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Comparative study of synchronization methods of fractional order chaotic systems

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Abstract: In this article, the active control method and the backstepping method are used during the synchronization of fractional order chaotic systems. The salient feature of the article is the analysis of time of synchronization between fractional order Chen and Qi systems using both the methods. Numerical simulation and graphical results clearly exhibit that backstepping approach is better than active control method for synchronization of the considered pair of systems, as it takes less time to synchronize while using the first one compare to second one.

Keywords: Fractional derivative, Chaotic systems, Chen system, Qi system, Synchronization, Active control method, Backstepping method

1 Introduction

The beginning of the fractional calculus [1] is considered to be the Leibniz's letter to L'Hospital in 1695, where the notation for differentiation of non-integer order 1/2 is discussed. In addition, Leibniz wrote, "Thus it follows that $d^{1/2}x$ will be equal to $x\sqrt{dx : x}$, and this is an apparent paradox from which, one day, useful consequences will be drawn." Nowadays, not only fractions but also arbitrary real and even complex numbers are considered as order of differentiation. The theory of fractional calculus gives us flexibility for the generalization of the order of the derivative and integration from integer to any real number. Nevertheless, the name "fractional calculus" is kept for the general theory. Again due to the non-local property of fractional order differential operator, it takes into account the fact that the future state depends upon the present state as well as all of the history of its previous states, the fractional calculus which was in earlier stage considered as mathematical curiosity now becomes the object for the extensive development of fractional order partial differential equations for its applications in various physical areas of science and engineering. Geometric and physical interpretations of fractional differentiation and fractional integration can be found in Podlubny's work [2].

Though the idea of chaos theory came from observing weather patterns [3], but eventually it has become applicable to a variety of other disciplines. Some areas in mathematics, computer science, microbiology, meteorology, biology, engineering, geology, finance, economics, algorithmic trading, politics, population dynamics, psychology, philosophy and robotics are already benefited by chaos theory.

Chaos synchronization is an important topic in the nonlinear science. Generally speaking, the synchronization phenomenon has the feature that the trajectories of two systems (drive and response systems) are identical, in spite of starting from different initial conditions. However, slight changes in the initial conditions may lead to completely different trajectories. Therefore, how to control two chaotic systems to be synchronized have received a great deal of interest in past two decades to the researchers working in the field of chaos theory.

Various forms of synchronization that have been observed in the literature are phase synchronization, lag synchronization, generalized synchronization and sequential synchronization. Notable among the various methods for achieving this aim include linear feed-back, adaptive synchronization, backstepping control, sliding mode control, active control method, OGY method, projective synchronization, inverse optimal control ([4–12]).

Active control method, proposed by Bai and Longren [13] has widely been accepted as an efficient technique for the synchronization of non-identical chaotic systems, a feature for which it has got advantage over other synchronization methods. After giving a generalized design of the method by Ho and Hung [14], the method had been treated as one of the most interesting control strategies for its simplicity. Despite of the fact that active control method is an expensive strategy as its takes comparative more time for synchronization as compared to other

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exiting methods, it is noticed that the method is effective and convenient to synchronize since Lyapunov exponents are not required mathematical calculation. The method has been well tested to many practical systems such as spatiotemporal dynamical systems, nonlinear bloch equations modeling, nuclear magnetic resonance, electric circuits modeling, complex dynamos, acoustic gravity waves, parametrically excited systems and RCL-shunted Josephson junctions etc. It is seen from literature that Vincent and Laoye [15] used active control based synchronization scheme for controlling directed transport arising from coexisting attractors in non-equilibrium physics.

Backstepping design and active control are both recognised as powerful design methods for chaos synchronization. Backstepping design can guarantee global stability, tracking and transient performance for a broad class of strict-feedback nonlinear systems ([16–18]). The method has been employed for controlling and synchronizing many chaotic systems as well as hyperchaotic systems. Some of the method is widely used for synchronization of chaotic systems for the advantage that it needs only one controller to realize synchronization between chaotic systems and finally there are no derivatives in the controller [19].

In the present article, active control method and backstepping approach are used to synchronize the fractional order chaotic systems. The fractional order Chen and Qi systems are taken to synchronize using both the methods. In 2008, both the methods are used by Vincent [21] during synchronization of identical integer order systems already considered by A.M. Harb and M.A. Zohdy. This has inspired the authors to extend it in fractional order system. The main feature of the article is a comparative study of time of synchronization through numerical simulation and graphical presentation.

