Hindawi Publishing Corporation
Journal of Mathematics
Volume 2016, Article ID 2480842, 7 pages
http://dx.doi.org/10.1155/2016/2480842

Research Article

Fuzzy Soft Compact Topological Spaces

Seema Mishra and Rekha Srivastava

Department of Mathematical Sciences, Indian Institute of Technology (Banaras Hindu University), Varanasi 221005, India

Correspondence should be addressed to Seema Mishra; seemamishra.rs.apm12@itbhu.ac.in

Received 26 July 2016; Accepted 16 November 2016

Academic Editor: Ali Jaballah

In this paper, we have studied compactness in fuzzy soft topological spaces which is a generalization of the corresponding concept by R. Lowen in the case of fuzzy topological spaces. Several basic desirable results have been established. In particular, we have proved the counterparts of Alexander's subbase lemma and Tychonoff theorem for fuzzy soft topological spaces.

1. Introduction

Molodtsov introduced the concept of a soft set in 1999 (cf. [1]) as a new approach to model uncertainties. He also applied his theory in several directions, for example, stability and regularization, game theory, and soft analysis (cf. [1]).

Maji et al. [2] introduced and studied fuzzy soft sets. Yang et al. [3] introduced interval valued fuzzy soft sets. Majumdar and Samanta [4] defined generalized fuzzy soft set.

Algebraic structures of soft sets and fuzzy soft sets have been studied by many researchers. Aktaş and Çağman [5] introduced soft groups. Feng et al. [6] gave the concept of a soft semiring and many related concepts. Aygünoğlu and Aygün [7] introduced fuzzy soft groups and Varol et al. [8] studied fuzzy soft rings.

Shabir and Naz [9] defined soft topological spaces and many related basic concepts. Aygünoğlu and Aygün [10] also studied soft topological spaces. Fuzzy soft topology was introduced by Tanay and Kandemir [11] and it was further studied by Varol and Aygün [12], Mahanta and Das [13], Mishra and Srivastava [14], and so forth.

In [15], Varol et al. have introduced a soft topology and an L-fuzzy soft topology in a different way, using Šostak approach [16], and studied soft compactness and L-fuzzy soft compactness.

In fuzzy topological spaces, compactness was first introduced by Chang [17], but it is well known by now that compactness in the sense of Chang does not satisfy the Tychonoff property. Lowen [18] introduced compactness in a fuzzy topological space, in another way which satisfies the Tychonoff property and has many other desirable properties.

Gain et al. [19], Osmanoğlu and Tokat [20], and Sreedevi and Ravi Shankar [21] have studied fuzzy soft compactness in a fuzzy soft topological space introduced by Tanay and Kandemir [11]. These authors have introduced fuzzy soft compactness as a generalization of Chang's fuzzy compactness.

In this paper, we have introduced and studied fuzzy soft compactness as a generalization of Lowen's fuzzy compactness, in a fuzzy soft topological space introduced by Varol and Aygün [12].

Several basic desirable results have been established. In particular, we have proved the counterparts of Alexander's subbase lemma and Tychonoff theorem for fuzzy soft topological spaces.

2. Preliminaries

Throughout this paper, X denotes a nonempty set, called the universe, E denotes the set of parameters for the universe X, and $A \subseteq E$.

Definition 1 (see [22]). A fuzzy set in X is a function $f : X \to [0, 1]$. Now we define some basic fuzzy set operations as follows.

Let f and g be fuzzy sets in X. Then

(1) $f = g$ if $f(x) = g(x)$, $\forall x \in X$.

(2) $f \subseteq g$ if $f(x) \leqslant g(x)$, $\forall x \in X$.

(3) $(f \cup g)(x) = \max\{f(x), g(x)\}$, $\forall x \in X$.

(4) $(f \cap g)(x) = \min\{f(x), g(x)\}$, $\forall x \in X$.

(5) $f^c(x) = 1 - f(x)$, $\forall x \in X$ (here f^c denotes the complement of f).

The constant fuzzy set in X, taking value $\alpha \in [0,1]$, will be denoted by α_X.

Definition 2 (see [23]). Let Ω be an index set and $\{f_i: i \in \Omega\}$ be a family of fuzzy sets in X. Then their union $\bigcup_{i\in\Omega} f_i$ and intersection $\bigcap_{i\in\Omega} f_i$ are defined, respectively, as follows:

(1) $(\bigcup_{i\in\Omega} f_i)(x) = \sup\{f_i(x): i \in \Omega\}$, $\forall x \in X$.

(2) $(\bigcap_{i\in\Omega} f_i)(x) = \inf\{f_i(x): i \in \Omega\}$, $\forall x \in X$.

Definition 3 (see [24]). A fuzzy point x_λ $(0 < \lambda < 1)$ in X is a fuzzy set in X given by

$$x_\lambda(x') = \begin{cases} \lambda, & \text{if } x' = x \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

Here x and λ are, respectively, called the *support* and the *value* of x_λ.

Definition 4 (see [18]). A fuzzy topological space is a pair $(X, \mathcal{T})$, where X is a nonempty set and $\mathcal{T}$ is a family of fuzzy sets in X such that the following conditions are satisfied:

(1) $\alpha_X \in \mathcal{T}$, $\forall \alpha \in [0,1]$.

(2) If $f, g \in \mathcal{T}$, then $f \cap g \in \mathcal{T}$.

(3) If $\{f_i: i \in \Omega\}$ is an arbitrary family of fuzzy sets in $\mathcal{T}$, then $\bigcup_{i\in\Omega} f_i \in \mathcal{T}$.

Then $\mathcal{T}$ is called a fuzzy topology on X and members of $\mathcal{T}$ are called fuzzy open sets. A fuzzy set in X is called fuzzy closed if $f^c \in \mathcal{T}$.

Definition 5 (see [18]). A fuzzy set f in X is said to be fuzzy compact if for any family $\beta \subseteq \mathcal{T}$ such that $\bigcup_{\mu\in\beta} \mu \supseteq f$ and for all $\epsilon > 0$, there exists a finite subfamily $\beta_o \subseteq \beta$ such that $\bigcup_{\mu\in\beta_o} \mu \supseteq f - \epsilon_X$.

