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A Legendre spectral finite difference method for the solution of non-linear space-time fractional Burger's–Huxley and reaction-diffusion equation with Atangana–Baleanu derivative

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In this research, we have solved non-linear reaction-diffusion equation and non-linear Burger's–Huxley equation with Atangana Baleanu Caputo derivative. We developed a numerical approximation for the ABC derivative of Legendre polynomial. A difference scheme is applied to deal with fractional differential term in the time direction of differential equation. We applied Legendre spectral method to deal with unknown function and spatial ABC derivatives. A formulation to deal with Dirichlet boundary condition is also included. After applying this spectral method our problem reduces to a system of fractional partial differential equation. To solve this system we developed finite difference scheme by which our FPDEs system reduces to a system of algebraic equations. Taking the help of initial conditions we solve this algebraic system and find the value of unknowns, To demonstrate the effectiveness and validity of our proposed method some numerical examples are also presented. We compare our obtained numerical results with the analytical results.

1. Introduction

Fractional calculus is an ancient topic of mathematics with history as ordinary or integer calculus [1]. It is developing progressively now. Theory of fractional calculus is developed by N. H. Abel and J. Liouville. The details can be found in [2]. Fractional calculus allows to generalize derivatives and integrals of integer order to real or variable order. It also can be considered as a branch of mathematical analysis that allows to investigate with real differential operators and equations where types of integral are convolution or weakly singular. It has a widely applications in control theory, stochastic process and special functions. Fractional calculus was considered as a esoteric theory without applications but in the last few years there has been a boost of research on its applications to economics, control system to finance. Many different forms of fractional order differential operators were introduced as the Hadamard, Caputo, Grunwald–Letnikov, Riemann–Liouville, Riesz and variable order operators. Due to its increasing applications, the researchers have paid their attention to find numerical and exact solutions of the fractional order differential equations. As there are many difficulties to solve a fractional order differential equation by analytic method so there is a need of seeking numerical solutions. There are many numerical methods available in literature viz., eigen-vector expansion, Adomain decomposition method [3], fractional differential transform method [4], homotopy perturbation method [5,6], homotopy perturbation transform method [7], predictor-corrector method [8] and generalized block pulse operational matrix method [9] etc. Some numerical methods based upon operational matrices of fractional order differentiation and integration with Legendre wavelets [10], Chebyshev wavelets [11], sine wavelets, Haar wavelets [12] have been developed to find the solutions of fractional order differential and integro-differential equations. The functions which are commonly used include Legendre polynomial [13], Laguerre polynomial [14], Chebyshev polynomial and semi-orthogonal polynomial as Genocchi polynomial [15]. The series solution for the time-fractional coupled mKdV equation using Homotopy analysis method is investigated by Francisco Gómez [16]. An analytic method i.e, Laplace homotopy analysis method for solving FPDEs having non-singular kernel is given in [17,18]. The numerical solution of Fractional Hunter–Saxton equation involving partial operators with bi-order in Riemann–Liouville and Liouville–Caputo sense is investigated in [19]. The Fengâs first integral method for the solution of non-linear mKdV space-time fractional partial differential equation is developed in article [20]. The modeling and simulations of real life problem is described by PDEs, integral and integro-differential equations. These equations have a lot of applications such as

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heat conduction in materials with memory, population dynamics, nuclear reactor dynamics, fluid dynamics, and compression of viscoelastic media. In many fields like as thermo elasticity and dynamics of nuclear reactor we have to depict the memory effect of the systems. When we modeled these systems using PDE, which included function at a given point of time and space, we ignored the effect of history. Therefore to include the memory effect of these systems an term of integration is added to this PDE. Partial integro differential equations have a widely applications in aerospace systems, chemical kinetics, biological models, control theory of financial mathematics and industrial mathematics. Apart of it many phenomena in physics like as viscoelastic mechanics, fluid dynamics, control theory and thermoelastic.

Reaction-diffusion process has been investigated from a long time. In the process of reaction- diffusion, reacting molecules are used to move through space due to diffusion. This definition excludes other modes of transports as convection, drift those may arise due to presence of externally imposed fields.

When a reaction occurs within an element of space, molecules can be created or consumed. These events are added to the diffusion equation and lead to reaction-diffusion equation of the form

$$\frac{\partial c}{\partial t} = D\nabla^2 c + R(c,t), \tag{1}$$

where $R(c, t)$ denotes reaction term at time t. The extension of the reaction-diffusion equation in fractional order system can be found in the articles [21]. In nature many of the beautiful systems in biology, physics,chemistry, and physiology can be described by reaction diffusion equations. For example, the distribution and organization of vegetation-like bushes in arid ecosystems [22], the stripes and spots on fish [23], snakes [24] and the skin or fur of mammals [25] have been studied by the standing waves which are produced by reaction-diffusion equations. Hodgkin and Huxley [26] proposed a model, known henceforth as the Huxley equation, to explain the ionic mechanisms underlying the initiation and propagation of action potentials in the squid giant axon. The most general form of the Huxley equation, known as the generalized BurgersHuxley equation. The generalized BurgersHuxley equation describes a wide class of physical nonlinear phenomena such as the interaction between reaction mechanisms, convection effects, and diffusion transports. It is used in many fields such as biology, chemistry, metallurgy,combustion, mathematics, and engineering.