2 Definition and lemma

Definition: There are several definitions for fractional order derivative ([22–24]). Since the Caputo's fractional derivative of a constant is zero, we choose Caputo's definition [25] which is written as

$$ {}_a^cD_t^q x(t) = \frac{d^q x(t)}{dt^q} = \begin{cases} \dfrac{1}{\Gamma(n-q)} \displaystyle\int_a^t \frac{x^{(n)}(\tau)}{(t-\tau)^{q+1-n}} d\tau\,, & n-1 < q < n \\ \dfrac{d^n x(t)}{dt^n}\,, & q = n\,, \end{cases} $$

where $0 < q \in R$, $n \in N$ and $\Gamma(\cdot)$ is the Gamma function.

Lemma: [26] Let $x(t) \in R$ be a continuous and derivable function. Then, for any time instant $t \geq a$,

$$ \frac{1}{2} {}_a^cD_t^q x^2(t) \leq x(t) {}_a^cD_t^q x(t), \qquad \forall\, q \in (0,\ 1). $$

3 Systems' descriptions

3.1 Fractional order Chen system

The fractional order Chen system [27] of order q, is given by

$$ \begin{aligned} \frac{d^q x_1}{dt^q} &= a_1 (x_2 - x_1)\,, \\ \frac{d^q x_2}{dt^q} &= (a_3 - a_1)x_1 - x_1x_3 + a_3x_2\,, \\ \frac{d^q x_3}{dt^q} &= x_1x_2 - a_2x_3\,, \quad 0 < q < 1, \end{aligned} \tag{1} $$

where a_1, a_2, a_3 are the parameters of the system. The phase portrait of the fractional order Chen system in $x_1 - x_2 - x_3$ space is depicted through Fig. 1 at $q = 0.96$ for the values of parameters $a_1 = 35$, $a_2 = 3$, $a_3 = 28$ and the initial condition $(10,\ 25,\ 36)$.

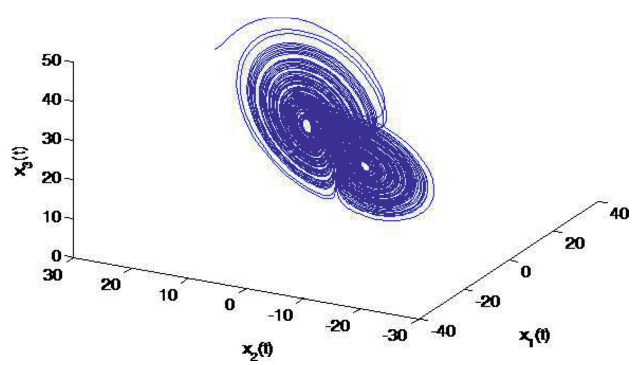


Fig. 1: Phase portrait of the Chen system in $x_1 - x_2 - x_3$ space at the order $q = 0.96$.

3.2 Fractional order Qi system

The fractional order Qi system [28] of order q, is given by

$$ \begin{aligned} \frac{d^q y_1}{dt^q} &= b_1 (y_2 - y_1) + y_2y_3\,, \\ \frac{d^q y_2}{dt^q} &= b_3y_1 - y_2 - y_1y_3\,, \\ \frac{d^q y_3}{dt^q} &= -b_2y_3 + y_1y_2\,, \quad 0 < q < 1, \end{aligned} \tag{2} $$

where b_1, b_2 and b_3 are the parameters of the system.

The phase portrait of system (2) at $q = 0.96$ is depicted through the Fig. 2 at the values of the parameters $b_1 = 35$, $b_2 = \frac{8}{3}$, $b_3 = 80$ and initial condition $(3, 2, 1)$.

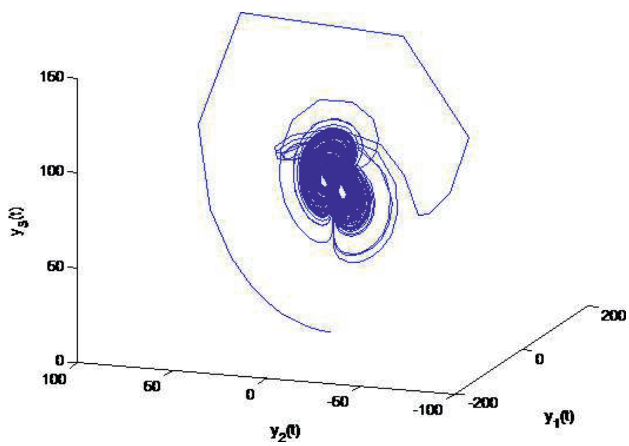


Fig. 2: Phase portrait of the Qi system in $y_1 - y_2 - y_3$ space for the order of derivative $q = 0.96$.