Definition 6 (see [18]). A fuzzy topological space $(X, \mathcal{T})$ is said to be fuzzy compact if each constant fuzzy set in X is fuzzy compact.

Definition 7 (see [2]). A pair (f, E) is called a fuzzy soft set over X if f is a mapping from E to I^X; that is, $f : E \to I^X$, where I^X is the collection of all fuzzy sets in X.

Definition 8 (see [12]). A fuzzy soft set f_A over X is a mapping from E to I^X; that is, $f_A : E \to I^X$ such that $f_A(e) \neq 0_X$, if $e \in A \subseteq E$ and $f_A(e) = 0_X$, otherwise, where 0_X denotes the constant fuzzy set in X taking value 0.

Definition 9 (see [12]). A constant fuzzy soft set α_E over X, where $\alpha \in [0,1]$, is the fuzzy soft set over X such that $\alpha_E(e) = \alpha_X$, $\forall e \in E$.

From here onwards, we will denote by $\mathcal{F}(X, E)$ the collection of all fuzzy soft sets over X, where E is the parameters set for X.

Definition 10 (see [12]). Let $f_A, g_B \in \mathcal{F}(X, E)$. Then

(1) f_A is said to be a *fuzzy soft subset* of g_B (or that f_A is contained in g_B), denoted by $f_A \sqsubseteq g_B$, if $f_A(e) \subseteq g_B(e)$, $\forall e \in E$

(2) f_A and g_B are said to be *equal*, denoted by $f_A = g_B$, if $f_A \sqsubseteq g_B$ and $g_B \sqsubseteq f_A$

(3) The *union* of f_A and g_B, denoted by $f_A \sqcup g_B$, is the fuzzy soft set over X defined by

$$(f_A \sqcup g_B)(e) = f_A(e) \cup g_B(e), \quad \forall e \in E \quad (2)$$

(4) The *intersection* of f_A and g_B, denoted by $f_A \sqcap g_B$, is the fuzzy soft set over X defined by

$$(f_A \sqcap g_B)(e) = f_A(e) \cap g_B(e), \quad \forall e \in E \quad (3)$$

Two fuzzy soft sets f_A and g_B over X are said to be *disjoint* if $f_A \sqcap g_B = 0_E$

(5) Let Ω be an index set and $\{f_{A_i}: i \in \Omega\}$ be a family of fuzzy soft sets over X. Then their *union* $\bigsqcup_{i\in\Omega} f_{A_i}$ and *intersection* $\bigsqcap_{i\in\Omega} f_{A_i}$ are defined, respectively, as follows:

(a) $(\bigsqcup_{i\in\Omega} f_{A_i})(e) = \bigcup_{i\in\Omega} f_{A_i}(e)$, $\forall e \in E$

(b) $(\bigsqcap_{i\in\Omega} f_{A_i})(e) = \bigcap_{i\in\Omega} f_{A_i}(e)$, $\forall e \in E$

(6) The *complement* of f_A, denoted by f_A^c, is the fuzzy soft set over X, defined by

$$f_A^c(e) = 1_X - f_A(e), \quad \forall e \in E \quad (4)$$

Definition 11 (see [7]). Let $\mathcal{F}(X, E)$ and $\mathcal{F}(Y, K)$ be the collections of all fuzzy soft sets over the universe sets X and Y, respectively, where E and K are the parameters sets for X and Y, respectively. Let $\varphi : X \to Y$ and $\psi : E \to K$ be two maps. Then the pair (φ, ψ) is called a fuzzy soft mapping from X to Y and is denoted by

$$(\varphi, \psi) : \mathcal{F}(X, E) \longrightarrow \mathcal{F}(Y, K). \quad (5)$$

(1) Let $f_A \in \mathcal{F}(X, E)$. Then the *image* of f_A under the fuzzy soft mapping (φ, ψ) is the fuzzy soft set over Y, denoted by $(\varphi, \psi) f_A$, and is defined as follows:

$$(\varphi, \psi) f_A(k)(y) = \begin{cases} \sup\limits_{\varphi(x)=y} \sup\limits_{\psi(e)=k} f_A(e)(x), & \text{if } \varphi^{-1}(y) \neq \phi,\ \psi^{-1}(k) \neq \phi \\ 0, & \text{otherwise,} \end{cases} \quad (6)$$

$$\forall k \in K,\ \forall y \in Y.$$

(2) Let $g_B \in \mathscr{F}(Y, K)$. Then the *inverse image* of g_B under the fuzzy soft mapping (φ, ψ) is the fuzzy soft set over X, denoted by $(\varphi, \psi)^{-1} g_B$, and is defined as follows:

$$(\varphi, \psi)^{-1} g_B(e)(x) = g_B(\psi(e))(\varphi(x)), \quad \forall e \in E,\ \forall x \in X. \tag{7}$$

Proposition 12 (see [25]). *Let $(\phi, \psi) : (X, \tau) \to (Y, \delta)$ be a fuzzy soft mapping and f_{A_1} and f_{A_2} be fuzzy soft sets over X such that $f_{A_1} \sqsubseteq f_{A_2}$. Then $(\phi, \psi) f_{A_1} \sqsubseteq (\phi, \psi) f_{A_2}$.*

Proposition 13 (see [25]). *Let $(\phi, \psi) : (X, \tau) \to (Y, \delta)$ be a fuzzy soft mapping and $\{f_{A_i} : i \in \Omega\}$ be a family of fuzzy soft sets over X. Then*

(1) $(\phi, \psi)(\bigsqcup_{i\in\Omega} f_{A_i}) = \bigsqcup_{i\in\Omega} (\phi, \psi) f_{A_i}$.

(2) $(\phi, \psi)(\bigsqcap_{i\in\Omega} f_{A_i}) \sqsubseteq \bigsqcap_{i\in\Omega} (\phi, \psi) f_{A_i}$.