Many mathematical functions arising in social sciences, engineering and natural sciences follow three mathematical laws namely power function, exponential decay law and Mittag–Leffler function. The fractional differentiation based upon these three laws are known as Riemann–Liouville and Caputo, Caputo–Fabrizio and Atangana–Baleanu derivative. A new fractional derivative with combination of power law, exponential law and Mittag–Leffler kernel is developed by Abdon Atangana and Francisco Gómez [27]. Some researchers gives a novel fractional conformable derivative of R-L type having order $\alpha = n - \epsilon$ where ϵ is a small quantity. These type of derivative is useful for the electrical circuits LC and RL and for the equation that characterize the motion of charged particle in electromagnetic field [28]. The normal distribution a common continuous probability distribution is highly used in social sciences and natural science to portray real-valued random variables [29]. The application of fractional operator having non index law to statistics and dynamical systems is shown in article [30] taking 4 examples of 3-D novel chaotic system, King cobra chaotic system, Ikeda delay system and chameleon system. Hyper chaotic behavior obtained via a nonlocal operator with exponential decay and Mittag–Leffler laws is given in article [31].

Our article is outlined as follows. In Section 2, we discussed about Caputo, RL and ABC fractional derivative and presented some properties of these derivatives. In Section 3, we develop a approximation formula of ABC derivative of the function x^k. It also continues definition and properties of Legendre polynomial with approximation of arbitrary function which has been written in linear combination of these Legendre polynomial.In Section 4, a spectral method to find out the numerical solution of Burger–Huxley equation and reaction-diffusion equation with combination of finite difference scheme is given. In Section 5, some numerical examples and results are presents. The last section includes the conclusion of all over work.

2. Preliminaries

Here, few definitions and important properties of fractional calculus have been introduced. It is well known that the Riemann–Liouville definition has disadvantages when it comes for modeling of real world problems. But definition of fractional differentiation given by M. Caputo is more reliable for application point of view. Nowadays new general type of fractional operators have been discovered. A brief description of ABC derivative is discussed here.

2.1. Definition of R-L order derivative and integration

Fractional order integration of Riemann–Liouville type of a given order ϑ of a function $h(t)$ is given by [32]

$$I^{\vartheta} h(t) = \frac{1}{\Gamma(\vartheta)} \int_0^t (t-\omega)^{\vartheta-1} h(\omega)\, d\omega, \;\; t > 0, \;\; \vartheta \in R^+. \tag{2}$$

Fractional order derivative of the Riemann–Liouville type of order $\vartheta > 0$ can be defined as

$$D_l^{\vartheta} h(t) = \left(\frac{d}{dt}\right)^m (I^{m-\vartheta} h)(t), \;\; (\vartheta > 0, \;\; m-1 < \vartheta < m). \tag{3}$$

2.2. Definition of Caputo derivative

Fractional derivative of order $\vartheta > 0$ in Caputo sense can be defined as

$$D_c^{\vartheta} h(t) = \begin{cases} \frac{d^l h(t)}{dt^l} & \vartheta = l \in N \\ \frac{1}{\Gamma(l-\vartheta)} \int_0^t (t-\eta)^{l-\vartheta-1} h^l(\eta)\, d\eta & l-1 < \vartheta < l. \end{cases} \tag{4}$$

Here, l is an integer, $t > 0$.

Basic properties of Caputo fractional derivative are:

$$D_c^{\vartheta} C = 0, \tag{5}$$

where C is a constant.

$$D_c^{\vartheta} t^{\sigma} = \begin{cases} 0, & \sigma \in N \cup 0 \;\; and \;\; \sigma < \lceil \vartheta \rceil \\ \dfrac{\Gamma\,(1+\sigma)}{\Gamma(1-\vartheta+\sigma)} t^{-\vartheta+\sigma} & \sigma \in N \cup 0 \;\; and \;\; \sigma \geq \lceil \vartheta \rceil \\ & or \; \sigma \notin N \;\; and \;\; \sigma > \lfloor \vartheta \rfloor, \end{cases} \tag{6}$$

where $\lfloor \vartheta \rfloor$ is floor function.

The operator D_c^{ϑ} is linear, since

$$D_c^{\vartheta} (ah(t) + bg(t)) = aD_c^{\vartheta} h(t) + bD_c^{\vartheta} g(t), \tag{7}$$

where $a\, and\, b$ are constants. Caputo operator and Riemann–Liouville operator have a relation:

$$(I^{\vartheta} D_c^{\vartheta} g)(t) = g(t) - \sum_{k=0}^{l-1} g^k(0^+) \frac{t^k}{k!}, \quad l-1 < \vartheta \leq l. \tag{8}$$

2.3. Definition of ABC derivative [33–35]

Let $g(t)$ be a function which belongs to Sobolev space $H^1(0, 1)$ then Atangana–Baleanu Caputo derivative in Caputo sense of order ϑ is defined by

$$ {}_0^{ABC}D_t^{\vartheta} g(t) = \frac{B(\vartheta)}{\lceil \vartheta \rceil - \vartheta} \int_0^t \frac{\partial^{n+1} g(s)}{\partial s^{n+1}} \times E_{\vartheta} \left[\frac{-\vartheta}{\lceil \vartheta \rceil - \vartheta} (t-s)^{\vartheta} \right] ds,$$

$$n < \vartheta \leq n+1, \tag{9}$$

where $B(\vartheta)$ is a normalization function such that $B(0)=B(1)=1$ and $E_\vartheta(z)$ is Mittag–Leffler function defined as

$$E_\vartheta(z)=\sum_{i=0}^{\infty}\frac{z^i}{\Gamma(i\vartheta+1)}.$$

3. Approximation of ABC derivative

Theorem 1. *If $\phi(x)\in H^1(a,b)$ is a function such that $\phi(x)=x^\gamma$. Then the approximation of ABC derivative w.r.t x of order $n<\vartheta\le n+1$ of this function can be defined as follows:*

$${}_0^{ABC}D_x^\vartheta x^\gamma = \frac{B(\vartheta)\Gamma(\gamma+1)}{\lceil\vartheta\rceil-\vartheta}\left(\sum_{j=0}^{\gamma-n-1}\frac{x^{\gamma+j\vartheta-n}}{\Gamma(\gamma+j\vartheta-n+1)}\times\left(\frac{-\vartheta}{\lceil\vartheta\rceil-\vartheta}\right)^j\right). \tag{10}$$