4 Synchronization of fractional order Chen and Qi systems using active control method

In this section, our aim is to achieve synchronization between fractional order Chen system and Qi system by using active control method. In this method, we assume that the Chen system (1) drives the Qi system (2). The response system is defined as

$$\begin{aligned}
\frac{d^q y_1}{dt^q} &= b_1(y_2 - y_1) + y_2 y_3 + u_1(t),\\
\frac{d^q y_2}{dt^q} &= b_3 y_1 - y_2 - y_1 y_3 + u_2(t),\\
\frac{d^q y_3}{dt^q} &= -b_2 y_3 + y_1 y_2 + u_3(t),
\end{aligned} \tag{3}$$

where $u_1(t)$, $u_2(t)$ and $u_3(t)$ are control functions.

Let us define the error $e = (e_1(t), e_2(t), e_3(t))^t$ as

$$e_i(t) = y_i - x_i, \quad i = 1, 2, 3. \tag{4}$$

Since the derivative operator is linear, we get the error system as

$$\begin{aligned}
\frac{d^q e_1}{dt^q} &= b_1(e_2 - e_1) + (b_1 - a_1)(x_2 - x_1) + y_2 y_3 + u_1(t),\\
\frac{d^q e_2}{dt^q} &= b_3 e_1 - e_2 + (b_3 - a_3 + a_1)x_1 + x_1 x_3 - y_1 y_3\\
&\quad - (1 + a_3)x_2 + u_2(t),\\
\frac{d^q e_3}{dt^q} &= -b_2 e_3 + (a_2 - b_2)x_3 + y_1 y_2 - x_1 x_2 + u_3(t).
\end{aligned} \tag{5}$$

To achieve our aim, we have to re-define the active control function as

$$\begin{aligned}
u_1(t) &= V_1(t) - (b_1 - a_1)(x_2 - x_1) - y_2 y_3,\\
u_2(t) &= V_2(t) - (b_3 - a_3 + a_1)x_1 - x_1 x_3 + y_1 y_3 + (1 + a_3)x_2,\\
u_3(t) &= V_3(t) - (a_2 - b_2)x_3 - y_1 y_2 + x_1 x_2.
\end{aligned}$$

Using above equations in system (5), we get the error system as

$$\begin{aligned}
\frac{d^q e_1}{dt^q} &= b_1(e_2 - e_1) + V_1(t),\\
\frac{d^q e_2}{dt^q} &= b_3 e_1 - e_2 + V_2(t),\\
\frac{d^q e_3}{dt^q} &= -b_2 e_3 + V_3(t).
\end{aligned} \tag{6}$$

To control the linear error system (6) with control inputs $V_1(t)$, $V_2(t)$ and $V_3(t)$ as functions of the error states $e_i(t)$, $i = 1, 2, 3$, our aim is to find the feedback control function in such a way that $e_i(t) \to 0$ as $t \to \infty$, $i = 1, 2, 3$, so that the systems (1) and (3) are globally synchronized. To achieve this, there are many choices for the control functions $V_1(t)$, $V_2(t)$ and $V_3(t)$. We choose

$$\begin{bmatrix} V_1(t) \\ V_2(t) \\ V_3(t) \end{bmatrix} = A \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix},$$

where A is a 3×3 constant matrix. In order to make the closed loop system stable, we choose the elements of the matrix A in such way that the error system must have all the eigenvalues with negative real parts.

Let the matrix A is chosen in the form

$$A = \begin{bmatrix} b_1 - 1 & -b_1 & 0 \\ -b_3 & 0 & 0 \\ 0 & 0 & b_2 - 1 \end{bmatrix}.$$

In this particular choice, the closed loop system (6) has the eigenvalues -1, -1 and -1. This choice leads to a stable system and thus the synchronization between fractional order Chen system and Qi system is achieved.

5 Synchronization of fractional order Chen and Qi systems using backstepping approach

The drive system is taken as fractional order Chen system (1) and fractional order Qi system considered as response system (3).