Proposition 14 (see [25]). *Let $(\phi, \psi) : (X, \tau) \to (Y, \delta)$ be a fuzzy soft mapping and $\{g_{B_i} : i \in \Omega\}$ be a family of fuzzy soft sets over Y. Then*

(1) $(\phi, \psi)^{-1}(\bigsqcup_{i\in\Omega} g_{B_i}) = \bigsqcup_{i\in\Omega} (\phi, \psi)^{-1} g_{B_i}$.

(2) $(\phi, \psi)^{-1}(\bigsqcap_{i\in\Omega} g_{B_i}) = \bigsqcap_{i\in\Omega} (\phi, \psi)^{-1} g_{B_i}$.

Definition 15. Let f_A and g_B be fuzzy soft sets over X such that $f_A \sqsupseteq g_B$. Then $f_A - g_B$ is the fuzzy soft set over X given by

$$(f_A - g_B)(e) = f_A(e) - g_B(e), \quad \forall e \in E. \tag{8}$$

Definition 16 (see [12]). Let $f_A \in \mathscr{F}(X, E)$ and $g_B \in \mathscr{F}(Y, K)$. Then the fuzzy soft product of f_A and g_B, denoted by $f_A \times g_B$, is the fuzzy soft set over $X \times Y$ and is defined by

$$(f_A \times g_B)(e, k) = f_A(e) \times g_B(k), \quad \forall (e, k) \in E \times K, \tag{9}$$

and, for $(x, y) \in X \times Y$,

$$(f_A(e) \times g_B(k))(x, y) = \min\{f_A(e)(x), g_B(k)(y)\}. \tag{10}$$

Definition 17 (see [11, 12]). A fuzzy soft topological space relative to the parameters set E is a pair (X, τ) consisting of a nonempty set X and a family τ of fuzzy soft sets over X satisfying the following conditions:

(1) $\alpha_E \in \tau$, $\forall \alpha \in [0, 1]$.

(2) If $f_A, g_B \in \tau$, then $f_A \sqcap g_B \in \tau$.

(3) If $f_{A_j} \in \tau$, $\forall j \in \Omega$, where Ω is some index set, then $\bigsqcup_{j\in\Omega} f_{A_j} \in \tau$.

Then τ is called a *fuzzy soft topology* over X and members of τ are called *fuzzy soft open* sets. A fuzzy soft set g_B over X is called *fuzzy soft closed* if $(g_B)^c \in \tau$.

We mention here that the fuzzy soft topology defined above has been called "enriched fuzzy soft topology" in [12].

Definition 18 (see [12]). Let (X, τ) be a fuzzy soft topological space. Then a subfamily $\mathscr{B}$ of τ is called a base for τ if every member of τ can be written as a union of members of $\mathscr{B}$.

Definition 19 (see [12]). Let (X, τ) be a fuzzy soft topological space. Then a subfamily $\mathscr{S}$ of τ is called a subbase for τ if the family of finite intersections of its members forms a base for τ.

Definition 20 (see [12]). A fuzzy soft topology τ over X is said to be generated by a subfamily $\mathscr{S}$ of fuzzy soft sets over X if every member of τ is a union of finite intersections of members of $\mathscr{S}$.

Definition 21 (see [12]). Let $\{(X_i, \tau_i)\}_{i\in\Omega}$ be a family of fuzzy soft topological spaces relative to the parameters sets E_i, respectively, and, for each $i \in \Omega$, let $(\varphi, \psi)_i : X \to (X_i, \tau_i)$ be a fuzzy soft mapping. Then the fuzzy soft topology τ over X is said to be initial with respect to the family $\{(\varphi, \psi)_i\}_{i\in\Omega}$ if τ has as subbase the set

$$\mathscr{S} = \left\{(\varphi, \psi)_i^{-1}(f_{A_i}) : i \in \Omega,\ f_{A_i} \in \tau_i\right\}; \tag{11}$$

that is, the fuzzy soft topology τ over X is generated by $\mathscr{S}$.

Definition 22 (see [12]). Let $\{(X_i, \tau_i)\}_{i\in\Omega}$ be a family of fuzzy soft topological spaces relative to the parameters sets E_i, respectively. Then their product is defined as the fuzzy soft topological space (X, τ) relative to the parameters set E, where $X = \prod_i X_i$, $E = \prod_i E_i$, and τ is the fuzzy soft topology over X which is initial with respect to the family $\{(p_{X_i}, q_{E_i})\}_{i\in\Omega}$, where $p_{X_i} : \prod_i X_i \to X_i$ and $q_{E_i} : \prod_i E_i \to E_i$ are the projection maps; that is, τ is generated by

$$\left\{(p_{X_i}, q_{E_i})^{-1}(f_{A_i}) : i \in \Omega,\ f_{A_i} \in \tau_i\right\}. \tag{12}$$

Definition 23 (see [14]). A fuzzy soft point e_{x_λ} over X is a fuzzy soft set over X defined as follows:

$$e_{x_\lambda}(e') = \begin{cases} x_\lambda, & \text{if } e' = e \\ 0_X, & \text{if } e' \in E - \{e\}, \end{cases} \tag{13}$$

where x_λ is the fuzzy point in X with support x and value λ, $\lambda \in (0, 1)$.

A fuzzy soft point e_{x_λ} is said to *belong* to a fuzzy soft set f_A, denoted by $e_{x_\lambda} \in f_A$, if $\lambda < f_A(e)(x)$ and two fuzzy soft points e_{x_λ} and e'_{y_s} are said to be *distinct* if $x \neq y$ or $e \neq e'$.

Proposition 24 (see [14]). *Let (X, τ) be a fuzzy soft topological space. Then a fuzzy soft set f_A is fuzzy soft open iff $\forall e_{x_r} \in f_A$; there exists a basic fuzzy soft open set g_B such that $e_{x_r} \in g_B \sqsubseteq f_A$.*

Definition 25 (see [26]). A family of sets is said to be of finite character iff each finite subset of a member of the family is also a member, and each set belongs to this family if each of its finite subsets belong to it.