Proof: Using the definition (9) of ABC derivative we have

$${}_0^{ABC}D_x^\vartheta x^\gamma = \frac{B(\vartheta)}{\lceil\vartheta\rceil-\vartheta}\int_0^x\frac{\partial^{n+1}}{\partial s^{n+1}}s^\gamma\times E_\vartheta\left[\frac{-\vartheta}{\lceil\vartheta\rceil-\vartheta}(x-s)^\vartheta\right]ds \tag{11}$$

$$=\frac{B(\vartheta)}{\lceil\vartheta\rceil-\vartheta}\int_0^x\frac{\Gamma(\gamma+1)s^{\gamma-n-1}}{\Gamma(\gamma-n)}\times\frac{(x-s)^{j\vartheta}}{\Gamma(\vartheta j+1)}\left(\frac{-\vartheta}{\lceil\vartheta\rceil-\vartheta}\right)^j ds \tag{12}$$

$$=\frac{B(\vartheta)\Gamma(\gamma+1)}{\lceil\vartheta\rceil-\vartheta}\sum_{j=0}^{\gamma-n-1}\frac{x^{\gamma+j\vartheta-n}}{\Gamma(\gamma+j\vartheta-n+1)}\left(\frac{-\vartheta}{\lceil\vartheta\rceil-\vartheta}\right)^j. \tag{13}$$

A series form of Legendre polynomial is defined as follows

$$\psi_j(x)=\sum_{k=0}^{j}(-1)^{j+k}\frac{(j+k)!}{(j-k)!(k!)^2}x^k. \tag{14}$$

Any function $u(x)\in L^2[0,1]$ can be written as linear combination of Legendre polynomials as

$$u(x)=\sum_{i=0}^{m}b_i\psi_i(x). \tag{15}$$

The coefficients b_i can be determined as follows

$$b_i=(2i+1)\int_0^1u(x)\psi_i(x)dx. \tag{16}$$

The lth order integer derivative of $\psi_i(x)$ with respect to x is as written as follows:

$${}_0D_x^l\psi_i(x)=\sum_{j=0}^{i}\frac{(-1)^{i+j}(i+j)!j(j-1).\cdots(j-l+1)}{(i-j)!(j!)^2}x^{j-l}. \tag{17}$$

Theorem 2. *The approximation of ${}_0^{ABC}D_x^\vartheta u(x)$ with the help of Eq. (15) is defined as*

$${}_0^{ABC}D_x^\vartheta u(x)=\sum_{i=\lceil\vartheta\rceil}^{m}\sum_{j=\lceil\vartheta\rceil}^{i}b_i\frac{(-1)^{i+j}(i+j)!}{(i-j)!(j!)^2}\frac{B(\vartheta)\Gamma(j+1)}{\lceil\vartheta\rceil-\vartheta}\Pi_{i,j,\vartheta}(x), \tag{18}$$

where,

$$\Pi_{i,j,\vartheta}(x)=\sum_{s=0}^{j-n-1}\frac{x^{j+s\vartheta-n}}{\Gamma(j+s\vartheta-n+1)}\left(\frac{-\vartheta}{\lceil\vartheta\rceil-\vartheta}\right)^s. \tag{19}$$

Proof: Operating ABC derivative of order ϑ w.r.t x on the both side of Eq. (15) and using the linear property of ABC derivative we get

$${}_0^{ABC}D_x^\vartheta u(x)=\sum_{i=0}^{m}b_i\times{}_0^{ABC}D_x(\psi_i(x)). \tag{20}$$

In the contrast of Theorem 1 we have

$${}_0^{ABC}D_x(\psi_i(x))=0,\quad i=0,1,\ldots,\lceil\vartheta\rceil-1. \tag{21}$$

Now for $i=\lceil\vartheta\rceil\ldots,m$ we obtain

$${}_0^{ABC}D_x(\psi_i(x))=\sum_{j=\lceil\vartheta\rceil}^{i}\frac{(-1)^{i+j}(i+j)!}{(i-j)!(j!)^2}\times\frac{B(\vartheta)\Gamma(j+1)}{\lceil\vartheta\rceil-\vartheta}\times\sum_{s=0}^{j-n-1}\frac{x^{j+s\vartheta-n}}{\Gamma(j+s\vartheta-n+1)}\left(\frac{-\vartheta}{\lceil\vartheta\rceil-\vartheta}\right)^s. \tag{22}$$

Now putting this value in Eq. (20) and considering the Eq. (21) we get the desired result.

4. Procedure for the solution of the problem

In this section we investigate the non-linear time and space fractional generalized Burger–Huxley equation involving ABC derivative and space-time fractional non-linear reaction-diffusion equation.