Defining error states as $e_1 = y_1 - x_1$, $e_2 = y_2 - x_2$, $e_3 = y_3 - x_3$, the error dynamical system reduces to

$$\begin{aligned}\frac{d^q e_1}{dt^q} &= b_1(e_2 - e_1) + e_2e_3 + e_2x_3 + e_3x_2 + \varphi_1 + u_1(t),\\ \frac{d^q e_2}{dt^q} &= b_3e_1 - e_2 - e_1e_3 - e_1x_3 - e_3x_1 + \varphi_2 + u_2(t),\\ \frac{d^q e_3}{dt^q} &= -b_2e_3 + e_1e_2 + e_1x_2 + e_2x_1 + \varphi_3 + u_3(t),\end{aligned} \tag{7}$$

where

$$\begin{aligned}\varphi_1 &= x_2x_3 + (b_1 - a_1)(x_2 - x_1),\\ \varphi_2 &= b_3x_1 - x_2 - (a_3 - a_1)x_1 - a_3x_2,\\ \varphi_3 &= (a_2 - b_2)x_3.\end{aligned}$$

Equation (7) can be considered as control problem where the system is to be controlled by the control functions $u_1(t)$, $u_2(t)$ and $u_3(t)$ which are the functions of error vectors $e_1(t)$, $e_2(t)$ and $e_3(t)$.

If the error states $e_1(t)$, $e_2(t)$ and $e_3(t)$ converge to zero as time t becomes large, then the systems (1) and (3) are said to be synchronized. Now our aim is to design the control functions for synchronization of fractional order chaotic systems using backstepping approach.

Theorem. If the control function are chosen as

$$\begin{aligned}u_1(t) &= -\varphi_1 - w_3x_2,\\ u_2(t) &= -\varphi_2 - b_3w_1 - b_1w_1,\\ u_3(t) &= -\varphi_3 - w_1x_2 - w_1w_2,\end{aligned}$$

where $w_1 = e_1$, $w_2 = e_2$, $w_3 = e_3$, then the drive system (1) will be synchronized with the response system (3).

Proof. To achieve our results we use backstepping procedure which has three steps.

Step-I: Defining $w_1 = e_1$, we have

$$\frac{d^q w_1}{dt^q} = b_1(e_2 - w_1) + e_2e_3 + e_2x_3 + e_3x_2 + \varphi_1 + u_1(t), \tag{8}$$

where $e_2 = \alpha_1(w_1)$ is regarded as an virtual controller. For the design of $\alpha_1(w_1)$ to stabilize w_1–subsystem, choose the Lyapunov function as

$$v_1 = \frac{1}{2}w_1^2.$$

The q–th order derivative of v_1 is

$$\frac{d^q v_1}{dt^q} = \frac{1}{2}\frac{d^q w_1^2}{dt^q} \le w_1\frac{d^q w_1}{dt^q} \quad \text{(using lemma)}$$

i.e.,

$$\frac{d^q v_1}{dt^q} \le w_1\left[b_1(\alpha_1(w_1) - w_1) + \alpha_1(w_1)e_3 + \alpha_1(w_1)x_3 + e_3x_2 + \varphi_1 + u_1(t)\right].$$

If we choose $u_1(t) = -\varphi_1 - e_3x_2$ and $\alpha_1(w_1) = 0$, then $\frac{d^q v_1}{dt^q} \le -b_1w_1^2 < 0$ is negative definite. This implies that the w_1–subsystem (8) is asymptotically stable.

Defining the error variable between e_2 and the estimative virtual controller $\alpha_1(w_1)$ as $w_2 = e_2 - \alpha_1(w_1)$, we can obtain the following (w_1, w_2)–subsystem as

$$\begin{aligned}\frac{d^q w_1}{dt^q} &= b_1(w_2 - w_1) + w_2e_3 + w_2x_3,\\ \frac{d^q w_2}{dt^q} &= b_3w_1 - w_2 - w_1e_3 - w_1x_3 - e_3x_1 + \varphi_2 + u_2(t),\end{aligned} \tag{9}$$

where $e_3 = \alpha_2(w_1, w_2)$ is a virtual controller.

