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Quasi wavelet numerical approach of non-linear reaction diffusion and integro reaction-diffusion equation with Atangana–Baleanu time fractional derivative

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ABSTRACT

In this presented paper, we investigate the novel numerical scheme for the non-linear reaction-diffusion equation and non-linear integro reaction-diffusion equation equipped with Atangana Baleanu derivative in Caputo sense (ABC). A difference scheme with the help of Taylor series is applied to deal with fractional differential term in the time direction of differential equation. We applied a numerical method based on quasi wavelet for discretization of unknown function and their spatial derivatives. A formulation to deal with Dirichlet boundary condition is also included. To demonstrate the effectiveness and validity of our proposed method some numerical examples are also presented. We compare our obtained numerical results with the analytical results and we conclude that quasi wavelet method achieve accurate results and this method has a distinctive local property. On the other hand the method is easy to apply on higher order fractional partial differential equation and integro differential equation.

1. Introduction

Fractional calculus is an ancient topic of mathematics with history as ordinary or integer calculus [1]. It is developing progressively now. Theory of fractional calculus is developed by N. H. Abel and J. Liouville. The details can be found in [2]. Fractional calculus allows to generalize derivatives and integrals of integer order to real or variable order. It also can be considered as a branch of mathematical analysis that allows to investigate with real differential operators and equations where types of integral are convolution or weakly singular. It has a widely applications in control theory, stochastic process and special functions. Fractional calculus was considered as a esoteric theory without applications but in the last few years there has been a boost of research on its applications to economics, control system to finance. Many different forms of fractional order differential operators were introduced as the Hadamard, Caputo, Grunwald-Letnikov, Riemann-Liouville, Riesz and variable order operators. Due to its increasing applications, the researchers have paid their attention to find numerical and exact solutions of the fractional order differential equations. As there are many difficulties to solve a fractional order differential equation by analytic method so there is a need of seeking numerical solutions. There are many numerical methods available in literature viz., eigen-vector expansion, Adomain decomposition method [3], fractional differential transform method [4], homotopy perturbation method [5], predictor-corrector method [6] and generalized block pulse operational matrix method [7] etc. Some numerical methods based upon operational matrices of fractional order differentiation and integration with Legendre wavelets [8], Chebyshev wavelets [9], sine wavelets, Haar wavelets [10] have been developed to find the solutions of fractional order differential and integro-differential equations. The functions which are commonly used include Legendre polynomial [11], Laguerre polynomial [12], Chebyshev polynomial and semi-orthogonal polynomial as Genocchi polynomial [13]. The modeling and simulations of real life problem is described by PDEs, integral and integro-differential equations. These equations have a lot of applications such as heat conduction in materials with memory, population dynamics, nuclear reactor dynamics, fluid dynamics, and compression of viscoelastic media. In many fields like as thermo elasticity and dynamics of nuclear reactor we have to depict the memory effect of the systems. When we modeled these systems using PDE, which included function at a given point of time and space, we ignored the effect of history. Therefore to include the memory effect of these systems an term of integration is added to this PDE. Partial integro differential equations have a widely applications in aerospace systems, chemical kinetics, biological models, control theory of financial mathematics and industrial mathematics. Apart

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of it many phenomena in physics like as viscoelastic mechanics, fluid dynamics, control theory and thermoelastic.

Reaction-diffusion process has been investigated from a long time. In the process of reaction- diffusion, reacting molecules are used to move through space due to diffusion. This definition excludes other modes of transports as convection, drift those may arise due to presence of externally imposed fields.

When a reaction occurs within an element of space, molecules can be created or consumed. These events are added to the diffusion equation and lead to reaction-diffusion equation of the form

$$\frac{\partial c}{\partial t} = D\nabla^2 c + R(c,t), \tag{1}$$

where $R(c, t)$ denotes reaction term at time t. The extension of the reaction-diffusion equation in fractional order system can be found in the articles [14]. In nature many of the beautiful systems in biology, physics,chemistry, and physiology can be described by reaction diffusion equations. For example, the distribution and organization of vegetation-like bushes in arid ecosystems [15], the stripes and spots on fish [16], snakes [17] and the skin or fur of mammals [18] have been studied by the standing waves which are produced by reaction-diffusion equations. Now we will discussed the motivation behind using ABC derivative. We know that interchange of operators is an important concept of mathematical analysis arising in physics, biology and engineering. Considering two operators P and Q such that $PQ = QP$, then we say both of operators commute. But this expression is always specially in physics, statistics and mathematics. Some examples of operator which follows non-commutative property are as follows:

- In quantum mechanics, linear operators like y and $\frac{d}{dy}$ does not commute on wave function $\psi(y)$ in the formulation of Schrodinger equation.
- Lie bracket of a Lie algebra.
- Lie bracket of Lie ring.
- Division operator as $\frac{1}{2} \neq \frac{2}{1}$.
- Matrix product.

The general fractional derivatives in R-L and Caputo sense can be defined as follows

$$ {}^{RL}_{0}D^{\vartheta}_{t} g(t) = \frac{d}{dt}\int_0^t \kappa(t-x)g(x)dx = \frac{d}{dt}\kappa * g,$$
$$ {}^{C}_{0}D^{\vartheta}_{t} g(t) = \int_0^t \kappa(t-x)\frac{d}{dx}g(x)dx = \kappa * \frac{d}{dt}g.$$

We have two types of kernel $\kappa(t-x) = \frac{1}{\Gamma(1-\alpha)}(t-x)^{-\alpha}$ and $\kappa(t-x) = \frac{M(\alpha)}{(1-\alpha)}E_\alpha(\frac{-\alpha}{1-\alpha}(t-x)^{-\alpha})$. First type of kernel generates fractional derivative with power kernel law and second type generates derivative with generalized Mittag-Leffler law with Dirac-Delta properties known as A-B derivative [19]. The power law kernel corresponds to the Pareto distribution describing the wealth in society and fitting the shape of a large portion of wealth for a small portion of the population. The ABC kernel corresponds to the Mittag-Leffler distribution. The wider applicability and properties of this kernel to be known in Pareto and Poisson statistics. If we apply ABC fractional derivative to chaotic problems it provides a infinite expectation. Since AB fractional derivative are non-commutative, they are able to model the dynamical system taking place in the space of Penrose tilling which has several remarkable properties including its non-periodicity. It is able to describe the fractal model in the space of irreducible unitary representation of a discrete group and the model in the space of leaves of foliation. This derivative can be used as a powerful tool for handling the brillouin zone in quantum hall effect and phase space in quantum mechanics. This is useful to the situation when distribution of a waiting time does not depend on elapsed time in a certain event [20]. Some properties of AB derivative are

1. The A-B distribution is a Gaussian to non-Gaussian crossover.
2. The A-B fraction derivative mean square displacement is a usual to sub-diffusion crossover.
3. The asymptotic behavior of A-B match the power law behavior and relate the old ides of fading memory with non-singular kernels [21].
4. The A-B derivative is stochastic as it is comparable to the Brownian motion.
5. The asymptotic behavior of A-B match the power law behavior and relate the old idea of fading memory with non-singular kernels.