Considering the following equation

$${}_0D_t^\alpha u(x,t)={}_0^{ABC}D_x^\vartheta u(x,t)-au^\delta(x,t)\frac{\partial u(x,t)}{\partial x}+bu(x,t)(1-u^\delta(x,t))(u^\delta-\eta), \tag{23}$$

where $a,\ b,\ \delta\ge0,\ 0<\alpha\le1$ and $1<\vartheta\le2$ along with initial and boundary conditions

$$u(x,0)=f_1(x),u(0,t)=g_1(t),u(1,t)=g_2(t), \tag{24}$$

and following reaction-diffusion equation

$${}_0D_t^\alpha u(x,t)={}_0^{ABC}D_x^\vartheta u(x,t)-du(x,t)(1-u(x,t))+f(x,t), \tag{25}$$

where $0<\alpha\le1$ and $1<\vartheta\le2$ along with initial and boundary conditions

$$u(x,0)=f_1(x),u(0,t)=g_1(t),u(1,t)=g_2(t). \tag{26}$$

To find numerical solution of above two model we use Legendre spectral method with combination of finite difference method. First we take an approximation of $u(x,t)$ as follows

$$u(x,t)=u_m(x,t)=\sum_{i=0}^{m}u_i(t)\psi_i(x). \tag{27}$$

Substituting the value of $u(x,t)$ in Eq. (23) we obtain

$$\sum_{i=0}^{m}\frac{\partial^\alpha u_i(t)}{\partial t^\alpha}\psi_i(x)=\sum_{i=0}^{m}{}_0^{ABC}D_x^\vartheta\psi_i(x)u_i(t)-a\left(\sum_{i=0}^{m}u_i(t)\psi_i(x)\right)^\delta\sum_{i=0}^{m}\frac{\partial\psi_i(x)}{\partial x}u_i(t)+b\left(\sum_{i=0}^{m}u_i(t)\psi_i(x)\right)\left(1-\left(\sum_{i=0}^{m}u_i(t)\psi_i(x)\right)^\delta\right)\times\left(\left(\sum_{i=0}^{m}u_i(t)\psi_i(x)\right)^\delta-\eta\right). \tag{28}$$

Now using the Theorem 2 we can rewrite the above equation as follows

$$\sum_{i=0}^{m}\frac{\partial^\alpha u_i(t)}{\partial t^\alpha}\psi_i(x)=\sum_{i=\lceil\vartheta\rceil}^{m}\sum_{j=\lceil\vartheta\rceil}^{i}\frac{(-1)^{i+j}(i+j)!}{(i-j)!(j!)^2}\frac{B(\vartheta)\Gamma(j+1)}{\lceil\vartheta\rceil-\vartheta}\Pi_{i,j,\vartheta}(x)u_i(t)$$
$$-a\left(\sum_{i=0}^{m}u_i(t)\psi_i(x)\right)^\delta\sum_{i=0}^{m}\sum_{j=0}^{i}\frac{(-1)^{i+j}(i+j)!x^{j-1}}{(i-j)!j!(j-1)!}u_i(t)$$
$$+b\left(\sum_{i=0}^{m}u_i(t)\psi_i(x)\right)\left(1-\left(\sum_{i=0}^{m}u_i(t)\psi_i(x)\right)^\delta\right)\times\left(\left(\sum_{i=0}^{m}u_i(t)\psi_i(x)\right)^\delta-\eta\right). \tag{29}$$

We collocate Eq. (31) at $m+1-\lceil\vartheta\rceil$ points $x=x_r$ with $r=0,1,\ldots,m-\lceil\vartheta\rceil$. So that we obtained a first order system of fractional ODEs:

$$\sum_{i=0}^{m}\frac{\partial^\alpha u_i(t)}{\partial t^\alpha}\psi_i(x_r)=\sum_{i=\lceil\vartheta\rceil}^{m}\sum_{j=\lceil\vartheta\rceil}^{i} \tag{30}$$
$$\times\frac{(-1)^{i+j}(i+j)!}{(i-j)!(j!)^2}\frac{B(\vartheta)\Gamma(j+1)}{\lceil\vartheta\rceil-\vartheta}\Pi_{i,j,\vartheta}(x_r)u_i(t)$$
$$-a\left(\sum_{i=0}^{m}u_i(t)\psi_i(x_r)\right)^\delta\sum_{i=0}^{m}\sum_{j=0}^{i}\frac{(-1)^{i+j}(i+j)!(x_r)^{j-1}}{(i-j)!j!(j-1)!}u_i(t)$$
$$+b\left(\sum_{i=0}^{m}u_i(t)\psi_i(x_r)\right)\left(1-\left(\sum_{i=0}^{m}u_i(t)\psi_i(x_r)\right)^\delta\right)\times\left(\left(\sum_{i=0}^{m}u_i(t)\psi_i(x_r)\right)^\delta-\eta\right). \tag{30}$$

In view of Eq. (27) boundary conditions (24) can be written as

$$\sum_{i=0}^{m}\psi_i(0)u_i(t)=g_1(t),$$
$$\sum_{i=0}^{m}\psi_i(1)u_i(t)=g_2(t). \tag{31}$$