Step-II: In this step, we define the following Lyapunov function v_2 to stabilize (w_1, w_2)–subsystem (9) as

$$v_2 = v_1 + \frac{1}{2}w_2^2 = \frac{1}{2}w_1^2 + \frac{1}{2}w_2^2.$$

The q–th order derivative of v_2 w.r. to t is

$$\frac{d^q v_2}{dt^q} = \frac{1}{2}\frac{d^q w_1^2}{dt^q} + \frac{1}{2}\frac{d^q w_2^2}{dt^q} \le w_1\frac{d^q w_1}{dt^q} + w_2\frac{d^q w_2}{dt^q}$$

(using lemma), i.e.,

$$\begin{aligned}\frac{d^q v_2}{dt^q} \le{}& -b_1w_1^2 - w_2^2 + w_1[b_1w_2 + w_2\alpha_2(w_1, w_2) + w_2x_3]\\ &+ w_2[b_3w_1 - w_1\alpha_2(w_1, w_2) - w_1x_3 - \alpha_2(w_1, w_2)x_1 + \varphi_2 + u_2(t)].\end{aligned}$$

If we choose $\alpha_2(w_1, w_2) = 0$ and $u_2(t) = -\varphi_2 - b_3w_1 - b_1w_1$, then $\frac{d^q v_2}{dt^q} \le -b_1w_1^2 - w_2^2 < 0$ is negative define which shows that (w_1, w_2)–subsystem (9) is asymptotically stable.

Similarly considering $w_3 = e_3 - \alpha_2(w_1, w_2)$, we get the following (w_1, w_2, w_3)–system as

$$\begin{aligned}\frac{d^q w_1}{dt^q} &= b_1(w_2 - w_1) + w_2w_3 + w_2x_3,\\ \frac{d^q w_2}{dt^q} &= -w_2 - w_1w_3 - w_1x_3 - w_3x_1 - b_1w_1,\\ \frac{d^q w_3}{dt^q} &= -b_2w_3 + w_1w_2 + w_1x_2 + w_2x_1 + \varphi_3 + u_3(t).\end{aligned} \tag{10}$$

Step-III: In order to stabilize (w_1, w_2, w_3)–system, define the Lyapunov function v_3 as

$$v_3 = v_2 + \frac{1}{2}w_3^2 = \frac{1}{2}w_1^2 + \frac{1}{2}w_2^2 + \frac{1}{2}w_3^2.$$

The q–th order fractional order derivative of v_3 w.r. to t is

$$\begin{aligned}\frac{d^q v_3}{dt^q} &= \frac{1}{2}\frac{d^q w_1^2}{dt^q} + \frac{1}{2}\frac{d^q w_2^2}{dt^q} + \frac{1}{2}\frac{d^q w_3^2}{dt^q}\\ &\le w_1\frac{d^q w_1}{dt^q} + w_2\frac{d^q w_2}{dt^q} + w_3\frac{d^q w_3}{dt^q} \quad \text{(using lemma)}\end{aligned}$$

i.e.,

$$\frac{d^q v_3}{dt^q} \le -b_1 w_1^2 - w_2^2 - b_2 w_3^2 - w_2 w_3 x_1 + w_3[w_1 w_2 + w_1 x_2 + w_2 x_1 + \varphi_3 + u_3(t)].$$

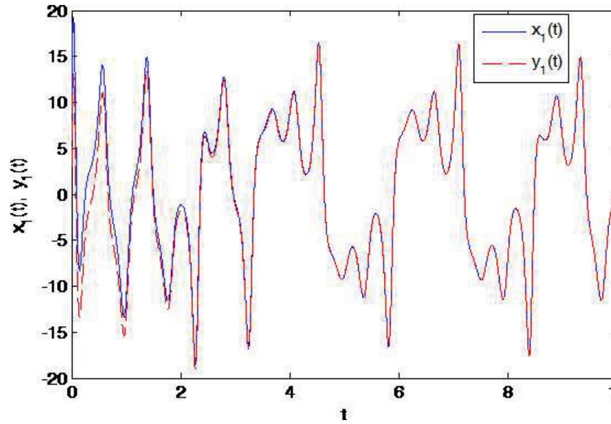
If we choose $u_3(t) = -\varphi_3 - w_1 x_2 - w_1 w_2$, then $\frac{d^q v_3}{dt^q} \le -b_1 w_1^2 - w_2^2 - b_2 w_3^2 < 0$, which implies that (w_1, w_2, w_3)–system is asymptotically stable. Thus for $w_1 = e_1$, $w_2 = e_2 - \alpha_1(w_1) = e_2$, $w_3 = e_3 - \alpha_2(w_1, w_2) = e_3$, the state errors e_1, e_2 and e_3 converge to zero after a finite period of time, confirm the synchronization between fractional order Chen system and Qi system. □

