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A Mathematical Model for Hyperbolic Space-Fractional Bioheat Transfer during Thermal Therapy

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Abstract

In this paper, we have developed a hyperbolic space-fractional bioheat transfer model based on the single-phase-lagging constitutive relation. The numerical solution of the present problem has been done using fractional backward finite difference scheme and Legendre wavelet Galerkin approach. The effect of fractional parabolic and hyperbolic bioheat transfer model on temperature profile within living biological tissues has been studied and compared with their respective standard cases. Numerical results are presented graphically in both standard and anomalous case for different values of parameters.

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Keywords: Hyperbolic fractional bioheat model; Thermal therapy; Fractional finite difference scheme; Legendre wavelet Galerkin method.

1. Introduction

Fractional calculus and fractals have found applications in improving the modelling accuracy of many phenomena in natural sciences. The process of development of models based on fractional order differential system has gained lot of attention due to its greater degree of freedom in the model and accurate explanations of physical phenomena. Fractional derivatives are non-local in nature as compared to local behaviour of integer derivatives. Fractional space derivatives are used to model anomalous diffusion or dispersion. There are some diffusion processes in nature for which the Fick's second law fails to describe the related transport behaviour. This phenomena is called anomalous behaviour, it differs according to the order of fractional derivative p.

The success of thermal therapy in the treatment of metastatic cancerous cell depends on the precise prediction and control of temperature. As a result, several technical areas related to the measurement, generation and modelling of

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heat inside living biological tissues came under intense studies. The Pennes bioheat transfer model [1] is used most commonly for the prediction of thermal data. The conduction term in this model is based on the classical Fourier's law:

$$q(r,t) = -k\nabla T(r,t)\,, \tag{1}$$

which implies unphysical infinite propagation speed of thermal disturbance. In order to overcome this unphysical behaviour, a modified constitutive relation was proposed by Cattaneo [2] and Vernotte [3] independently, by introducing a phase lag time in Fourier's law:

$$q(r,t+\tau) = -k\nabla T(r,t)\big[0,1\big]\,, \tag{2}$$

which is most widely accepted. The phase lag time τ captures the micro-scale responses in time.

Numerical solution of fractional bioheat transfer equation has been done by many researchers. Singh et al., [4] done the solution of fractional bioheat equation by finite difference and homotopy perturbation method. Damor et al., [5] solve the space-fractional bioheat transfer equation using finite difference scheme. Jiang and Qi [6] analytically solved the thermal wave model of fractional bioheat transfer equation. Wavelet based numerical methods are recently developed tool in applied mathematics. It reduces the computational complexity and localizes small scale variations of solutions. Recently, Kumar et al., [7,8] studied the dual-phase-lag model of bioheat transfer using finite element wavelet Galerkin method.

In this paper, a hyperbolic space-fractional bioheat transfer model has been developed. We use fractional backward finite difference scheme and wavelet Galerkin method to solve the present problem. The dimensionless temperature profiles are obtained for parabolic and hyperbolic bioheat transfer model for different order of fractional derivative *p*. Fractional derivative of order *p* is used in the Caputo sense and defined for m to be the smallest integer that just exceeds *p* as:

$$\frac{\partial^p f(x,t)}{\partial x^p} = \frac{1}{\Gamma(m-p)}\int (x-\sigma)^{m-p-1}\frac{\partial^m f(x,\sigma)}{\partial \sigma^m}d\sigma, \quad \text{for } m\text{-}1<p<m \tag{3}$$

Details about the fractional derivatives and their properties can be found in the book by Podlubny [9].

2. Mathematical formulation

In this paper, the body tissue which is initially at a constant temperature $T_o = 37^o C$ is heated by electromagnetic radiation using 1-D Cartesian coordinate and constant temperature boundary condition. The single-phase-lagging constitutive relation is used as:

$$q(r,t) + \tau\frac{\partial q(r,t)}{\partial t} = -k\nabla T(r,t). \tag{4}$$

The one-dimensional energy equation of the present problem is

$$\rho c\frac{\partial T}{\partial t} = -\frac{\partial^\alpha q}{\partial r^\alpha} + \omega_b c_b(T_b - T) + Q_m + Q_r\,, \tag{5}$$

where ρ, c, T, t, r represents density, specific heat, temperature, time and distance respectively; α is the space fractional derivative; the subscripts *t* and *b* are for the tissue and blood respectively. ω_b is the blood perfusion rate, Q_m is the metabolic heat generation which is assumed to be linear function of temperature in the present problem and is defined as:

$$Q_m = Q_{mo}\left(1+\frac{(T-T_o)}{10}\right). \tag{6}$$

Q_r is the external heat source due to electromagnetic radiation and is defined as:

$$Q_r = \rho SP\exp(a(\overline{r}-0.01)), \tag{7}$$

where S and a are the antenna constants, P is the transmitted power which may be varied according to the requirement.