Now to solve the system of FPDEs (30) and (31) we apply the finite difference scheme and will find out the unknowns $u_i(t)$ for $i=0,1,\ldots,m$. So first we discretized the time fractional derivative. Let the time interval is divided into N parts and $t_n=n\Delta t$, $n=0,1,\ldots,N$. The value of $u_i(t)$ at point $t=t_n$ is denoted by u_i^n and using the definition of Caputo's definition (4) we have

$$\frac{\partial^\alpha u_i(t_n)}{\partial t^\alpha}=\frac{1}{\Gamma(1-\alpha)}\int_0^{t_n}(t_n-s)^{-\alpha}\frac{\partial u_i}{\partial s}ds$$
$$=\frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{n-1}\frac{u_i^{j+1}-u_i^j}{\Delta t}\int_{t_j}^{t_{j+1}}(t_n-s)^{-\alpha}ds$$
$$=\frac{1}{\Gamma(2-\alpha)}\sum_{j=0}^{n-1}\frac{u_i^{j+1}-u_i^j}{\Delta t}\left[(t_n-t_j)^{1-\alpha}-(t_n-t_{j+1})^{1-\alpha}\right]. \tag{32}$$

Considering the system of FPDEs (30) and (31) at point $t=t_n$ and using Eq. (32) we get a set of non-linear algebraic equations

$$\frac{1}{\Gamma(2-\alpha)}\sum_{i=0}^{m}\sum_{j=0}^{n-1}\frac{u_i^{j+1}-u_i^j}{\Delta t}\left[(t_n-t_j)^{1-\alpha}-(t_n-t_{j+1})^{1-\alpha}\right]\psi_i(x_r)$$
$$=\sum_{i=\lceil\vartheta\rceil}^{m}\sum_{j=\lceil\vartheta\rceil}^{i}\frac{(-1)^{i+j}(i+j)!}{(i-j)!(j!)^2}\frac{B(\vartheta)\Gamma(j+1)}{\lceil\vartheta\rceil-\vartheta}\Pi_{i,j,\vartheta}(x_r)u_i^n$$
$$-a\left(\sum_{i=0}^{m}u_i^n\psi_i(x_r)\right)^\delta\sum_{i=0}^{m}\sum_{j=0}^{i}\frac{(-1)^{i+j}(i+j)!(x_r)^{j-1}}{(i-j)!j!(j-1)!}u_i^n$$
$$+b\left(\sum_{i=0}^{m}u_i^n\psi_i(x_r)\right)\left(1-\left(\sum_{i=0}^{m}u_i^n\psi_i(x_r)\right)^\delta\right)\times\left(\left(\sum_{i=0}^{m}u_i^n\psi_i(x_r)\right)^\delta-\eta\right), \tag{33}$$

and by the boundary conditions

$$\sum_{i=0}^{m}\psi_i(0)u_i^n=g_1(t_n),$$
$$\sum_{i=0}^{m}\psi_i(1)u_i^n=g_2(t_n). \tag{34}$$

Now to find the initial approximation, we will take the help of initial condition as follows

$$\sum_{i=0}^{m}u_i^0\psi_i(x_r)=f_1(x_r). \tag{35}$$

Collocating this equation at collocation points x_r with $r=0,1,\ldots,m+1-\lceil\vartheta\rceil$ we get a system of algebraic equations. By solving this system we get the value of initial approximation and using this initial approximation in FPDEs (33)–(34) we can find the value of unknowns $u_i(t)$. In a similar way we can find the solution of reaction-diffusion Eq. (25) under initial and boundary conditions (26).

5. Results and discussion

In this section our aim is to show validity and effectiveness of our proposed method. All numerical computations are done with Wolfram Mathematica version-11.3.

Example 1. Considering $a=0,\ \alpha=1,\ \eta=0.5,$ and $b=1,$ we get the following time-space fractional Fisher's equation with ABC derivative

$${}_0D_t u(t,x)={}_0^{ABC}D_x^{1.5}u(t,x)+u(t,x)(1-u(t,x))(u-\eta), \tag{36}$$

with the aid of following initial and boundary conditions

$$u(x,0)=\frac{1}{2}(1+\eta)+\frac{1}{2}(1-\eta)\tanh\left(\sqrt{\frac{1}{8}}(1-\eta)x\right),$$
$$u(0,t)=\frac{1}{2}(1+\eta)+\frac{1}{2}(1-\eta)\tanh\left(\frac{1}{4}(1-\eta^2)t\right),$$
$$u(1,t)=\frac{1}{2}(1+\eta)+\frac{1}{2}(1-\eta)\tanh\left(\sqrt{\frac{1}{8}}(1-\eta)+\frac{1}{4}(1-\eta^2)t\right). \tag{37}$$

The exact analytical solution of above problem is $u(x,t)=\frac{1}{2}(1+\eta)+\frac{1}{2}(1-\eta)\tanh(\sqrt{\frac{1}{8}}(1-\eta)x+\frac{1}{4}(1-\eta^2)t)$.

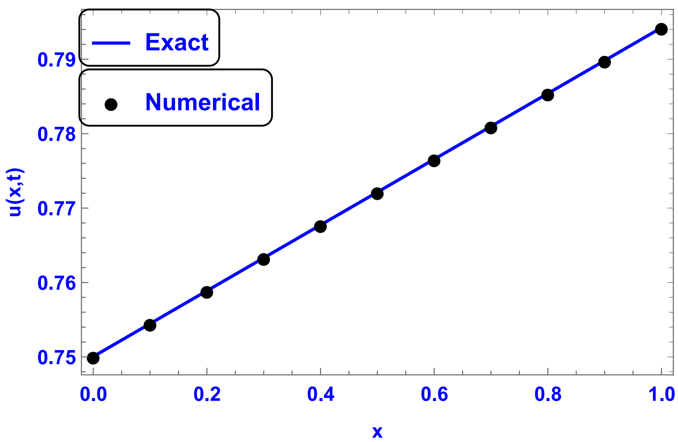

Fig. 1. Plots of $u(x,\ t)$ for $m = 4$ and $\Delta t = 0.0001$ in case of numerical and exact solution.

