,x,\tau)\right] - \theta_{,n0}(y,t-\tau)L_2\left[\theta^{(2)}(y,x,\tau)\right]\right\}dA(y)\,d\tau - d\int_0^t$$
$$\times\int_{B_5}\left\{P_0(y,t-\tau)L_3\left[P_{,n}^{(2)}(y,x,\tau)\right] - P_{,n}(y,t-\tau)L_3\left[P_0^{(2)}(y,x,\tau)\right]\right\}dA(y)\,d\tau - d\int_0^t$$
$$\times\int_{B_6}\left\{P(y,t-\tau)L_3\left[P_{,n0}^{(2)}(y,x,\tau)\right] - P_{,n0}(y,t-\tau)L_3\left[P^{(2)}(y,x,\tau)\right]\right\}dA(y)\,d\tau$$

In the above expressions, we used the following notations:

$$L_1[f] = \left[1+(\tau_0+\tau_1)\frac{\partial}{\partial t}\right]f,\quad L_2[f] = \left[1+\tau_1\frac{\partial}{\partial t}\right]f,\quad L_3[f] = \left[1+\tau_0\frac{\partial}{\partial t}\right]f,$$
$$L_1^*[f] = \int_0^t f(y,x,\tau)\,d\tau + (\tau_0+\tau_1)f(y,x,t),\quad L_2^*[f] = \int_0^t f(y,x,\tau)\,d\tau + \tau_1 f(y,x,t) \text{ and } L_3^*[f] = \int_0^t f(y,x,\tau)\,d\tau + \tau_0 f(y,x,t)$$

The solution of Eqs. (106)–(108) can be written as:

$$\Delta(x)\theta(x,t) = \frac{T_0}{\tau_0+\tau_1}e^{-t/(\tau_0+\tau_1)}\int_0^t e^{\tau/(\tau_0+\tau_1)}W_1(x,\tau)\,d\tau \tag{109}$$

$$\Delta(x)u_j(x,t) = \frac{1}{\tau_0+\tau_1}e^{-t/(\tau_0+\tau_1)}\int_0^t e^{\tau/(\tau_0+\tau_1)}W_2(x,\tau)\,d\tau \tag{110}$$

$$\Delta(x)P(x,t) = \frac{1}{\tau_0+\tau_1}e^{-t/(\tau_0+\tau_1)}\int_0^t e^{\tau/(\tau_0+\tau_1)}W_3(x,\tau)\,d\tau \tag{111}$$

Now, letting $x \to \xi, \xi \in B$ and using (103) in Eqs. (109)–(111), we finally obtain

$$\theta(\xi,t) = \frac{2T_0}{\tau_0+\tau_1}e^{-t/(\tau_0+\tau_1)}\int_0^t e^{\tau/(\tau_0+\tau_1)}W_1(\xi,\tau)\,d\tau \tag{112a}$$

$$u_j(\xi,t) = \frac{2}{\tau_0+\tau_1}e^{-t/(\tau_0+\tau_1)}\int_0^t e^{\tau/(\tau_0+\tau_1)}W_2(\xi,\tau)\,d\tau \tag{112b}$$

$$P(\xi,t) = \frac{2}{\tau_0+\tau_1}e^{-t/(\tau_0+\tau_1)}\int_0^t e^{\tau/(\tau_0+\tau_1)}W_3(\xi,\tau)\,d\tau \tag{112c}$$

Eq. 112(a–c) together with the boundary conditions 12(a–f) and the limiting behavior of the solutions can be used to set up a linear system of equations for the boundary integral equation method (BIEM). This completes our formulation.

7. Conclusion

The Green functions (fundamental solutions) are derived for linear theory of generalized thermoelastic diffusion for homogenous and isotropic elastic medium. By deriving a reciprocal relation, the direct formulation of boundary integral equations for an initial, mixed boundary value problem is carried out. The authors believe that this formulation could be helpful for the solution of problems in the present theory by using boundary element method.

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