The C-F fractional operator attract the interest of researchers in few last decades. The applications of this derivatives can be found in flow of complex rheological media, Keller Segel equation, ground water flow mass-spring damped system electric circuit and elasticity [22]. The application of non-singular kernel to the Burger's equation is presented in article [23]. The series form of newly derived fractional operator involving Mittag-Leffler operator is given in [24].The application of these non-singular kernel to optimal control of tumor-immune surveillance is discussed in the article [25].

Our article is outlined as follows. In Section 1, we discussed about Caputo, RL and ABC fractional derivative and presented a short introduction to the numerical method based on quasi wavelet and fractional calculus in Section 2. In Section 3, we develop a detailed difference numerical algorithm based on Taylor series expansion to discretize the time derivative and quasi wavelet based algorithm for spatial-temporal discretization. In Section 4, some numerical examples and results are presents. The last section includes the conclusion of all over work.

2. Preliminaries

Here, few definitions and important properties of fractional calculus have been introduced. It is well known that the Riemann-Liouville definition has disadvantages when it comes for modeling of real world problems. But definition of fractional differentiation given by M. Caputo is more reliable for application point of view. Nowadays new general type of fractional operators have been discovered. A brief description of Caputo-Fabrizio and ABC derivative is discussed here.

2.1. Definition of R-L order derivative and integration

Fractional order integration of Riemann-Liouville type of a given order ϑ of a function $h(t)$ is given by[26]

$$I^{\vartheta}h(t) = \frac{1}{\Gamma(\vartheta)}\int_0^t (t-\omega)^{\vartheta-1}h(\omega)\,d\omega, \quad t>0, \quad \vartheta \in R^{+}. \tag{2}$$

Fractional order derivative of the Riemann-Liouville type of order $\vartheta > 0$ can be defined as

$$D^{\vartheta}_{l}h(t) = \left(\frac{d}{dt}\right)^{m}(I^{m-\vartheta}h)(t), \ (\vartheta > 0, \ m-1 < \vartheta < m). \tag{3}$$

2.2. Definition of Caputo derivative

Fractional derivative of order $\vartheta > 0$ in Caputo sense can be defined as

$$D^{\vartheta}_{c}h(t) = \begin{cases} \frac{d^l h(t)}{dt^l} & \vartheta = l \in N \\ \frac{1}{\Gamma(\vartheta)}\int_0^t (t-\eta)^{l-\vartheta-1}h^{l}(\eta)\ d\eta & l-1 < \vartheta < l. \end{cases} \tag{4}$$

Here, l is an integer, $t > 0$.

Basic properties of Caputo fractional derivative are:

$$D_c^{\vartheta} C = 0, \tag{5}$$

where C is a constant.

$$D_c^{\vartheta} t^{\sigma} = \begin{cases} 0, & \sigma \in N \cup 0 \text{ and } \sigma < \lceil \vartheta \rceil \\ \dfrac{\Gamma\,(1+\sigma)}{\Gamma(1-\vartheta+\sigma)} t^{-\vartheta+\sigma} & \sigma \in N \cup 0 \text{ and } \sigma \geq \lceil \vartheta \rceil \\ & \text{or } \sigma \notin N \text{ and } \sigma > \lfloor \vartheta \rfloor, \end{cases} \tag{6}$$

where $\lfloor \vartheta \rfloor$ is floor function.

The operator D_c^{ϑ} is linear, since

$$D_c^{\vartheta}(ah(t) + bg(t)) = aD_c^{\vartheta} h(t) + bD_c^{\vartheta} g(t), \tag{7}$$

where a and b are constants.

Caputo operator and Riemann-Liouville operator have a relation:

$$(I^{\vartheta} D_c^{\vartheta} g)(t) = g(t) - \sum_{k=0}^{l-1} g^k(0^+)\frac{t^k}{k!}, \ \ l-1 < \vartheta \leq l. \tag{8}$$

2.3. Definition of Caputo-Fabrizio derivative

Let $g(t)$ be a function which belongs to Sobolev space $H^1(0, 1)$ then Caputo-Fabrizio derivative in Caputo sense of order ϑ is defined by

$$ {}_0^{CFC}D_t^{\vartheta} g(t) = \frac{B(\vartheta)}{n-\vartheta}\int_0^t \frac{\partial g(s)}{\partial s} \times \exp\left[\frac{-\vartheta}{n-\vartheta}(t-s)\right] ds, \quad n-1 < \vartheta \leq n, \tag{9}$$

where $B(\vartheta)$ is a normalization function such that $B(0) = B(1) = 1$.

2.4. Definition of ABC derivative [27–29]

Let $g(t)$ be a function which belongs to Sobolev space $H^1(0, 1)$ then Atangana–Baleanu Caputo derivative in Caputo sense of order ϑ is defined by

$$ {}_0^{ABC}D_t^{\vartheta} g(t) = \frac{B(\vartheta)}{n-\vartheta}\int_0^t \frac{\partial g(s)}{\partial s} \times E_{\vartheta}\left[\frac{-\vartheta}{n-\vartheta}(t-s)^{\vartheta}\right] ds, \quad n-1 < \vartheta \leq n, \tag{10}$$

where $B(\vartheta)$ is a normalization function such that $B(0) = B(1) = 1$ and $E_{\vartheta}(z)$ is Mittag-Leffler function defined as

$$E_{\vartheta}(z) = \sum_{i=0}^{\infty} \frac{z^i}{\Gamma(i\vartheta + 1)}.$$

2.5. Introduction to the numerical method based on quasi-wavelets

This Quasi-wavelet based algorithm has been emerging as a new local spectral collocation method for finding numerical solution of fractional PDEs and integro differential equation. Firstly, we define the discrete singular convolution which is a special mathematical transformation having significant importance in science and engineering. A singular convolution is defined in the distribution theory as

$$\Psi(v) = (G * h)(v) = \int_{-\infty}^{\infty} G(v-x)h(x)dx, \tag{11}$$

where G is a singular kernel and $h(x)$ is a test function. A family of wavelets can be constructed from a single function called mother wavelet φ using operations of translation and dilation

$$\varphi_{\beta,\gamma}(x) = \beta^{\frac{-1}{2}} \varphi\left(\frac{x-\gamma}{\beta}\right). \tag{12}$$