We plot the graph of exact and numerical solution with $\eta = 0.5,\ m = 4,\ \Delta t = 0.0001$ which is depict by Fig. 1. The absolute error between exact and numerical results for various x and Δt is presented by Table 1. The absolute error obtained in article [36] for Example 1 is of range $(0, 10^{-3})$, while from Table 1 we can easily see that our error range is $(0, 10^{-9})$. It demonstrate the superiority of our method.

Example 2. If we consider $a = 0,\ \alpha = 0.9,\ \vartheta = 1.5$ and $b = 1$ so that our model (23) is reduced to

$$ {}_0D_t^{0.9}u(t,x) = {}^{ABC}_{0}D_x^{1.5}u(t,x) + u(t,x)(1-u(t,x))(u-\eta) + f(x,t), \tag{38}$$

which under the prescribed initial and boundary conditions

$$\begin{aligned} u(0,x) &= 0,\\ u(t,0) &= 0,\\ u(t,1) &= t, \end{aligned} \tag{39}$$

gives the exact solution of above the problem is $u(x,t) = x^2 t$ with suitable force function $f(x,\ t)$.

The graph of numerical and exact solution with $m = 4, \Delta t = 0.0001$ is depict by Fig. 2. The absolute error for various x and Δt is presented by Table 2, which clearly predict that our numerical results are in complete agreement with the existing results.

Example 3. Consider the following equation with $a = 1,\ \alpha = 0.9,\ \vartheta = 1.5,\ b = 1,\ \delta = 1$ and $\eta = 0.5$

$$\begin{aligned} {}_0D_t^{0.9}u(t,x) = {}^{ABC}_{0}D_x^{1.5}u(t,x) - u(x,t)\frac{\partial u}{\partial x} \\ + u(t,x)(1-u(t,x))(u-\eta) + f(x,t). \end{aligned} \tag{40}$$

The Eq. (40) with initial and boundary conditions

$$u(x,0) = (1-x)^2x^2, u(0,t) = 0, u(1,t) = 0, \tag{41}$$

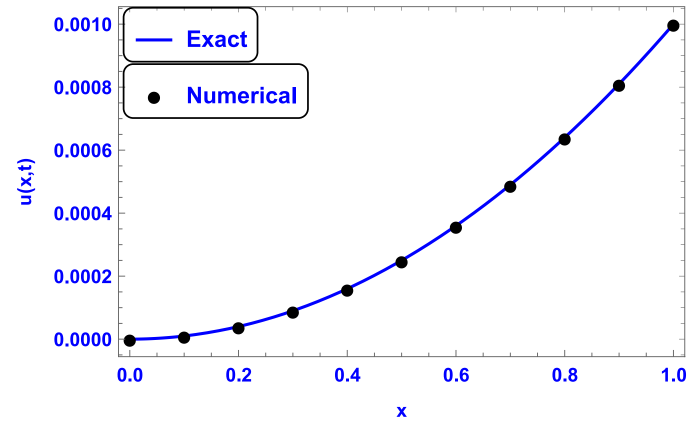

Fig. 2. Plots of $u(x,\ t)$ for $m = 4$ and $\Delta t = 0.0001$ in case of numerical and exact solution.

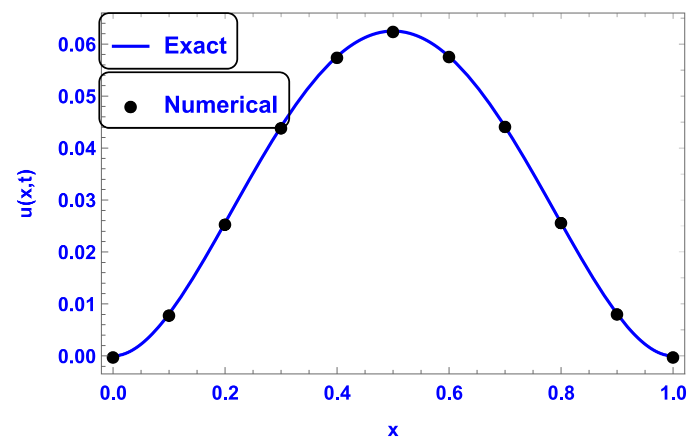

Fig. 3. Plots of $u(x,\ t)$ for $m = 4$ and $\Delta t = 0.0001$ in case of numerical and exact solution.

gives the exact solution $u(x,t) = (1-x)^2x^2e^t$ with suitable force function $f(x,\ t)$.

We plot the graph of exact and numerical solution with $m = 4, \Delta t = 0.0001$ which is depicted by Fig. 3. The absolute error for various x and Δt is presented in Table 3 which clearly predict that our numerical results are in complete agreement with the existing results.

Example 4. Considering the following reaction-diffusion equation a particular case of model (25)

$$ {}_0D_t^{0.5}u(t,x) = {}^{ABC}_{0}D_x^{1.5}u(t,x) + u(t,x)(1-u(t,x)) + f(x,t), \tag{42}$$

The Eq. (42) with initial and boundary conditions

$$\begin{aligned} u(x,0) &= 0,\\ u(0,t) &= 0,\\ u(1,t) &= t\sin 1, \end{aligned} \tag{43}$$

gives the exact solution $u(x,t) = t\sin(x)$.