Lemma 26 (TUKEY, [26]). *Each nonempty family of sets of finite character has a maximal element.*

3. Fuzzy Soft Compact Topological Spaces

Definition 27. Let (X, τ) be a fuzzy soft topological space relative to the parameters set E. Then a fuzzy soft set f_A over X is said to be fuzzy soft compact if, for any family $\beta \subseteq \tau$ such that $\bigsqcup_{g_B \in \beta} g_B \sqsupseteq f_A$ and $\forall \epsilon$ such that $f_A \sqsupseteq \epsilon_E$, there exists a finite subfamily β_o of β such that $\bigsqcup_{g_B \in \beta_o} g_B \sqsupseteq f_A - \epsilon_E$.

Definition 28. A fuzzy soft topological space (X, τ) relative to the parameters set E is said to be fuzzy soft compact if each constant fuzzy soft set over X is fuzzy soft compact; that is, for $\alpha \in [0,1]$, if there exists a family β of fuzzy soft open sets over X such that $\bigsqcup_{f_A \in \beta} f_A \sqsupseteq \alpha_E$, then $\forall \epsilon \in (0, \alpha)$; there exists a finite subfamily β_o of β such that $\bigsqcup_{f_A \in \beta_o} f_A \sqsupseteq (\alpha - \epsilon)_E$.

Proposition 29. *Let (X, τ) and (Y, δ) be fuzzy soft topological spaces relative to parameters sets E and K, respectively, $(\phi, \psi) : (X, \tau) \to (Y, \delta)$ be a fuzzy soft continuous mapping, and $f_A \in \mathcal{F}(X, E)$ be fuzzy soft compact. Then $(\phi, \psi) f_A$ is fuzzy soft compact.*

Proof. Let $\beta \subseteq \delta$ be such that

$$\bigsqcup_{g_B \in \beta} g_B \sqsupseteq (\phi, \psi) f_A \Longrightarrow \bigsqcup_{g_B \in \beta} (\phi, \psi)^{-1} g_B \sqsupseteq f_A. \tag{14}$$

Since $\{(\phi, \psi)^{-1} g_B\}_{g_B \in \beta}$ is a family of fuzzy soft open sets over X and f_A is fuzzy soft compact, so, $\forall \epsilon$ such that $f_A \sqsupseteq \epsilon_E$, there exists a finite subfamily $\beta_o \subseteq \beta$ such that

$$\bigsqcup_{g_B \in \beta_o} (\phi, \psi)^{-1} g_B \sqsupseteq f_A - \epsilon_E. \tag{15}$$

Then applying (ϕ, ψ) on both sides, we get

$$\bigsqcup_{g_B \in \beta_o} g_B \sqsupseteq (\phi, \psi)(f_A - \epsilon_E) = (\phi, \psi) f_A - \epsilon_K, \tag{16}$$

which implies that $(\phi, \psi) f_A$ is fuzzy soft compact. □

From the fact that (ϕ, ψ) is surjective if ϕ and ψ both are surjective (cf. [12]), each constant fuzzy soft set α_K over Y is the image of constant fuzzy soft set α_E over X. Hence we have the following result.

Corollary 30. *Let (X, τ) and (Y, δ) be fuzzy soft topological spaces where (X, τ) is fuzzy soft compact and (ϕ, ψ) be a surjective fuzzy soft continuous mapping from (X, τ) to (Y, δ). Then (Y, δ) is fuzzy soft compact.*

As in the case of soft topological spaces [10], here we have the following.

Definition 31. Let (X, τ) be a fuzzy soft topological space relative to the parameters set E. Then, for $e \in E$, the e-parameter fuzzy topological spaces are given by (X, τ_e), where $\tau_e = \{f_A(e) : f_A \in \tau\}$.

The following proposition is a counterpart of Theorem 4.1 in [10].

Proposition 32. *Let (X, τ) be a fuzzy soft topological space relative to the parameters set E, which is finite. Then (X, τ) is fuzzy soft compact if each e-parameter fuzzy topological space is fuzzy compact.*

Proof. Suppose that each e-parameter fuzzy topological space is fuzzy compact. Then to show that α_E, $\alpha \in [0,1]$ is fuzzy soft compact, consider a family β of fuzzy soft open sets over X such that

$$\begin{aligned} & \alpha_E \sqsubseteq \bigsqcup_{f_A \in \beta} f_A \\ \Longrightarrow \quad & \alpha_E(e) \subseteq \bigcup_{f_A \in \beta} f_A(e), \quad \forall e \in E \\ \Longrightarrow \quad & \alpha_X \subseteq \bigcup_{f_A \in \beta} f_A(e), \quad \forall e \in E. \end{aligned} \tag{17}$$

Then, for $e \in E$, by fuzzy compactness of (X, τ_e), for $\epsilon \in (0, \alpha)$, there exists a finite subfamily β_o^e of β such that $(\alpha - \epsilon)_X \subseteq \bigcup_{f_A \in \beta_o^e} f_A(e)$.

Now set $\beta_o = \bigcup_{e \in E} \beta_o^e$. Then β_o is a finite subfamily of β such that $(\alpha - \epsilon)_X \subseteq \bigcup_{f_A \in \beta_o} f_A(e)$, $\forall e \in E$. Hence $(\alpha - \epsilon)_E \sqsubseteq \bigsqcup_{f_A \in \beta_o} f_A$, which shows that (X, τ) is fuzzy soft compact. □

Now we consider the mappings (cf. [27]) $h : \mathcal{F}(X, E) \to I^{X \times E}$, where $I^{X \times E}$ is the set of all fuzzy sets in $X \times E$, defined as follows:

$$h(f_A)(x, e) = f_A(e)(x), \quad \forall f_A \in \mathcal{F}(X, E), \tag{18}$$

and $g : I^{X \times E} \to \mathcal{F}(X, E)$ as follows:

$$\begin{aligned} & g(U) = f_E^U, \\ & \forall U \in I^{X \times E}, \text{ where } \forall e \in E, \ \forall x \in X, \ f_E^U(e)(x) = U(x, e). \end{aligned} \tag{19}$$

In view of the above, we state the following theorem proved in [27].