Here β is used in dilation and γ in translation. An arbitrary wavelet subspace is generated by the help of orthonormal wavelet bases, which can be constructed by the corresponding orthonormal scaling functions. In this work, we shall used Shannon's delta sequence kernel which has the following form

$$\delta_{\alpha}(t) = \frac{1}{\pi}\int_0^{\pi} \cos(ty)dy = \frac{\sin(\alpha t)}{\pi t}, \tag{13}$$

where $\lim_{\alpha\to\alpha_0} \delta_{\alpha}(t) = \delta(t)$. Dirac first discussed about δ in his text on quantum mechanics, so we called δ as Dirac delta function. Walter and Blum[30] gave the numerical use of delta sequences as probability density estimators. Specially when $\alpha = \pi, \delta_{\pi}(t)$ is known as Shannon's wavelet scaling function. For a given $\alpha > 0$, Shannon's delta sequence kernel generates a basis for the Paley-Wiener reproducing kernel Hilbert space $\mathbf{B}_{\alpha}^2$[31] that is a subspace of $\mathbf{L}^2(R)$. A function $g(y) \in \mathbf{B}_{\alpha}^2$ can be uniquely reproduced by

$$g(y) = \int_{-\infty}^{\infty} g(y)\delta_{\alpha}(y-t)dt = \int_{-\infty}^{\infty} g(y)\frac{\sin(\alpha(y-t))}{\pi(y-t)}dt, \ \ \forall g \in \mathbf{B}_{\alpha}^2 \tag{14}$$

However, an another form of this sampling scaling function in the Paley-Wiener reproducing kernel,

$$\delta_{\alpha,k} = \delta_{\alpha}(x-x_k) = \frac{\sin((x-x_k)\alpha)}{(x-x_k)\pi}, \tag{15}$$

here $\{x_k\}$ is considered as a set of sampling points centered around x. Using Eqs. (11) and (12) every function $\forall g \in \mathbf{B}_{\alpha}^2$ can be represented in discrete form

$$g(y) = \sum_{k=-\infty}^{\infty} g(y_k)\delta_{\alpha}(y-y_k). \tag{16}$$

The Shannnon sampling theorem states that the uniformly spatial discrete samples for a given bandlimited signal in B_{α}^2 can shown it if we sample at the Nyquist frequency α. Here $\alpha = \frac{\pi}{\Delta}$ and Δ is the grid size in the spatial direction. Now we have

$$g(y) = \sum_{k=-\infty}^{\infty} f(y_k)\delta_{\alpha}(y-y_k) = \sum_{k=-\infty}^{\infty} g(y_k)\frac{\sin(\frac{\pi(y-y_k)}{\Delta})}{\frac{\pi(y-y_k)}{\Delta}} \tag{17}$$

Wan proposed a method improve the localized asymptotic behavior of Dirichlet's delta sequence kernel. By introducing a regularizer $R_{\sigma}(y)$ we increases its regularity such as

$$\delta_{\alpha}(y) \to \delta_{\alpha,\sigma} = \delta_{\alpha}(y)R_{\sigma}(y) \tag{18}$$

where regularizer R_{σ} satisfies

$$\lim_{\sigma\to\infty} R_{\sigma}(y) = 1$$

and

$$\int_{-\infty}^{\infty} \lim_{\sigma\to\infty} R_{\sigma}(y)\delta_{\alpha}(y)dy = R_{\sigma}(0) = 1.$$

However, there are many regularizers satisfying above two conditions but a commonly used regularizer is the Gaussian type

$$R_{\sigma}(y) = \exp\left(\frac{-y^2}{2\sigma^2}\right), \ \ \sigma > 0 \tag{19}$$

where σ denotes the width parameter of Gaussian envelope. The relation between Δ and $\sigma$is $\sigma = r \times \Delta$, where r is parameter which will be chosen in computation. Using the Gaussian regularizer $R_{\sigma}(x)$, gaussian regularized orthogonal sampling scaling function is defined as

$$\delta_{\Delta,\sigma}(x) = \frac{\sin(\frac{\pi x}{\Delta})}{\frac{\pi x}{\Delta}} \exp\left(\frac{-x^2}{2\sigma^2}\right). \tag{20}$$

Here

$$\lim_{\sigma\to\infty} \delta_{\Delta,\sigma}(x) = \frac{\sin(\frac{\pi x}{\Delta})}{\frac{\pi x}{\Delta}},$$

Gaussian regularized sampling scaling function is a quasi scaling function as it does not follow the property of exact orthonormal wavelet scaling function.

2.6. Description of the proposed method

An arbitrary function $g \in \mathbf{B}^2_\alpha$ can be written as by using quasi scaling function

$$g(y) = \sum_{k=-\infty}^{\infty} g(y_k)\delta_\alpha(y-y_k) = \sum_{k=-\infty}^{\infty} g(y_k)\delta_\alpha(y-y_k)R_\alpha(y-y_k) \tag{21}$$

We can clearly see that the taking infinite sampling points is not possible for computation. Thus we have to restrict our computational domain to finite sampling points close to x perform numerical calculations. In practical numerical computation, we select the $(2W+1)$ sampling points for this problem. Therefore Eq. (18) can be simplified as

$$g(y) = \sum_{k=-W}^{W} g(y_k)\delta_{\Delta,\sigma}(y-y_k), \tag{22}$$

for approximation of n^{th} order derivatives of $f(x)$

$$g^n(y) = \sum_{k=-W}^{W} g(y_k)\delta^n_{\Delta,\sigma}(y-y_k), n = 1, 2, \cdots \tag{23}$$

where computational bandwidth is equal to $(2W+1)$, centered around x and superscript (n) denotes the nth-order derivative with respect to x. For the calculation purpose, we give the following detailed formulas of $\delta_{\Delta,\sigma}$, $\delta^1_{\Delta,\sigma}$ and $\delta^2_{\Delta,\sigma}$ [32],

$$\delta_{\Delta,\sigma}(x) = \begin{cases} \dfrac{\exp\{-\frac{x^2}{2\sigma^2}\}\sin\left(\frac{x\pi}{\Delta}\right)}{\frac{\pi x}{\Delta}}, & x \neq 0 \\ 1 & x = 0. \end{cases} \tag{24}$$