Table 1
Variations of absolute error for different value of x at $\Delta t = 0.0001$ and $\Delta t = 0.00001$.

x ↓	$\Delta t = 0.0001$	$\Delta t = 0.00001$
$\frac{1}{10}$	3.1×10^{-7}	3.0×10^{-9}
$\frac{3}{10}$	6.7×10^{-7}	6.6×10^{-9}
$\frac{5}{10}$	1.0×10^{-6}	1.0×10^{-8}
$\frac{7}{10}$	1.2×10^{-6}	1.2×10^{-8}
$\frac{9}{10}$	1.1×10^{-6}	1.1×10^{-8}

Table 2
Variations of absolute error for different value of x at $\Delta t = 0.0001$ and $\Delta t = 0.00001$.

x ↓	$\Delta t = 0.0001$	$\Delta t = 0.00001$
$\frac{1}{10}$	4.1×10^{-7}	6.6×10^{-11}
$\frac{3}{10}$	9.4×10^{-7}	1.5×10^{-10}
$\frac{5}{10}$	1.3×10^{-6}	2.1×10^{-10}
$\frac{7}{10}$	1.4×10^{-6}	2.3×10^{-10}
$\frac{9}{10}$	1.24×10^{-6}	1.9×10^{-10}

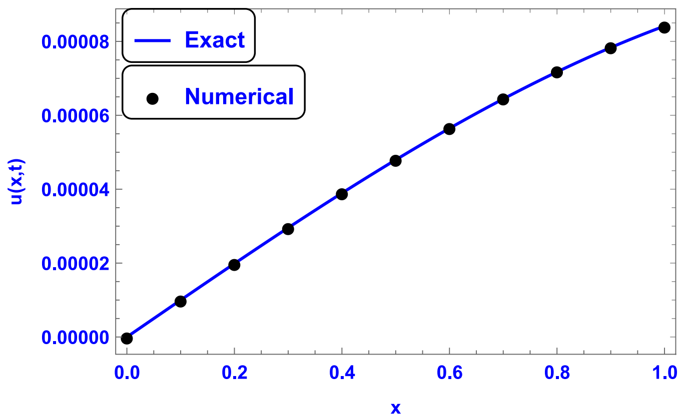

Fig. 4. Plots of $u(x, t)$ for $m = 4$ and $\Delta t = 0.0001$ in case of numerical and exact solution.

Table 3
Variations of absolute error for different value of x at $\Delta t = 0.0001$ and $\Delta t = 0.00001$.

x ↓	$\Delta t = 0.0001$	$\Delta t = 0.00001$
$\frac{1}{10}$	3.9×10^{-4}	6.3×10^{-6}
$\frac{3}{10}$	4×10^{-4}	6.5×10^{-6}
$\frac{5}{10}$	1.6×10^{-3}	2.5×10^{-5}
$\frac{7}{10}$	2.1×10^{-3}	3.1×10^{-5}
$\frac{9}{10}$	1.4×10^{-3}	2.2×10^{-5}

Table 4
Variations of absolute error for different value of x at $\Delta t = 0.0001$ and $\Delta t = 0.00001$.

x ↓	$\Delta t = 0.0001$	$\Delta t = 0.00001$
$\frac{1}{10}$	6.1×10^{-7}	5.1×10^{-10}
$\frac{3}{10}$	1.4×10^{-6}	1.8×10^{-9}
$\frac{5}{10}$	4.5×10^{-6}	3.1×10^{-9}
$\frac{7}{10}$	8.5×10^{-6}	8.9×10^{-9}
$\frac{9}{10}$	6.7×10^{-6}	4.9×10^{-9}

The graph of numerical and exact solution for $m = 4, \Delta t = 0.0001$ is shown in Fig. 4 and the absolute error is shown in Table 4. The results clearly predict that our numerical results are in complete agreement with the existing results.

6. Conclusion

In this presented article, we have developed a approximation formula for the ABC derivative of Legendre polynomial. Here we solved non-linear Burger's–Huxley equation and reaction-diffusion equation having spatial ABC derivative with Caputo time fractional derivative. First we apply Legendre spectral method to the models. And we presented a finite difference scheme to solve obtained system of FPDEs from the model. The graph of numerical and exact solution shows the effectiveness of our numerical method. The accuracy of our method can be easily shown by the errors Tables. The error Tables depicted that our numerical results are in excellent agreement with the exact ones.

Declaration of Competing Interest

None.