6 Numerical simulation and results

In this section, we take the earlier considered values of the parameters of both the systems. The initial conditions of drive and response systems are taken as $(x_1(0), x_2(0), x_3(0)) = (10, 25, 36)$ and $(y_1(0), y_2(0), y_3(0)) = (3, 2, 1)$, respectively. Hence the initial conditions of error system is $(e_1(0), e_2(0), e_3(0)) = (-7, -23, -35)$. During synchronization of the systems, the time step size is taken as 0.005. The synchronization between $x_1 - y_1$, $x_2 - y_2$ and $x_3 - y_3$ are depicted through Fig. 3 and Fig. 5 at $q = 0.96$,for active control method and backstepping approach respectively. The error functions are depicted through Fig. 4 and Fig. 6 for active control method and backstepping approach respectively at $q = 0.92, 0.96, 1$. It is clear from the figures that in both the cases it takes less time to synchronize as the systems pair approaches from standard order to fractional order. Also it is found from the figures that it takes less time for synchronization for the backstepping method compared to active control method.

7 Conclusion

The theme of the present research article is to investigate the synchronization between two non-identical fractional order chaotic systems using active control method and backstepping method. Based on stability analysis, the required synchronization of the chaotic systems, viz., Chen and Qi systems has been achieved. The components of error systems tending to zero as time progresses is attempt through proper choices of control functions. This helps to get the time required for synchronization. The novelty of the article is the finding that less time is necessary for synchronization, when the system pair approaches to fractional order from the standard order upon application of both the methods, exhibited through graphical presentations. Additionally a comparison of time requirement for



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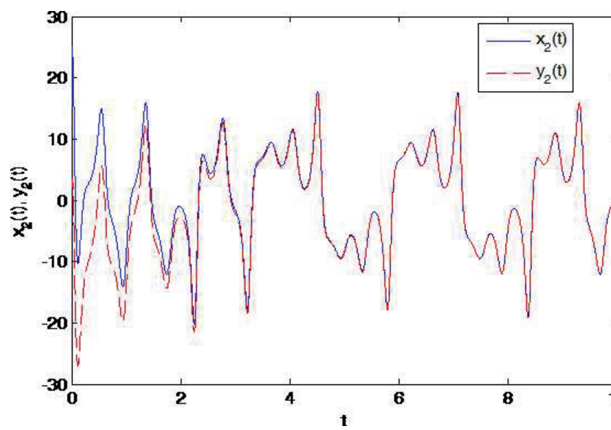

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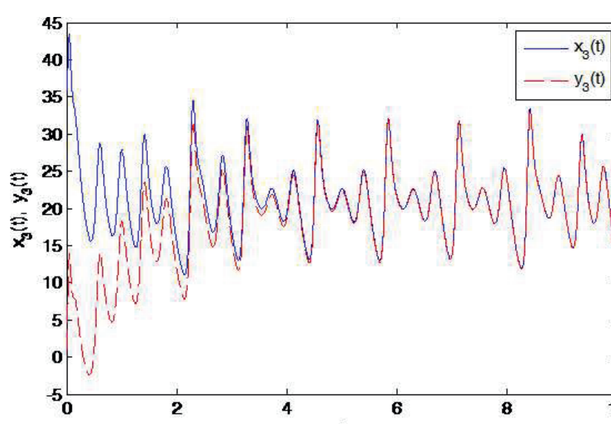
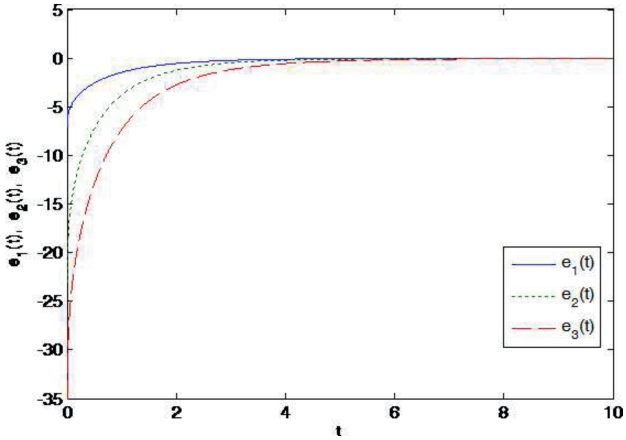
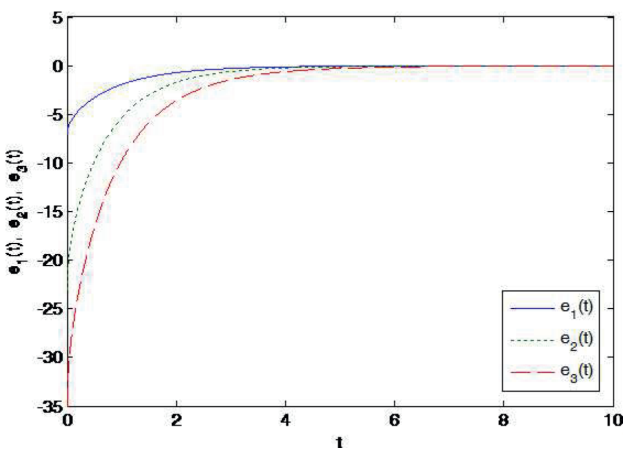