$$\delta^1_{\Delta,\sigma}(x) = \begin{cases} \left(-\dfrac{\sin\left(\frac{\pi x}{\Delta}\right)}{\frac{\pi x^2}{\Delta}} - \dfrac{\Delta\sin\left(\frac{\pi x}{\Delta}\right)}{\frac{\pi\sigma^2}{\Delta}} + \dfrac{\cos\left(\frac{\pi x}{\Delta}\right)}{x}\right)\exp\left(-\frac{x^2}{2\sigma^2}\right) & x \neq 0, \\ 0 & x = 0. \end{cases} \tag{25}$$

$$\delta^2_{\Delta,\sigma}(x) = \begin{cases} \left(\dfrac{2\Delta\sin\left(\frac{x\pi}{\Delta}\right)}{\pi x^3} - \dfrac{2\cos\left(\frac{\pi x}{\Delta}\right)}{x^2} + \dfrac{\Delta x\sin\left(\frac{x\pi}{\Delta}\right)}{\pi\sigma^4}\right. \\ \left. + \dfrac{\Delta\sin\left(\frac{\pi x}{\Delta}\right)}{\pi\sigma^2 x} - \dfrac{2\cos\left(\frac{\pi x}{\Delta}\right)}{\sigma^2} - \dfrac{\pi\sin\left(\frac{x\pi}{\Delta}\right)}{x\Delta}\right)\exp\left(-\frac{x^2}{2\sigma^2}\right) & x \neq 0, \\ 0 & x = 0. \end{cases} \tag{26}$$

3. Proposed method

We presented the quasi wavelet based numerical method for solving ABC time fractional non-linear reaction-diffusion and integro integro-diffusion equation. We have taken the following type of model

$$ {}^{ABC}_{0}D^\alpha_t u(t,x) = \frac{\partial^2 u}{\partial x^2} + au(t,x)(1-u(t,x)) + bu(t,x)\int_0^t \kappa(t-s,x)u(s,x)ds + f(x,t), \tag{27}$$

along with boundary conditions

$$u(0,t) = g_1(t), \quad u(1,t) = g_2(t), \tag{28}$$

and the initial condition

$$u(x,0) = g_3(x). \tag{29}$$

Here $\kappa(x, t)$ is known as kernel of above integro fractional partial differential equation and $f(x, t)$ is known as forced term.

3.0.1. Approximation of time fractional ABC derivative

We shall take the help of Taylor series expansion to discretize the Atangana–Baleanu time fractional derivative in the Caputo sense. Let us assume step length in time is denoted by Δt and $t_n = n\times\Delta t$ where $n = 0, 1, \ldots M$. At point $t_n = n\times\Delta t$ the values of $u(x, t)$ and $f(x, t)$ are denoted by respectively u^n and f^n.

Taylor series expansion of a function $g(t)$ around the point t_n in the interval (t_n, t_{n+1})

$$g'(t) = g'(t_n) + g''(t_n)(t-t_n) + g'''(t_n)\frac{(t-t_n)^2}{2!} + O((t-t_n)^3). \tag{30}$$

By using Taylor series we can easily find the following,

$$g'(t_n) = \frac{g(t_{n+1}) - g(t_{n-1})}{2\Delta t} - \frac{g^{(3)}(t_n)}{3!}\times(\Delta t)^2 + O(\Delta t)^4, \tag{31}$$

and

$$g''(t_n) = \frac{g(t_{n+1}) - 2g(t_n) + g(t_{n-1})}{(\Delta t)^2} - \frac{g^{(4)}(t_n)}{4!}\times(\Delta t)^2 + O(\Delta t)^4. \tag{32}$$

Using the value of $g'(t)$ and $g''(t_n)$ in the Eq. (30) we get

$$\begin{aligned} g'(t) = {} & \frac{g(t_{n+1}) - g(t_{n-1})}{2\Delta t} + \frac{g(t_{n+1}) - 2g(t_n) + g(t_{n-1})}{(\Delta t)^2}(t-t_n) \\ & - \frac{g^{(3)}(t_n)}{3!}\times(\Delta t)^2 - \frac{g^{(4)}(t_n)}{4!}\times(\Delta t)^2 \\ & + g'''(t_n)\frac{(t-t_n)^2}{2!} + O((t-t_n)^3). \end{aligned} \tag{33}$$

Now by the definition of ABC derivative approximation of $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ at grid point (x, t_n) can be approximated by following quadrature formula:

$$\begin{aligned} & \frac{\partial^\alpha u(x,t_n)}{\partial t^\alpha} \\ & = \frac{B(\alpha)}{\Gamma(-\alpha+1)}\int_0^{t_n} E_\alpha\left[\frac{-\alpha}{1-\alpha}(t_n-s)^\alpha\right]\times\frac{\partial u(x,s)}{\partial s}ds \\ & = \frac{B(\alpha)}{\Gamma(-\alpha+1)}\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}\left(\frac{u(x,t_{j+1}) - 2u(x,t_j) + u(x,t_{j-1})}{(\Delta t)^2}(s-t_n)ds\right. \\ & \left. + \frac{u(x,t_{j+1}) - u(x,t_{j-1})}{2\Delta t}\right)E_\alpha\left[\frac{(t_n-s)^\alpha(-\alpha)}{1-\alpha}\right] \\ & = \frac{B(\alpha)}{\Gamma(-\alpha+1)}\sum_{j=0}^{n-1}\frac{u^{j+1}-u^{j-1}}{2\Delta t}\int_{t_j}^{t_{j+1}} E_\alpha\left[\frac{-\alpha}{1-\alpha}(t_n-s)^\alpha\right]ds \\ & + \sum_{j=0}^{n-1}\frac{u(x,t_{j+1}) - 2u(x,t_j) + u(x,t_{j-1})}{(\Delta t)^2} \\ & \times\int_{t_j}^{t_{j+1}}(s-t_n)E_\alpha\left[\frac{-\alpha(t_n-s)^\alpha}{-\alpha+1}\right]ds \\ & = \frac{B(\alpha)}{\Gamma(-\alpha+1)}\sum_{j=0}^{n-1}\frac{u^{j+1}-u^{j-1}}{2\Delta t}\left[(t_n-t_j)E_{\alpha,2}\left[\frac{(t_n-t_j)^\alpha(-\alpha)}{1-\alpha}\right]\right. \\ & \left. - (t_n-t_{j+1})E_{\alpha,2}\left[\frac{(t_n-t_{j+1})^\alpha(-\alpha)}{-\alpha+1}\right]\right] \\ & + \frac{B(\alpha)}{\Gamma(-\alpha+1)}\sum_{j=0}^{n-1}\frac{u^{j+1}-2u^j+u^{j-1}}{(\Delta t)^2} \\ & \times\int_{t_j}^{t_{j+1}}(s-t_n)E_\alpha\left[\frac{-\alpha}{1-\alpha}(t_n-s)^\alpha\right]ds. \end{aligned}$$