References

[1] Machado JT, Kiryakova V, Mainardi F. Recent history of fractional calculus. Commun Nonlinear Sci Numer simulation 2011;16(3):1140–53.
[2] Kilbas A, Srivastava H, J Trujillo J. Theory and applications of the fractional differential equations, Vol. 204. Elsevier (North- Holland), Amsterdam Amsterdam; 2006.
[3] Wazwaz AM. A reliable modification of Adomian decomposition method. Appl Math Comput 1999;102(1):77–86.
[4] Darania P, Ebadian A. A method for the numerical solution of the integro-differential equations 188, 657–668. 2007.
[5] Hashim I, Abdulaziz O, Momani S. Homotopy analysis method for fractional ivps. Commun Nonlinear Sci Numer Simul 2009;14(3):674–84.
[6] Pandey P, Kumar S, Das S. Approximate analytical solution of coupled fractional order reaction-advection-diffusion equations. Eur Phys J Plus 2019;134(7):364.
[7] Gómez-Aguilar J, Yépez-Martınez H, Torres-Jiménez J, Córdova-Fraga T, Escobar-Jiménez R, Olivares-Peregrino V. Homotopy perturbation transform method for nonlinear differential equations involving to fractional operator with exponential kernel. Adv Differ Equ 2017;2017(1):68.
[8] Diethelm K, Ford NJ, Freed AD. A predictor-corrector approach for the numerical solution of fractional differential equations. Nonlinear Dyn 2002;29(1–4):3–22.
[9] Li Y, Sun N. Numerical solution of fractional differential equations using the generalized block pulse operational matrix. Comput Math Appl 2011;62(3):1046–54.
[10] Jafari H, Yousefi S, Firoozjaee M, Momani S, Khalique CM. Application of Legendre wavelets for solving fractional differential equations. Comput Math Appl 2011;62(3):1038–45.
[11] Yuanlu L. Solving a nonlinear fractional differential equation using Chebyshev wavelets. Commun Nonlinear Sci Numer Simul 2010;15(9):2284–92.
[12] Li Y, Zhao W. Haar wavelet operational matrix of fractional order integration and its applications in solving the fractional order differential equations. Appl Math Comput 2010;216(8):2276–85.
[13] Odibat Z. On legendre polynomial approximation with the vim or ham for numerical treatment of nonlinear fractional differential equations. J Comput Appl Math 2011;235(9):2956–68.
[14] Gürbüz B, Sezer M. Laguerre polynomial solutions of a class of initial and boundary value problems arising in science and engineering fields. Acta Physica Polonica A 2016;130(1):194–7.
[15] Araci S. Novel identities for -Genocchi numbers and polynomials. J Funct Spaces Appl 2012.
[16] Gómez-Aguilar J, Yépez-Martınez H, Escobar-Jiménez R, Olivares-Peregrino V, Reyes J, Sosa I. Series solution for the time-fractional coupled mkdv equation using the homotopy analysis method. Mathematical problems in engineering; 2016.
[17] Morales-Delgado VF, Gómez-Aguilar JF, Yépez-Martınez H, Baleanu D, Escobar-Jimenez RF, Olivares-Peregrino VH. Laplace homotopy analysis method for solving linear partial differential equations using a fractional derivative with and without kernel singular. Adv Differ Equ 2016;2016(1):164.
[18] Yépez-Martínez H, Gómez-Aguilar J. Numerical and analytical solutions of nonlinear differential equations involving fractional operators with power and mittag-leffler kernel. Math Model Nat Phenomena 2018;13(1):13.
[19] Gómez-Aguilar J, Atangana A. Fractional hunter-saxton equation involving partial operators with bi-order in Riemann-Liouville and Liouville-Caputo sense. Eur Phys J Plus 2017;132(2):100.
[20] Yépez-Martínez H, Gómez-Aguilar F, Sosa I, Reyes J, Torres-Jiménez J. The fengs first integral method applied to the nonlinear mkdv space–time fractional partial differential equation. Revista mexicana de física 2016;62(4):310–16.
[21] Das S, Vishal K, Gupta P. Solution of the nonlinear fractional diffusion equation with absorbent term and external force. Appl Math Model 2011;35(8):3970–9.
[22] Couteron P, Lejeune O. Periodic spotted patterns in semi-arid vegetation explained by a propagation-inhibition model. J Ecol 2001;89(4):616–28.
[23] Kondo S, Asai R. A reaction–diffusion wave on the skin of the marine angelfish pomacanthus. Nature 1995;376(6543):765.
[24] Murray JD. A pre-pattern formation mechanism for animal coat markings. J Theor Biol 1981;88(1):161–99.
[25] Kondo S. How animals get their skin patterns: fish pigment pattern as a live turing wave. In: Systems Biology. Springer; 2009. p. 37–46.
[26] Hodgkin A. A quantitative description of ion currents and its application and excitation in nerve membranes. J Physiol 1952;117:500–44.
[27] Gómez-Aguilar J, Atangana A. New insight in fractional differentiation: power, exponential decay and Mittag-Leffler laws and applications. Eur Phys J Plus 2017;132(1):13.
[28] Morales-Delgado V, Gómez-Aguilar J, Escobar-Jiménez R, Taneco-Hernández M. Fractional conformable derivatives of Liouville–Caputo type with low-fractionality. Physica A 2018;503:424–38.
[29] Atangana A, Gómez-Aguilar J. A new derivative with normal distribution kernel: theory, methods and applications. Physica A 2017;476:1–14.
[30] Atangana A, Gómez-Aguilar J. Fractional derivatives with no-index law property: application to chaos and statistics. Chaos Solitons Fractals 2018;114:516–35.
[31] Atangana A, Gómez-Aguilar J. Hyperchaotic behaviour obtained via a nonlocal operator with exponential decay and Mittag-Leffler laws. Chaos Solitons Fractals 2017;102:285–94.

[32] Tavares D, Almeida R, Torres DF. Caputo derivatives of fractional variable order: numerical approximations. Commun Nonlinear Sci Numer Simul 2016;35:69–87.
[33] Atangana A, Koca I. Chaos in a simple nonlinear system with Atangana–Baleanu derivatives with fractional order. Chaos Solitons Fractals 2016;89:447–54.
[34] Atangana A, Gómez-Aguilar J. Numerical approximation of Riemann-Liouville definition of fractional derivative: from Riemann-Liouville to Atangana-Baleanu. Numer Methods Part Differ Equ 2018;34(5):1502–23.
[35] Atangana A, Khan MA. Validity of fractal derivative to capturing chaotic attractors. Chaos Solitons Fractals 2019;126:50–9.
[36] Saad K, Khader M, Gómez-Aguilar J, Baleanu D. Numerical solutions of the fractional Fisherâ's type equations with Atangana-Baleanu fractional derivative by using spectral collocation methods. Chaos 2019;29(2):023116.