(c)

Fig. 3: State trajectories of drive system (1) and response system (3) for fractional order $q = 0.96$, (a) between x_1 and y_1, (b) between x_2 and y_2, (c) between x_3 and y_3 using active control method.



(a)



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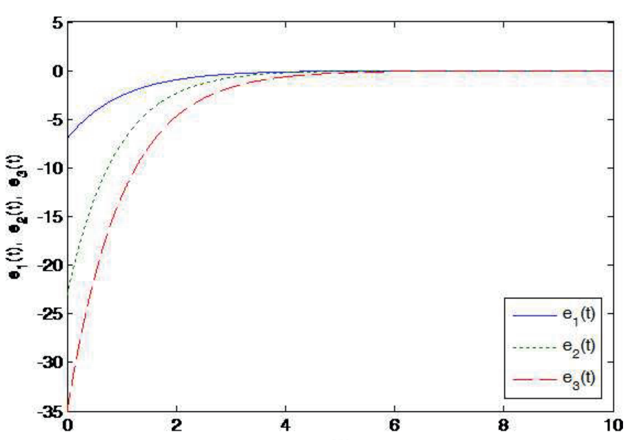
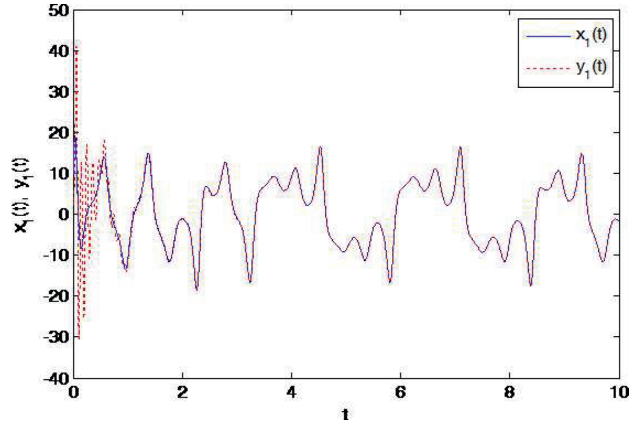


(c)

Fig. 4: The evolution of the error functions $e_1(t)$, $e_2(t)$ and $e_3(t)$: (a) $q = 0.92$, (b) $q = 0.96$, (c) $q = 1$, using active control method.



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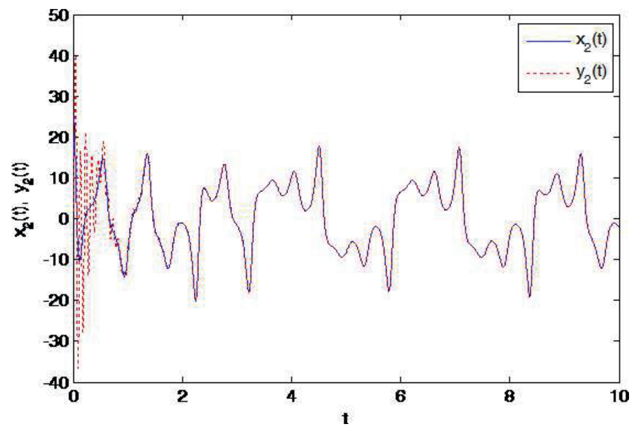