Theorem 33 (see [27]). *$(\mathcal{F}(X, E), \sqcup, \sqcap, ^c)$ is isomorphic to $(I^{X \times E} \cup, \cap, ^c)$, where $I^{X \times E}$ denotes the set of all fuzzy sets in $X \times E$.*

Then it is easy to verify that if (X, τ) is a fuzzy soft topological space relative to the parameters set E, then $(X \times E, h(\tau))$ is a fuzzy topological space and also if $(X \times E, \mathcal{T})$ is a fuzzy topological space, then $(X, g(\mathcal{T}))$ is a fuzzy soft topological space relative to the parameters set E, where $h(\tau) = \{h(f_A) : f_A \in \tau\}$ and $g(\mathcal{T}) = \{g(U) : U \in \mathcal{T}\}$ (cf. [27]).

Proposition 34. *A fuzzy soft topological space (X, τ) relative to the parameters set E is fuzzy soft compact iff $(X \times E, h(\tau))$ is fuzzy compact.*

Proof. First, suppose that (X,τ) is fuzzy soft compact. Then to show that $(X\times E, h(\tau))$ is fuzzy compact, consider a family $\beta \subseteq h(\tau)$ of fuzzy open sets in $X\times E$ such that

$$\begin{aligned} &\alpha_{X\times E} \subseteq \bigcup_{\nu\in\beta}\nu \\ \Longrightarrow\quad & g\left(\alpha_{X\times E}\right) \sqsubseteq g\left(\bigcup_{\nu\in\beta}\nu\right) \\ \Longrightarrow\quad & \alpha_E \sqsubseteq \bigsqcup_{\nu\in\beta} g(\nu). \end{aligned} \tag{20}$$

Note that if $\nu \in \beta \subseteq h(\tau)$, then $g(\nu) \in \tau$. So by fuzzy soft compactness of (X,τ), for $\epsilon \in (0,\alpha)$, there exists a finite subfamily β_o of β such that

$$\begin{aligned} &(\alpha-\epsilon)_E \sqsubseteq \bigsqcup_{\nu\in\beta_o} g(\nu) \\ \Longrightarrow\quad & h\left((\alpha-\epsilon)_E\right) \subseteq h\left(\bigsqcup_{\nu\in\beta_o} g(\nu)\right) \\ \Longrightarrow\quad & (\alpha-\epsilon)_{X\times E} \subseteq \bigcup_{\nu\in\beta_o}(hog)(\nu) \\ \Longrightarrow\quad & (\alpha-\epsilon)_{X\times E} \subseteq \bigcup_{\nu\in\beta_o}\nu, \end{aligned} \tag{21}$$

which proves the fuzzy compactness of $(X\times E, h(\tau))$.

Conversely, assume that $(X\times E, h(\tau))$ is fuzzy compact. To show that (X,τ) is fuzzy soft compact, we have to show that α_E, $\alpha\in[0,1]$ is fuzzy soft compact. Let $\beta\subseteq\tau$ such that

$$\begin{aligned} &\alpha_E \sqsubseteq \bigsqcup_{f_A\in\beta} f_A \\ \Longrightarrow\quad & h\left(\alpha_E\right) \subseteq h\left(\bigsqcup_{f_A\in\beta} f_A\right) \\ \Longrightarrow\quad & \alpha_{X\times E} \subseteq \bigcup_{f_A\in\beta} h\left(f_A\right). \end{aligned} \tag{22}$$

Then by the fuzzy compactness of $(X\times E, h(\tau))$, for $\epsilon\in(0,\alpha)$, there exists a finite subfamily β_o of β such that

$$\begin{aligned} &(\alpha-\epsilon)_{X\times E} \subseteq \bigcup_{f_A\in\beta_o} h\left(f_A\right) \\ \Longrightarrow\quad & g\left((\alpha-\epsilon)_{X\times E}\right) \sqsubseteq g\left(\bigcup_{f_A\in\beta_o} h\left(f_A\right)\right) \\ \Longrightarrow\quad & (\alpha-\epsilon)_E \sqsubseteq \bigsqcup_{f_A\in\beta_o}(goh)\left(f_A\right) \\ \Longrightarrow\quad & (\alpha-\epsilon)_E \sqsubseteq \bigsqcup_{f_A\in\beta_o} f_A, \end{aligned} \tag{23}$$

which proves that the fuzzy soft topological space (X,τ) is fuzzy soft compact. □

Now we prove the counterparts of the well known Alexander's subbase lemma and the Tychonoff theorem for fuzzy soft topological spaces, the proofs of which are based on the proofs of the corresponding results given in [18] and [28], respectively.

Theorem 35. *Let (X,τ) be a fuzzy soft topological space relative to the parameters set E. Then (X,τ) is fuzzy soft compact iff, for any subbase σ for τ, if there is a family $\beta\subseteq\sigma$ such that $\bigsqcup_{f_A\in\beta} f_A \sqsupseteq \alpha_E$, then, for $\epsilon\in(0,\alpha)$, there exists a finite subfamily β_o of β such that $\bigsqcup_{f_A\in\beta_o} f_A \sqsupseteq (\alpha-\epsilon)_E$.*

Proof. First assume that (X,τ) is fuzzy soft compact. Choose $\beta\subseteq\sigma$ such that $\bigsqcup_{f_A\in\beta} f_A \sqsupseteq \alpha_E$, $\alpha\in[0,1]$. Now since $\sigma\subseteq\tau$ and (X,τ) is fuzzy soft compact, there exists a finite subfamily β_o of β such that $\bigsqcup_{f_A\in\beta_o} f_A \sqsupseteq (\alpha-\epsilon)_E$.

Conversely, to show that (X,τ) is fuzzy soft compact, we have to show that if, for $\beta\subseteq\tau$, there exist α and $\epsilon\in(0,\alpha)$ such that there does not exist any finite subfamily β_o of β, such that $\bigsqcup_{f_A\in\beta_o} f_A \sqsupseteq (\alpha-\epsilon)_E$, then it must follow that $\bigsqcup_{f_A\in\beta} f_A$ does not contain α_E.