Now eliminating q from (4) and (5), we get the hyperbolic space-fractional bioheat transfer equation as follows:

$$\rho c\tau\frac{\partial^2 T}{\partial t^2}+\left(\rho c+\omega_b c_b\tau-\frac{Q_{mo}}{10}\right)\frac{\partial T}{\partial t}+\left(\omega_b c_b-\frac{Q_{mo}}{10}\right)T=k\frac{\partial^p T}{\partial r^p}+\omega_b c_b T_b+Q_{mo}\left(1-\frac{T_o}{10}\right)+\rho SPe^{a(\bar{r}-0.01)},$$

where $1<p=2\alpha\le 2$. (8)

With initial condition as

$$T(r,o)=T_o,\quad \frac{\partial T(r,0)}{\partial t}=0 \qquad . \tag{9}$$

Boundary condition

$$T(R,t)=T_w, \tag{10}$$

and symmetric condition as

$$\frac{\partial T(0,t)}{\partial r}=0\ , \tag{11}$$

where T_w is the tissue wall temperature. Eqn. (8) reduces to fractional parabolic bioheat equation by setting $\tau=0$.

3. Solution of the Problem

Introducing the dimensionless variables and similarity criteria as

$$x=\frac{r}{R},F_o=\frac{kt}{\rho cR^2},F_{o\tau}=\frac{k\tau}{\rho cR^2},\theta=\frac{T-T_o}{T_o},\theta_b=\frac{T_b-T_o}{T_o},\theta_w=\frac{T_w-T_o}{T_o}$$

$$P_{mo}=\frac{Q_m}{kT_o}R^2,P_{ro}=\frac{\rho SP}{kT_o}R^2,P_f=\sqrt{\frac{\omega_b c_b}{k}}R,a^*=aR,x^*=\frac{R-0.01}{R}. \tag{12}$$

The system of Eqns.(8)-(11) reduces in the following form

$$F_{o\tau}\frac{\partial^2\theta}{\partial F_o^2}+\left(1+P_f^2F_{o\tau}-P_{mo}F_{o\tau}d\right)\frac{\partial\theta}{\partial F_o}+\left(P_f^2-P_{mo}d\right)\theta=\frac{\partial^p\theta}{\partial x^p}+P_f^2\theta_b+P_{mo}+P_{ro}e^{-a^*(x-x^*)}, \tag{13}$$

With initial condition as

$$\theta(x,0)=0,\frac{\partial\theta(x,0)}{\partial F_o}=0 \tag{14}$$

boundary condition

$$\theta(1,F_o)=\theta_w, \tag{15}$$

and symmetric condition as

$$\frac{\partial\theta(0,F_o)}{\partial x}=0. \tag{16}$$

In order to establish the numerical approximation scheme, let $x_i=ih,(i=0,1,2...n)$ be the space direction and $0\le x_i\le 1,h=\frac{1}{n}>0$ is the grid size. The value of θ at grid point x_i is denoted by $\theta_i(F_o)$. We take the shifted-Grunwald backward finite difference scheme for space fractional derivative $\frac{\partial^p\theta(x,F_o)}{\partial x^p}$ as

$$\frac{\partial^p\theta(x,F_o)}{\partial x^p}=\frac{1}{h^p}\sum_{j=0}^{i}g_{\ j}\,\theta(x-jh,F_o)+o(h) \tag{17}$$

Where $g_o=1,g_1=-p,g_j=(-1)^jF[p,j]$ and it is defined as

$$F[p,j]=\frac{\Gamma(j+1)}{\Gamma(p+1)\Gamma(j+1-p)}\ ,\ j=1,2,3...n. \tag{18}$$

Applying eqn. (17) in Eqns. (13)-(16), we can write the eqns. by taking *n = 10, h = 0.1* in the vector matrix form as

$$F_{o\tau}\frac{d^2\theta}{dF_o^2}+\left(1+P_f^2F_{o\tau}-P_{mo}F_{o\tau}d\right)\frac{d\theta}{dF_o}=A\theta+N, \tag{19}$$