Now on simplifying

$$\frac{\partial^\alpha u(x,t_n)}{\partial t^\alpha} = \frac{B(\alpha)}{\Gamma(-\alpha+1)}\sum_{j=0}^{n-1}\frac{u^{j+1}-u^{j-1}}{2\Delta t} \times \Bigg[(t_n-t_j)E_{\alpha,2}\Bigg[\frac{(t_n-t_j)^\alpha(-\alpha)}{1-\alpha}\Bigg] - (t_n-t_{j+1})E_{\alpha,2}\Bigg[\frac{(t_n-t_{j+1})^\alpha(-\alpha)}{-\alpha+1}\Bigg]\Bigg] + \frac{B(\alpha)}{\Gamma(-\alpha+1)}\sum_{j=0}^{n-1}\frac{u^{j+1}-2u^j+u^{j-1}}{(\Delta t)^2} \times \Bigg[-(t_n-t_{j+1})\Delta t E_{\alpha,2}\Bigg[\frac{(t_n-t_{j+1})^\alpha(-\alpha)}{-\alpha+1}\Bigg] - (t_n-t_{j+1})^2E_{\alpha,3}\Bigg[\frac{(t_n-t_{j+1})^\alpha(-\alpha)}{-\alpha+1}\Bigg] + (t_n-t_j)^2E_{\alpha,3}\Bigg[\frac{(t_n-t_j)^\alpha(-\alpha)}{1-\alpha}\Bigg]\Bigg] + R_n, \quad (34)$$

where

$$R_n = \frac{B(\alpha)}{\Gamma(1-\alpha)}\sum_{j=0}^{n-1}\Bigg[-\frac{g^{(3)}(t_n)}{3!}\times(\Delta t)^2 + g^{(3)}(t_n)\frac{(s-t_j)^2}{2!}\Bigg] \times E_\alpha\Bigg[\frac{-\alpha}{1-\alpha}(t_n-s)^\alpha\Bigg]ds$$
$$= \frac{B(\alpha)}{\Gamma(1-\alpha)}\sum_{j=0}^{n-1}-\frac{g^{(3)}(t_n)}{3!} \times (\Delta t)^2\Big((t_n-t_j)E_\alpha\Big[\frac{-\alpha}{1-\alpha}(t_n-t_j)^\alpha\Big]\Big) - (t_n-t_{j+1})E_\alpha\Big[\frac{-\alpha}{1-\alpha}(t_n-t_{j+1})^\alpha\Big]ds$$
$$+ \frac{B(\alpha)}{\Gamma(1-\alpha)}\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}-\frac{g^{(3)}(t_n)}{2!}\times\Big((\Delta t)^2(t_n-t_{j+1}) \times E_{\alpha,2}\Big[\frac{-\alpha}{1-\alpha}(t_n-t_{j+1})^\alpha\Big] - 2(\Delta t)(t_n-t_{j+1})^2E_{\alpha,2}\Big[\frac{-\alpha}{1-\alpha}(t_n-t_{j+1})^\alpha\Big] - 2(t_n-t_{j+1})^3E_\alpha\Big[\frac{-\alpha}{1-\alpha}(t_n-t_{j+1})^\alpha\Big] + 2(t_n-t_j)^3E_\alpha\Big[\frac{-\alpha}{1-\alpha}(t_n-t_j)^\alpha\Big]\Big)ds.$$

After simplifying the value of R_n we get

$$R_n \le \frac{C_1B(\alpha)}{\Gamma(1-\alpha)}\max_{0\le x\le 1}|g^{(3)}(t)|\Delta t^3,$$

where C_1 is a constant.

Now to approximate the another term $\int_0^t \kappa(t-s,x)u(s,x)ds$ we use product trapezoidal technique

$$\int_0^{t_{n+\frac{1}{2}}}\kappa(t-s,x)u(x,s)ds = \frac{1}{2}\Bigg\{\int_0^{t_n}\kappa(t-s,x)u(x,s)ds + \int_0^{t_{n+1}}\kappa(t-s,x)u(x,s)ds\Bigg\}.$$

If we take $\kappa(t-s,x)=\exp\{-(t-s)\}$ then

$$\int_0^{t_n}\kappa(t-s,x)u(s,x)ds = \sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}\exp\{-(t_n-s)\}u(x,s)ds$$
$$= \sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}\exp\{-(t_n-s)\}\{u(x,t_{j+1})\frac{s-t_j}{\Delta t}+u(x,t_j)\frac{t_{j+1}-s}{\Delta t}\}ds$$
$$= \frac{e^{-t_n}}{\Delta t}\sum_{j=0}^{n-1}\{u^{j+1}(e^{t_{j+1}}\Delta t - e^{t_{j+1}}+e^{t_j}) + u^j(-e^{t_j}\Delta t + e^{t_{j+1}} - e^{t_j})\} \quad (35)$$

Similarly,

$$\int_0^{t_{n+1}}\kappa(t-s,x)u(x,s)ds = \frac{e^{-t_{n+1}}}{\Delta t}\{u^{n+1}(e^{t_{n+1}}\Delta t - e^{t_{n+1}} + e^{t_n}) + u^n(-e^{t_n}\Delta t + e^{t_{n+1}} - e^{t_n})\} + \frac{e^{-t_{n+1}}}{\Delta t}\sum_{j=0}^{n-1}\{u^{j+1}(e^{t_{j+1}}\Delta t - e^{t_{j+1}} + e^{t_j}) + u^j(-e^{t_j}\Delta t + e^{t_{j+1}} - e^{t_j})\}. \quad (36)$$