Consider the family

$$\mathscr{C} = \left\{\beta\subseteq\tau\text{: there does not exist any finite subfamily } \beta_o \text{ of } \beta \text{ such that } \bigsqcup_{f_A\in\beta_o} f_A \sqsupseteq (\alpha-\epsilon)_E\right\}. \tag{24}$$

Then $\mathscr{C}$ is of finite character. Now it follows from Tukey's lemma that, $\forall\beta\in\mathscr{C}$, there exists a maximal element $\beta'\in\mathscr{C}$ containing β.

Next, we show that if $f_A\in\beta'$ and $f_{A_1}, f_{A_2}, \ldots, f_{A_n}\in\tau$ such that $f_A \sqsupseteq f_{A_1}\sqcap f_{A_2}\sqcap\cdots\sqcap f_{A_n}$, then there exists some k, $k=1,2,\ldots,n$ such that $f_{A_k}\in\beta'$. For this we proceed as follows.

If we take $f_{A_1}\in\tau$ such that $f_{A_1}\notin\beta'$, then since β' is maximal, for the family $\{f_{A_1}\}\cup\beta'$, there exists a finite subfamily $\{f_{A_1}, g_{B_1}, g_{B_2}, \ldots, g_{B_p}\}$, where $g_{B_i}\in\beta'$, $\forall i = 1,2,\ldots,p$, such that $f_{A_1}\sqcup g_{B_1}\sqcup g_{B_2}\sqcup\cdots\sqcup g_{B_p} \sqsupseteq (\alpha-\epsilon)_E$, which implies that if, for $e\in E$ and $x\in X$,

$$\sup\left\{g_{B_1}(e)(x), g_{B_2}(e)(x), \ldots, g_{B_p}(e)(x)\right\} < \alpha-\epsilon, \tag{25}$$

then

$$f_{A_1}(e)(x) \geq \alpha-\epsilon. \tag{26}$$

Similarly, if we take another $f_{A_2} \in \tau$ such that $f_{A_2} \notin \beta'$, then again since β' is maximal, for the family $\{f_{A_2}\} \cup \beta'$, there exists a finite subfamily $\{f_{A_2}, g'_{B_1}, g'_{B_2}, \ldots, g'_{B_q}\}$, where $g'_{B_i} \in \beta'$, $\forall i = 1, 2, \ldots, q$, such that $f_{A_2} \sqcup g'_{B_1} \sqcup g'_{B_2} \sqcup \cdots \sqcup g'_{B_q} \sqsupseteq (\alpha - \epsilon)_E$, which implies that if, for $e' \in E$ and $x' \in X$,

$$\sup \left\{ g_{B_1}\left(e'\right)\left(x'\right), g_{B_2}\left(e'\right)\left(x'\right), \ldots, g_{B_q}\left(e'\right)\left(x'\right) \right\} < \alpha - \epsilon, \tag{27}$$

then

$$f_{A_2}\left(e'\right)\left(x'\right) \geq \alpha - \epsilon. \tag{28}$$

Now we show that

$$\left(f_{A_1} \sqcap f_{A_2}\right) \sqcup g_{B_1} \sqcup g_{B_2} \sqcup \cdots \sqcup g_{B_p} \sqcup g'_{B_1} \sqcup g'_{B_2} \sqcup \cdots \sqcup g'_{B_q} \sqsupseteq (\alpha - \epsilon)_E, \tag{29}$$

as follows.

If for $k \in E$ and $y \in X$, $(g_{B_1} \sqcup g_{B_2} \sqcup \cdots \sqcup g_{B_p} \sqcup g'_{B_1} \sqcup g'_{B_2} \sqcup \cdots \sqcup g'_{B_q})(k)(y) < \alpha - \epsilon$ which implies that $\sup_{j=1,2,\ldots,p} g_{B_j}(k)(y) < \alpha - \epsilon$ and $\sup_{i=1,2,\ldots,q} g'_{B_i}(k)(y) < \alpha - \epsilon$, then, from (26) and (28), we get $f_{A_1}(k)(y) \geq \alpha - \epsilon$ and $f_{A_2}(k)(y) \geq \alpha - \epsilon$. This implies that $(f_{A_1} \sqcap f_{A_2})(k)(y) \geq \alpha - \epsilon$ and hence $f_{A_1} \sqcap f_{A_2} \notin \beta'$. Thus, in general, if $f_{A_1}, f_{A_2}, \ldots, f_{A_n}$ do not belong to β', then $f_{A_1} \sqcap f_{A_2} \sqcap \cdots \sqcap f_{A_n}$ does not belong to β' implying that there is no fuzzy soft open set containing $f_{A_1} \sqcap f_{A_2} \sqcap \cdots \sqcap f_{A_n}$ which belong to β'. Thus we have shown that if f_{A_i}, $i = 1, 2, \ldots, n$ do not belong to β', then no fuzzy soft open set f_A such that $f_A \sqsupseteq f_{A_1} \sqcap f_{A_2} \sqcap \cdots \sqcap f_{A_n}$ belongs to β'. Equivalently, if $f_A \in \beta'$ such that $f_A \sqsupseteq f_{A_1} \sqcap f_{A_2} \sqcap \cdots \sqcap f_{A_n}$, then there exists some k, $k = 1, 2, \ldots, n$ such that $f_{A_k} \in \beta'$.