Where

$\theta(x)=\left[\theta_1,\theta_2,\theta_3,\theta_4,\theta_5,\theta_6,\theta_7,\theta_8,\theta_9\right]^T$,

$N=\left[P_f^2\theta_b+P_{mo}+P_{ro}e^{-a^*(x_i-x^*)}\right]^T$,

$$A=-\begin{bmatrix} a_{0,1}^{11}+c & F_{12}\times F_1 & F_{13}\times F_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{1,2}^{11} & a_{0,2}^{12}+c & F_{13}\times F_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{2,3}^{11} & a_{1,3}^{12} & a_{0,3}^{13}+c & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{3,4}^{11} & a_{2,4}^{12} & a_{1,4}^{13} & c+a_0 & 0 & 0 & 0 & 0 & 0 \\ a_{4,5}^{11} & a_{3,5}^{12} & a_{2,5}^{13} & a_1 & c+a_0 & 0 & 0 & 0 & 0 \\ a_{5,6}^{11} & a_{4,6}^{12} & a_{3,6}^{13} & a_2 & a_1 & c+a_0 & 0 & 0 & 0 \\ a_{6,7}^{11} & a_{5,7}^{12} & a_{4,7}^{13} & a_3 & a_2 & a_1 & c+a_0 & 0 & 0 \\ a_{7,8}^{11} & a_{6,8}^{12} & a_{5,8}^{13} & a_4 & a_3 & a_2 & a_1 & c+a_0 & 0 \\ a_{8,9}^{11} & a_{7,9}^{12} & a_{6,9}^{13} & a_5 & a_4 & a_3 & a_2 & a_1 & c+a_0 \end{bmatrix},$$

Where $F_i=(-1)^i F[p,i], F_{11}=\frac{13}{21h^p}, F_{12}=\frac{17}{21h^p}, F_{13}=\frac{3}{7h^p}, a_j=\frac{F_j}{h^p}, a_{j,i}^{1k}=a_j+F_{1i}\times F_k, k=1,2,3;\ j=0,1,2,...,8; i=1,2,3,...,9.$

3.1 Legendre Wavelet Galerkin Method

Let us assume that $\frac{d^2\theta(F_o)}{dF_o^2}=C^T\psi(F_o)$, (20)

where $C^T\psi(\mathrm{F_o})=\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{n,m}\,\psi_{\mathrm{n,m}}(\mathrm{F_o})\approx f(\mathrm{F_o})$ and

$c_{n,m}=\int_0^1 f(\mathrm{F_o})\psi_{\mathrm{n,m}}(\mathrm{F_o})\mathrm{dF_o}$. (21)

C and $\psi(\mathrm{F_o})$ are $2^{k-1}M\times 1$ matrices given by

$C=[c_{1,0},c_{1,1},\ldots,c_{1,M-1},c_{2,0},c_{2,1}\ldots,c_{2,M-1},c_{2^{k-1},0}c_{2^{k-1},1},\ldots,c_{2^{k-1},M-1}]$ and (22)

$\psi(\mathrm{F_o})=[\psi_{1,0}(\mathrm{F_o}),\psi_{1,1}(\mathrm{F_o}),\ldots,\psi_{1,M-1}(\mathrm{F_o}),\psi_{2,0}(\mathrm{F_o}),\ldots,\psi_{2,M-1}(\mathrm{F_o}),\ldots,\psi_{2^{k-1},0}(\mathrm{F_o}),$
$\psi_{2^{k-1},1}(\mathrm{F_o}),\ldots,\psi_{2^{k-1},M-1}(\mathrm{F_o})]$ (23)

The Legendre wavelets [10]

$$\psi_{\mathrm{n,m}}(\mathrm{F_o})=\begin{cases}\sqrt{(m+1/2)}\,2^{\frac{k}{2}}P_m(2^k\mathrm{F_o}-\hat{n}), & \frac{\hat{n}-1}{2^k}\le \mathrm{F_o}<\frac{\hat{n}+1}{2^k}\\ 0, & otherwise\end{cases} \tag{24}$$

where m is the order of Legendre polynomials , $\hat{n}=2n-1$, n = 1, 2,…, 2^{k-1}, k is any positive integer and $\mathrm{F_o}$ is defined on the interval [0, 1]. Here $P_m(F_o)$ is the well known Legendre polynomials of order m and is given as:

$P_0(F_o)=1,\ P_1(F_o)=F_o,\ P_{m+1}(F_o)=\frac{2m+1}{m+1}F_o\,P_m(F_o)-\frac{m}{m+1}\,P_{m-1}(F_o),\ m=1,2,3,\ldots,M-1$ (25)

Integrating equation (20) with respect to F_o from 0 to F_o, we have

$\frac{d\theta(F_o)}{dF_o}=C^TP\psi(F_o)$, (26)

where P is an operational matrix of integration(Razzaghi and Yusufi [11]) of order $2^{k-1}M\times 2^{k-1}M$ and is given as follow:

$$P=\frac{1}{2}\begin{bmatrix} 1 & \frac{1}{\sqrt{3}} & 0 & 0 & & & \\ \frac{-1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{15}} & 0 & \cdots & 0 & 0 \\ 0 & \frac{-1}{\sqrt{15}} & 0 & \frac{1}{\sqrt{35}} & & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{35}} & 0 & & & \\ & & \vdots & & \ddots & \vdots & \\ 0 & 0 & 0 & 0 & & 0 & \frac{\sqrt{2M-3}}{(2M-3)\sqrt{2M-1}} \\ 0 & 0 & 0 & 0 & \cdots & \frac{-\sqrt{2M-1}}{(2M-1)\sqrt{2M-3}} & 0 \end{bmatrix}. \quad (27)$$

Again integrating Eq. (26) with respect to F_o from 0 to F_o, we have

$$\theta(F_o) = C^T P^2 \psi(F_o)\,. \quad (28)$$

Substituting the values of $\frac{d^2\theta}{dF_o^2}(F_o)$, $\frac{d\theta}{dF_o}(F_o)$ and $\theta(F_o)$ in Eqn.(19), we get

$$AC^T M^T - C^T + B = 0\,, \quad (29)$$

Where

$M = \left(P^2 \left(F_{oq} I + \left(1 + F_{o\tau} P_f^2 - F_{o\tau} P_{mo} d\right) P \right)^{-1} \right)^T$, B=$N\mathrm{d}^T \left(F_{oq} I + \left(1 + F_{o\tau} P_f^2 - F_{o\tau} P_{mo} d\right) P \right)^{-1}$.

Solving Sylvester matrix Eqn. (29), we obtain value of *C*, which in turns gives dimensionless temperature.

4. Results and discussion

Unless otherwise mentioned, the following thermo-physical properties for living biological tissue are used in numerical analysis:

$\rho = 1000 Kgm^{-3}, k = 0.5 Wm^{-1}K^{-1}, c = 4.18\times10^3 JKg^{-1}K^{-1}$ .Blood temperature and its thermo-physical properties are: $T_b = 37^o C, c_b = 3.5\times10^3 JKg^{-1}K^{-1}, \omega_b = 0.5\times10^3 Kgm^{-3}s^{-1}$. Basal metabolic heat generation rate is $Q_{mo} = 1091 Wm^{-3}$; thickness of tissue is R = 0.05m; initial temperature is $T_o = 37^o C$ and wall temperature is $T_w = 37^o C$. The source term parameter values are taken as: $S = 11.5 Kg^{-1}, a = -140 m^{-1}, P = 30 W$. In order to take account of effects of finite heat propagation speed, values of phase lag time τ has been taken in the range of 0-20s. Some parameter values are possibly adjusted for comparison and discussion and are noted in each figure. MATLAB 2011a software has been used for all computational work.

Dimensionless temperature profiles at different time have been shown in Fig. 1. As time increases from 15 min. to 25 min., the temperature at the hyperthermia position (x = 0.9) rises from $41^o C$ to $46^o C$, which is required range for thermal treatment.

Fig. 2 and 3 show temperature profile in case of Parabolic and Hyperbolic model of bioheat transfer for different values of *p*. It can be seen from these figures that as *p* increases, the temperature at the hyperthermia position decreases in both cases. Moreover, Fractional bioheat models causes greater temperature rise as compared to standard bioheat models in both parabolic and hyperbolic cases.

The phase lag time τ is the inherent property of living biological tissues. Fig. 4 shows that for fractional bioheat models as phase lag time τ increases, temperature decreases at the hyperthermia position.