Using Eqs. (34)–(36), and into given model we get the temporal semi-discrete form

$$\frac{B(\alpha)}{\Gamma(-\alpha+1)}\sum_{j=0}^{n-1}\frac{u^{j+1}-u^{j-1}}{2\Delta t}\Big[(t_n-t_j)E_{\alpha,2}\Big[\frac{-\alpha}{1-\alpha}(t_n-t_j)^\alpha\Big] - (t_n-t_{j+1})E_{\alpha,2}\Big[\frac{-\alpha}{1-\alpha}(t_n-t_{j+1})^\alpha\Big]\Big] + \frac{B(\alpha)}{\Gamma(-\alpha+1)}\sum_{j=0}^{n-1}\frac{u^{j+1}-2u^j+u^{j-1}}{(\Delta t)^2}\big[-(t_n-t_{j+1})\Delta t E_{\alpha,2} \times \Big[\frac{-\alpha}{1-\alpha}(t_n-t_{j+1})^\alpha\Big] - (t_n-t_{j+1})^2E_{\alpha,3}\Big[\frac{-\alpha}{1-\alpha}(t_n-t_{j+1})^\alpha\Big] + (t_n-t_j)^2E_{\alpha,3} \times \Big[\frac{-\alpha}{1-\alpha}(t_n-t_j)^\alpha\Big]\Big]$$
$$= bu^n(x)\frac{e^{-t_n}+e^{-t_{n+1}}}{2\Delta t}\sum_{j=0}^{n-1}\big\{u_i^{j+1}(e^{t_{j+1}}\Delta t - e^{t_{j+1}} + e^{t_j}) + u^j(-e^{t_j}\Delta t + e^{t_{j+1}} - e^{t_j})\big\} + bu^n(x)\frac{e^{-t_{n+1}}}{2\Delta t}\big\{u^{n+1}(e^{t_{n+1}}\Delta t - e^{t_{n+1}} + e^{t_n}) + u^n(-e^{t_n}\Delta t + e^{t_{n+1}} - e^{t_n})\big\} + \frac{\partial^2 u^n(x)}{\partial x^2} + au^n(x)(1-u^n(x)). \quad (37)$$

3.1. Discretization in space using quasi wavelet based numerical method

Now we apply quasi wavelet method given in Section 2 to discretize the spatial derivative. Let $\Delta x = \frac{1}{N}$ denote the spatial step. We assume u_i^n be the approximation of $u(x,\ t)$ at point $(x_i,\ t_n)$ where $n = 0, 1, \cdots M$ and $i = 0, 1, \cdots N$. According to the quasi wavelet based numerical method to approximate the spatial derivative the value of function at $2W$ neighboring points around that point itself inside the computational domain or outside the domain are applied. For example, n^{th} order derivative $u_x^{(n)}(x_i)$ of a function $u(x)$ at the point x_i is approximated by

$$u^n(x_i) = \sum_{p=i-W}^{i+W}u(x_k,t_n)\delta^n_{\Delta,\sigma}(x_i-x_k),$$
$$i = 0, 1, \cdots, N-1, n = 0, 1, 2, \cdots. \quad (38)$$

Considering Eq. (37) at point $x = x_i$ and using Eq. (38)

$$\frac{B(\alpha)}{\Gamma(-\alpha+1)}\sum_{j=0}^{n-1}\frac{u_i^{j+1}-u_i^{j-1}}{2\Delta t}\Big[(t_n-t_j)E_{\alpha,2}\Big[\frac{-\alpha}{1-\alpha}(t_n-t_j)^{\alpha}\Big]$$
$$-(t_n-t_{j+1})E_{\alpha,2}\Big[\frac{-\alpha}{1-\alpha}(t_n-t_{j+1})^{\alpha}\Big]\Big]$$
$$+\frac{B(\alpha)}{\Gamma(-\alpha+1)}\sum_{j=0}^{n-1}\frac{u_i^{j+1}-2u_i^j+u_i^{j-1}}{(\Delta t)^2}$$
$$\times\Big[-(t_n-t_{j+1})\Delta t E_{\alpha,2}\Big[\frac{-\alpha}{1-\alpha}(t_n-t_{j+1})^{\alpha}\Big]$$
$$-(t_n-t_{j+1})^2E_{\alpha,3}\Big[\frac{-\alpha}{1-\alpha}(t_n-t_{j+1})^{\alpha}\Big]+(t_n-t_j)^2E_{\alpha,3}$$
$$\times\Big[\frac{-\alpha}{1-\alpha}(t_n-t_j)^{\alpha}\Big]\Big]$$
$$=b\sum_{k=i-W}^{i+W}u(x_k,t_n)\delta_{\Delta,\sigma}(x_i-x_k)\frac{e^{-t_n}+e^{-t_{n+1}}}{2\Delta t}$$
$$\times\sum_{j=0}^{n-1}\{u^{j+1}(e^{t_{j+1}}\Delta t-e^{t_{j+1}}+e^{t_j})$$
$$+u^j(-e^{t_j}\Delta t+e^{t_{j+1}}-e^{t_j})\}$$
$$+b\sum_{p=i-W}^{i+W}u(x_k,t_n)\delta_{\Delta,\sigma}(x_i-x_k)\frac{e^{-t_{n+1}}}{2\Delta t}\{u^{n+1}(e^{t_{n+1}}\Delta t-e^{t_{n+1}}+e^{t_n})$$
$$+u^n(-e^{t_n}\Delta t+e^{t_{n+1}}-e^{t_n})\}+\sum_{p=i-W}^{i+W}u(x_k,t_n)\delta^2_{\Delta,\sigma}(x_i-x_k)$$
$$+a\sum_{p=i-W}^{i+W}u(x_k,t_n)\delta_{\Delta,\sigma}(x_i-x_k)\left(1-\sum_{p=i-W}^{i+W}u(x_k,t_n)\delta_{\Delta,\sigma}(x_i-x_k)\right).\tag{39}$$