Next, consider $\beta' \cap \sigma$. Then, from the assumption of the theorem, we have that $\bigsqcup_{f_A \in \beta' \cap \sigma} f_A$ does not contain α_E. Now we show $\bigsqcup_{f_A \in \beta'} f_A \sqsubseteq \bigsqcup_{g_B \in \beta' \cap \sigma} g_B$. Since, $\forall f_A \in \beta'$, $\forall e \in E$ and $\forall x \in X$ such that $f_A(e)(x) > 0$ and $\forall a < f_A(e)(x)$, then $e_{f_A(e)(x) - a} \in f_A$ and hence, using Proposition 24, there exist $f^a_{A_1}, f^a_{A_2}, \ldots, f^a_{A_n} \in \sigma$ such that $f^a_{A_1} \sqcap f^a_{A_2} \sqcap \cdots \sqcap f^a_{A_n} \sqsubseteq f_A$ and $f^a_{A_1}(e)(x) \wedge f^a_{A_2}(e)(x) \wedge \cdots \wedge f^a_{A_n}(e)(x) > f_A(e)(x) - a$. Since $f_A \in \beta'$ and β' is maximal, it follows that there exists some k, $k = 1, 2, \ldots, n$ such that $f^a_{A_k} \in \beta'$. Thus, $\forall a > 0$, there exists $f^a_{A_k}$ such that $f^a_{A_k}(e)(x) > f_A(e)(x) - a$, $\forall e \in E$, $\forall x \in X$ and $f^a_{A_k} \in \beta' \cap \sigma$. Now fix e and x both. Then $\forall f_A \in \beta'$ such that $f_A(e)(x) > 0$ and $a < f_A(e)(x)$, where $a > 0$, there exists $f^a_A \in \beta' \cap \sigma$ such that

$$\begin{aligned} & f^a_A(e)(x) > f_A(e)(x) - a \\ \Longrightarrow\ & \left(\bigsqcup_{g_B \in \beta' \cap \sigma} g_B \right)(e)(x) \geq f_A(e)(x) \\ \Longrightarrow\ & \bigsqcup_{f_A \in \beta'} f_A \sqsubseteq \bigsqcup_{g_B \in \beta' \cap \sigma} g_B. \end{aligned} \tag{30}$$

This implies that $\bigsqcup_{f_A \in \beta'} f_A$ does not contain α_E and hence $\bigsqcup_{f_A \in \beta} f_A$ does not contain α_E. Thus, (X, τ) is fuzzy soft compact. □

Theorem 36. *If $\{(X_i, \tau_i): i \in \Omega\}$ is a family of fuzzy soft topological spaces relative to the parameters sets E_i, respectively, then the product of fuzzy soft topological space,*

$$(X, \tau) = \prod_{i \in \Omega} (X_i, \tau_i), \tag{31}$$

is fuzzy soft compact if and only if each coordinate fuzzy soft topological space (X_i, τ_i) is fuzzy soft compact.

Proof. Let us first assume that each coordinate space (X_i, τ_i), $i \in \Omega$, is fuzzy soft compact. From the previous theorem, to show that (X, τ) is fuzzy soft compact, it is sufficient to show that for any family $\beta \subseteq \sigma = \{(p_{X_i}, q_{E_i})^{-1}(f_{A_i}): i \in \Omega,\ f_{A_i} \in \tau_i\}$, if there exist α, ϵ, $\alpha > \epsilon > 0$ such that $\forall$ finite subfamily β_o of β, $\bigsqcup_{f_A \in \beta_o} f_A$ does not contain $(\alpha - \epsilon)_E$, then it must follow that $\bigsqcup_{f_A \in \beta} f_A$ does not contain α_E.

Let β be such a family. Then, $\forall j \in \Omega$, put $\beta_j = \{f_{A_j} \in \tau_j : (p_{X_j}, q_{E_j})^{-1}(f_{A_j}) \in \beta\}$. Then, $\forall$ finite subfamily $(\beta_j)_o$ of β_j, $\{(p_{X_j}, q_{E_j})^{-1}(f_{A_j}) : f_{A_j} \in (\beta_j)_o\}$ is a finite subfamily of β. Hence, from our assumption there exist some $e \in E$ and $x \in X$ such that

$$\begin{aligned} & \left(\bigsqcup_{f_{A_j} \in (\beta_j)_o} \left(p_{X_j}, q_{E_j}\right)^{-1} f_{A_j} \right)(e)(x) < \alpha - \epsilon \\ \Longrightarrow\ & \sup_{f_{A_j} \in (\beta_j)_o} f_{A_j}\left(e_j\right)\left(x_j\right) < \left(\alpha - \frac{\epsilon}{2}\right) - \frac{\epsilon}{2}. \end{aligned} \tag{32}$$

Since the above inequality holds $\forall$ finite subfamily $(\beta_j)_o$ of β_j, so, from the fuzzy soft compactness of (X_j, τ_j), there exist $e'_j \in E_j$ and $x'_j \in X_j$ such that

$$\sup_{f_{A_j} \in \beta_j} f_{A_j}\left(e'_j\right)\left(x'_j\right) < \left(\alpha - \frac{\epsilon}{2}\right). \tag{33}$$

The same inequality holds for all $j \in \Omega$. Finally, if we set $e = (e'_j)_{j \in \Omega}$ and $x = (x'_j)_{j \in \Omega}$, then

$$\begin{aligned} & \left(\bigsqcup_{f_A \in \beta} f_A \right)(e)(x) = \sup_{f_A \in \beta} f_A(e)(x) \\ & = \sup_{j \in \Omega} \sup_{f_A \in \beta \cap (p_{X_j}, q_{E_j})^{-1}(\tau_j)} f_A(e)(x) \\ & = \sup_{j \in \Omega} \sup_{f_{A_j} \in \beta_j} \left(\left(p_{X_j}, q_{E_j}\right)^{-1} f_{A_j} \right)(e)(x) \\ & = \sup_{j \in \Omega} \sup_{f_{A_j} \in \beta_j} f_{A_j}\left(e'_j\right)\left(x'_j\right) \\ & \leq \alpha - \frac{\epsilon}{2} \\ & < \alpha \\ \Longrightarrow\ & \bigsqcup_{f_A \in \beta} f_A \text{ does not contain } \alpha_E. \end{aligned} \tag{34}$$

The converse part follows using Corollary 30 as well as the fact that (p_{X_j}, q_{E_j}) are fuzzy soft continuous maps, $\forall j \in \Omega$. □

4. Conclusion

Soft sets were introduced by Molodtsov in 1999 [1]. Maji et al. [2] introduced and studied fuzzy soft sets. The theory of fuzzy soft topology was initiated by Tanay and Kandemir [11] and further studied by Varol and Aygün [12], Mahanta and Das [13], and so forth. In this paper we have introduced fuzzy soft compactness in a fuzzy soft topological space, which is an extension of Lowen's concept of fuzzy compactness in the case of fuzzy topological spaces. Several basic desirable results have been obtained. In particular the counterparts of Alexander's subbase lemma and the Tychonoff theorem for fuzzy soft topological spaces have been proved.