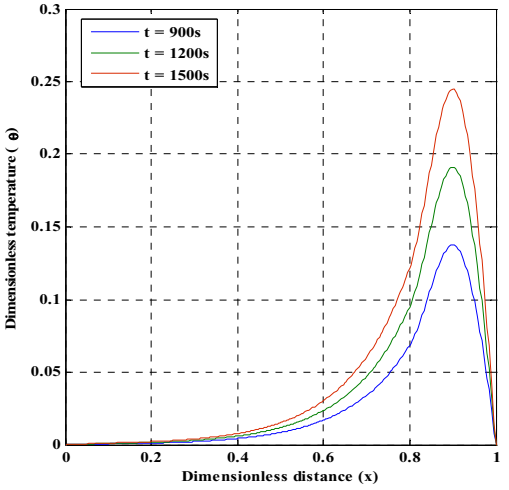

Fig.1 Plot of dimensionless temperature vs. distance at different time levels.

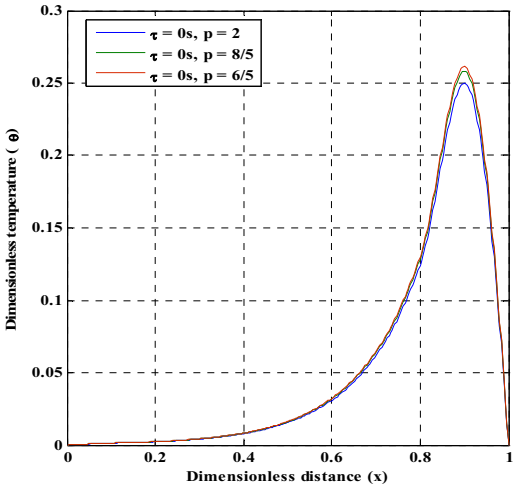

Fig. 2 Plot of dimensionless temperature vs. distance for fractional parabolic bioheat equation at different values of p.

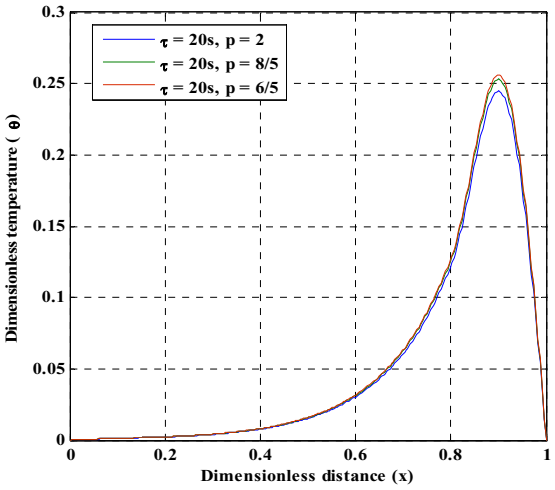

Fig.3. Plot of dimensionless temperature vs. distance for fractional hyperbolic bioheat equation at different values of p.

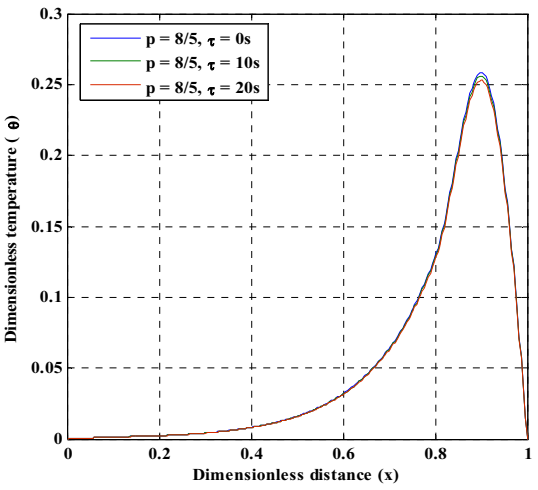

Fig.4. Plot of dimensionless temperature vs. distance for fractional bioheat equation at different values of phase lag time.

5. Conclusion

The hyperbolic space-fractional bioheat transfer model is derived and their effect is shown in thermal therapy applications. It can be concluded from the above study that the fractional bioheat models causes greater temperature rise as compared to standard bioheat models in both parabolic and hyperbolic cases. Standard parabolic and hyperbolic bioheat models show normal diffusion, whereas fractional bioheat model shows anomalous diffusion. Increase in phase lag time τ results in decrease of temperature at hyperthermia position.

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