Let $x_p-x_i=x_k$, then above equation can be re-written as

$$\frac{B(\alpha)}{\Gamma(-\alpha+1)}\sum_{j=0}^{n-1}\frac{u_i^{j+1}-u_i^{j-1}}{2\Delta t}\Big[(t_n-t_j)E_{\alpha,2}\Big[\frac{-\alpha}{1-\alpha}(t_n-t_j)^{\alpha}\Big]$$
$$-(t_n-t_{j+1})E_{\alpha,2}\Big[\frac{-\alpha}{1-\alpha}(t_n-t_{j+1})^{\alpha}\Big]\Big]$$
$$+\frac{B(\alpha)}{\Gamma(-\alpha+1)}\sum_{j=0}^{n-1}\frac{u_i^{j+1}-2u_i^j+u_i^{j-1}}{(\Delta t)^2}$$
$$\times\Big[-(t_n-t_{j+1})\Delta t E_{\alpha,2}\Big[\frac{-\alpha}{1-\alpha}(t_n-t_{j+1})^{\alpha}\Big]$$
$$-(t_n-t_{j+1})^2E_{\alpha,3}\Big[\frac{-\alpha}{1-\alpha}(t_n-t_{j+1})^{\alpha}\Big]+(t_n-t_j)^2E_{\alpha,3}$$
$$\times\Big[\frac{-\alpha}{1-\alpha}(t_n-t_j)^{\alpha}\Big]\Big]$$
$$=b\sum_{k=-W}^{+W}u_{k+i}^n\delta_{\Delta,\sigma}(-k\Delta x)$$
$$\times\frac{e^{-t_n}+e^{-t_{n+1}}}{2\Delta t}\sum_{j=0}^{n-1}\{u_i^{j+1}(e^{t_{j+1}}\Delta t-e^{t_{j+1}}+e^{t_j})$$
$$+u_i^j(-e^{t_j}\Delta t+e^{t_{j+1}}-e^{t_j})\}$$
$$+b\sum_{k=-W}^{+W}u_{k+i}^n\delta_{\Delta,\sigma}(-k\Delta x)\times\frac{e^{-t_{n+1}}}{2\Delta t}\{u_i^{n+1}(e^{t_{n+1}}\Delta t-e^{t_{n+1}}+e^{t_n})$$
$$+u_i^n(-e^{t_n}\Delta t+e^{t_{n+1}}-e^{t_n})\}+\sum_{k=-W}^{+W}u_{k+i}^n\delta^2_{\Delta,\sigma}(x_i-x_k)$$
$$+a\sum_{k=-W}^{i+W}u_{k+i}^n\delta_{\Delta,\sigma}(-k\Delta x)\left(1-\sum_{k=-W}^{+W}u_{k+i}^n\delta_{\Delta,\sigma}(-k\Delta x)\right).\tag{40}$$

Hence we get the full discrete form of our model when kernel $\kappa(x,t)=e^{-t}$. Similarly, full discrete form of our model when kernel $\kappa(x,t)=1$ and $\kappa(x,t)=e^{-x^2t}$ is given by following equations respectively

$$\frac{B(\alpha)}{\Gamma(-\alpha+1)}\sum_{j=0}^{n-1}\frac{u_i^{j+1}-u_i^{j-1}}{2\Delta t}\Big[(t_n-t_j)E_{\alpha,2}\Big[\frac{-\alpha}{1-\alpha}(t_n-t_j)^{\alpha}\Big]$$
$$-(t_n-t_{j+1})E_{\alpha,2}\Big[\frac{-\alpha}{1-\alpha}(t_n-t_{j+1})^{\alpha}\Big]\Big]$$
$$+\frac{B(\alpha)}{\Gamma(-\alpha+1)}\sum_{j=0}^{n-1}\frac{u_i^{j+1}-2u_i^j+u_i^{j-1}}{(\Delta t)^2}$$
$$\times\Big[-(t_n-t_{j+1})\Delta t E_{\alpha,2}\Big[\frac{-\alpha}{1-\alpha}(t_n-t_{j+1})^{\alpha}\Big]$$
$$-(t_n-t_{j+1})^2E_{\alpha,3}\Big[\frac{-\alpha}{1-\alpha}(t_n-t_{j+1})^{\alpha}\Big]+(t_n-t_j)^2E_{\alpha,3}$$
$$\times\Big[\frac{-\alpha}{1-\alpha}(t_n-t_j)^{\alpha}\Big]\Big]$$
$$=b\left(\frac{\Delta t}{4}\sum_{j=0}^{n-1}(u_i^j+u_i^{j+1})+\frac{\Delta t}{4}\sum_{j=0}^{n}(u_i^j+u_i^{j+1})\right)$$
$$+\sum_{k=-W}^{+W}u_{k+i}^n\delta^2_{\Delta,\sigma}(x_i-x_k)$$
$$+a\sum_{k=-W}^{i+W}u_{k+i}^n\delta_{\Delta,\sigma}(-k\Delta x)\left(1-\sum_{k=-W}^{+W}u_{k+i}^n\delta_{\Delta,\sigma}(-k\Delta x)\right),\tag{41}$$

$$\frac{B(\alpha)}{\Gamma(-\alpha+1)}\sum_{j=0}^{n-1}\frac{u_i^{j+1}-u_i^{j-1}}{2\Delta t}\Big[(t_n-t_j)E_{\alpha,2}\Big[\frac{-\alpha}{1-\alpha}(t_n-t_j)^{\alpha}\Big]$$
$$-(t_n-t_{j+1})E_{\alpha,2}\Big[\frac{-\alpha}{1-\alpha}(t_n-t_{j+1})^{\alpha}\Big]\Big]$$
$$+\frac{B(\alpha)}{\Gamma(-\alpha+1)}\sum_{j=0}^{n-1}\frac{u_i^{j+1}-2u_i^j+u_i^{j-1}}{(\Delta t)^2}$$
$$\times\Big[-(t_n-t_{j+1})\Delta t E_{\alpha,2}\Big[\frac{-\alpha}{1-\alpha}(t_n-t_{j+1})^{\alpha}\Big]$$
$$-(t_n-t_{j+1})^2E_{\alpha,3}\Big[\frac{-\alpha}{1-\alpha}(t_n-t_{j+1})^{\alpha}\Big]+(t_n-t_j)^2E_{\alpha,3}$$
$$\times\Big[\frac{-\alpha}{1-\alpha}(t_n-t_j)^{\alpha}\Big]\Big]$$
$$=b\sum_{k=-W}^{+W}u_{k+i}^n\delta_{\Delta,\sigma}(-k\Delta x)\times\frac{e^{-t_{n+1}x_i^2}}{2\Delta t}$$
$$\times\sum_{j=0}^{n-1}\left\{u_i^{j+1}\left(\frac{e^{t_{j+1}x_i^2}}{x_i^2}\Delta t-\frac{e^{t_{j+1}x_i^2}}{x_i^4}+\frac{e^{t_jx_i^2}}{x_i^4}\right)\right.$$
$$\left.+u_i^j\left(\frac{-e^{t_{j+1}x_i^2}}{x_i^2}\Delta t+\frac{e^{t_{j+1}x_i^2}}{x_i^4}-\frac{e^{t_jx_i^2}}{x_i^4}\right)\right\}$$
$$+\sum_{k=-W}^{+W}u_{k+i}^n\delta^2_{\Delta,\sigma}(x_i-x_k)+a\sum_{k=-W}^{i+W}u_{k+i}^n\delta_{\Delta,\sigma}(-k\Delta x)$$
$$\times\left(1-\sum_{k=-W}^{+W}u_{k+i}^n\delta_{\Delta,\sigma}(-k\Delta x)\right).\tag{42}$$