Competing Interests

The authors declare that they have no competing interests.

Acknowledgments

Seema Mishra gratefully acknowledges the financial support in the form of scholarship, given by Council of Scientific and Industrial Research, New Delhi (Award no. 09/013(0544)/2014-EMR-I) and Indian Institute of Technology (Banaras Hindu University).

References

[1] D. Molodtsov, "Soft set theory—first results," *Computers & Mathematics with Applications*, vol. 37, no. 4-5, pp. 19–31, 1999.

[2] P. K. Maji, R. Biswas, and A. R. Roy, "Fuzzy soft sets," *Journal of Fuzzy Mathematics*, vol. 9, no. 3, pp. 589–602, 2001.

[3] X. Yang, T. Y. Lin, J. Yang, Y. Li, and D. Yu, "Combination of interval-valued fuzzy set and soft set," *Computers & Mathematics with Applications*, vol. 58, no. 3, pp. 521–527, 2009.

[4] P. Majumdar and S. K. Samanta, "Generalised fuzzy soft sets," *Computers & Mathematics with Applications*, vol. 59, no. 4, pp. 1425–1432, 2010.

[5] H. Aktaş and N. Çağman, "Soft sets and soft groups," *Information Sciences*, vol. 177, no. 13, pp. 2726–2735, 2007.

[6] F. Feng, Y. B. Jun, and X. Zhao, "Soft semirings," *Computers & Mathematics with Applications*, vol. 56, no. 10, pp. 2621–2628, 2008.

[7] A. Aygünoğlu and H. Aygün, "Introduction to fuzzy soft groups," *Computers & Mathematics with Applications*, vol. 58, no. 6, pp. 1279–1286, 2009.

[8] B. P. Varol, A. Aygünoğlu, and H. Aygün, "On fuzzy soft rings," *Journal of Hyperstructures*, vol. 1, no. 2, pp. 1–15, 2012.

[9] M. Shabir and M. Naz, "On soft topological spaces," *Computers & Mathematics with Applications*, vol. 61, no. 7, pp. 1786–1799, 2011.

[10] A. Aygünoğlu and H. Aygün, "Some notes on soft topological spaces," *Neural Computing and Applications*, vol. 21, no. 1, pp. S113–S119, 2012.

[11] B. Tanay and M. B. Kandemir, "Topological structure of fuzzy soft sets," *Computers & Mathematics with Applications*, vol. 61, no. 10, pp. 2952–2957, 2011.

[12] B. P. Varol and H. Aygün, "Fuzzy soft topology," *Hacettepe Journal of Mathematics and Statistics*, vol. 41, no. 3, pp. 407–419, 2012.

[13] J. Mahanta and P. K. Das, "Results on fuzzy soft topological spaces," https://arxiv.org/abs/1203.0634.

[14] S. Mishra and R. Srivastava, "Hausdorff fuzzy soft topological spaces," *Annals of Fuzzy Mathematics and Informatics*, vol. 9, no. 2, pp. 247–260, 2015.

[15] B. P. Varol, A. Shostak, and H. Aygün, "A new approach to soft topology," *Hacettepe Journal of Mathematics and Statistics*, vol. 41, no. 5, pp. 731–741, 2012.

[16] A. P. Šostak, "On fuzzy topological structure," *Rendiconti del Circolo Matematico di Palermo*, vol. 11, pp. 89–103, 1985.

[17] C. L. Chang, "Fuzzy topological spaces," *Journal of Mathematical Analysis and Applications*, vol. 24, pp. 182–190, 1968.

[18] R. Lowen, "Fuzzy topological spaces and fuzzy compactness," *Journal of Mathematical Analysis and Applications*, vol. 56, no. 3, pp. 621–633, 1976.

[19] P. K. Gain, R. P. Chakraborty, and M. Pal, "On compact and semicompact fuzzy soft topological spaces," *Journal of Mathematics and Computer Science*, vol. 4, no. 2, pp. 425–445, 2014.

[20] İ. Osmanoğlu and D. Tokat, "Compact fuzzy soft spaces," *Annals of Fuzzy Mathematics and Informatics*, vol. 7, no. 1, pp. 45–51, 2014.

[21] A. Sreedevi and N. Ravi Shankar, "A novel fuzzy soft theory on compact spaces for particular issues," *International Journal of Basic Sciences and Applied Computing*, vol. 1, no. 1, pp. 10–14, 2014.

[22] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.

[23] P. M. Pu and Y. M. Liu, "Fuzzy topology. I. Neighborhood structure of a fuzzy point and Moore-Smith convergence," *Journal of Mathematical Analysis and Applications*, vol. 76, no. 2, pp. 571–599, 1980.

[24] R. Srivastava, S. N. Lal, and A. K. Srivastava, "Fuzzy Hausdorff topological spaces," *Journal of Mathematical Analysis and Applications*, vol. 81, no. 2, pp. 497–506, 1981.

[25] A. Kharal and B. Ahmad, "Mappings on fuzzy soft classes," *Advances in Fuzzy Systems*, vol. 2009, Article ID 407890, 6 pages, 2009.

[26] S. Willard, *General Topology*, Addison-Wesley, 1970.

[27] F.-G. Shi and B. Pang, "Redundancy of fuzzy soft topological spaces," *Journal of Intelligent and Fuzzy Systems*, vol. 27, no. 4, pp. 1757–1760, 2014.

[28] R. Lowen, "Initial and final fuzzy topologies and the fuzzy Tychonoff theorem," *Journal of Mathematical Analysis and Applications*, vol. 58, no. 1, pp. 11–21, 1977.