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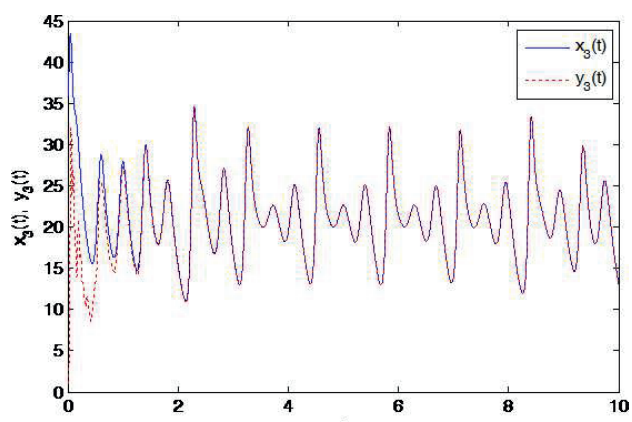

(c)

Fig. 5: State trajectories of drive system (1) and response system (3) for fractional order $q = 0.96$: (a) between x_1 and y_1, (b) between x_2 and y_2, (c) between x_3 and y_3 using backstepping method.

both standard and fractional order derivatives applying the active control method and backstepping method has been done.

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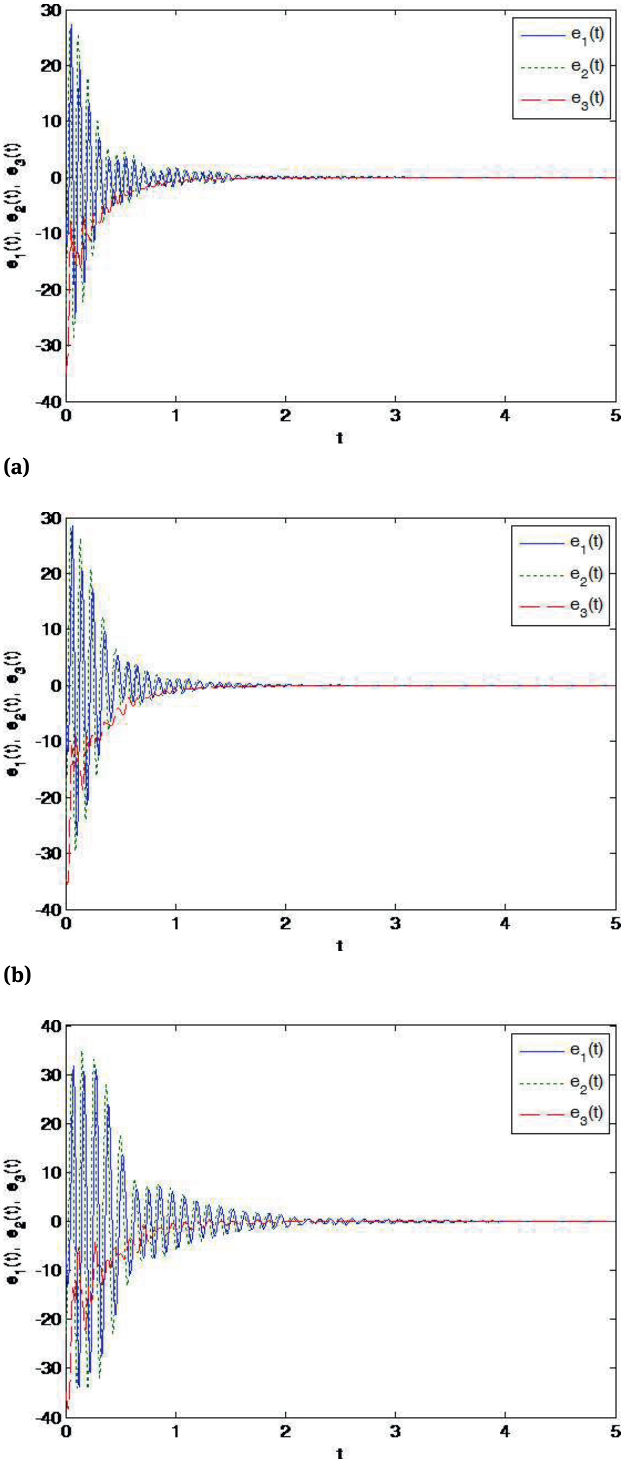

Fig. 6: The evolution of the error functions $e_1(t)$, $e_2(t)$ and $e_3(t)$: (a) $q = 0.92$, (b) $q = 0.96$, (c) $q = 1$, using backstepping method.

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