Now for discretization of boundary condition we apply a technique because function value $u(x_k)$ are undefined outside the domain [0,1]. Since our boundary condition are Dirichlet type so we applied our zero boundary conditions and discretize as follows

$$u(i,n)=u_i^n=0,\, i<0,\, i>M,\, n=0,1,\cdots N.\tag{43}$$

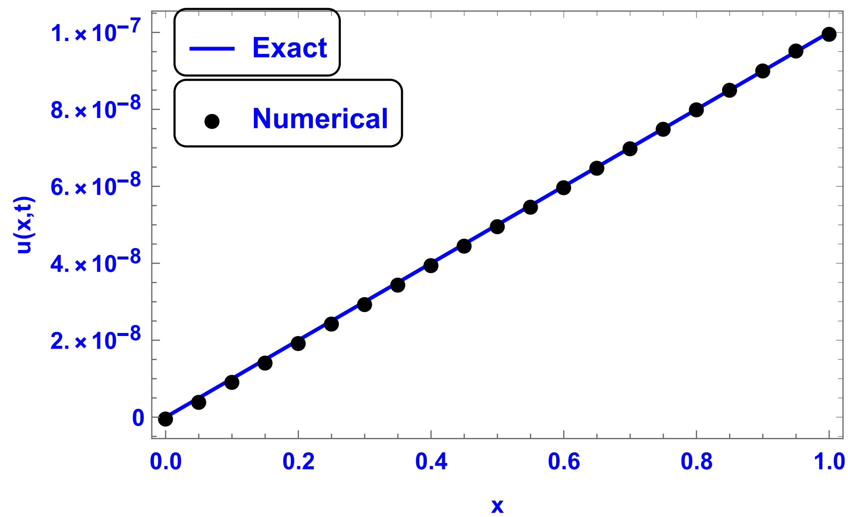

Fig. 1. Plots of $u(x, t)$ for $M = 20,\ \alpha = 0.9,\ W = 20,\ \Delta t = 0.00001$ and $r = 3.2$ in case of numerical and exact solution.

In addition discretization of initial condition is as follows

$$u_i^0 = f_3(x_i), i = 0, 1, \cdots M. \tag{44}$$

3.2. Theorem: [33]

If $f(x)$ be a function which belongs to the space $L_\infty \bigcap L_2(\Omega) \bigcap C^s(\Omega)$ and band-limited to B, $s \in Z^+$, $\sigma = \Delta r$, $W \in N$, $W \geq \frac{rs}{\sqrt{2}}$. Then we have

$$\left\| f^s - \sum_{k=-W}^{W} \delta^s_{\sigma,\Delta}(x - x_k) f(x_k) \right\| \leq \beta \times \exp\left(\frac{-\gamma^2}{2r^2} \right) \tag{45}$$

where

$$\gamma = \min(r^2(\pi - B\Delta), W),$$
$$\beta = (\sqrt{2B}\|f\|_{L_s(\Omega)} + 2r\|f\|_{L_\infty(\Omega)}) \times \frac{e^\pi (s+1)! r}{\gamma \pi \Delta^s}. \tag{46}$$

4. Results and discussion

In this section our aim is to show performances of our proposed method. All numerical computations are done with Wolfram Mathematica version-11.3.

Example 1. Considering $a = 1,\ \alpha = 0.9$ and $b = 0$ we get the following time fractional non-linear reaction-diffusion equation

$$^{ABC}_{0}D_t^{0.9} u(t, x) = \frac{\partial^2 u(t,x)}{\partial x^2} + \quad u(t, x)(1 - u(t, x)) + f(t, x), \tag{47}$$

with the aid of following initial and boundary conditions

$$u(x, 0) = 0, u(0, t) = 0, u(1, t) = \sin t, \tag{48}$$

The force function is $f(x, t)$ is such that the exact analytical solution of above problem is $u(x, t) = x \sin t$.

We plot the graph of exact and numerical solution with $N = 10, M = 20, \Delta t = 0.00001$ which is depict by Fig. 1. The absolute error between exact and numerical results for various M and Δt is presented by Table 1.

Example 2. If we consider $\kappa(x, t) = 1,\ a = 0,\ \alpha = 0.5$ and $b = 1$ so that our model (27) is reduced to

$$^{ABC}_{0}D_t^{0.5} u(t, x) = \frac{\partial^2 u(t, x)}{\partial x^2} - \int_0^t u(s, x) ds + f(x, t). \tag{49}$$

which under the prescribed initial and boundary conditions

Table 1
variations of absolute error for different value of x at $\Delta t = 0.00001$.

x ↓	$\Delta t = 0.00001$
0.2	1.2×10^{-5}
0.4	5.9×10^{-6}
0.6	7.0×10^{-7}
0.8	4.8×10^{-6}
1	1.1×10^{-5}

Table 2
variations of absolute error for different value of x at $\Delta t = 0.00001$.

x ↓	$\Delta t = 0.00001$
0.2	1.5×10^{-3}
0.4	4.2×10^{-3}
0.6	6.8×10^{-3}
0.8	9.6×10^{-3}
1	1.1×10^{-3}

$$u(0, x) = \frac{1}{2}\left(1 - x^2\right),$$
$$u(t, 0) = \frac{\cosh(t)}{\sinh^2(t) + 2},$$
$$u(t, 1) = 0, \tag{50}$$

gives the exact solution of above the problem is $u(x, t) = \frac{(1-x^2)\cosh(t)}{\sinh^2(t)+2}$ with suitable force function $f(x, t)$.

The graph of numerical and exact solution with $N = 10, M = 20, \Delta t = 0.00001$ is depict by Fig. 2. The absolute error for various M and Δt is presented by Table 2, which clearly predict that our numerical results are in complete agreement with the existing results.

Example 3. Consider the following integro reaction diffusion equation with $a = 1,\ \alpha = 0.9,\ b = 1$and kernel $\kappa(t, x) = e^{-x^2 t}$,

$$^{ABC}_{0}D_t^{0.9} u(t, x) = \frac{\partial^2 u(t, x)}{\partial x^2} + u(t, x)(1 - u(t, x)) + u(t, x) \int_0^t e^{-x^2(t-s)} u(s, x) ds + f(t, x). \tag{51}$$

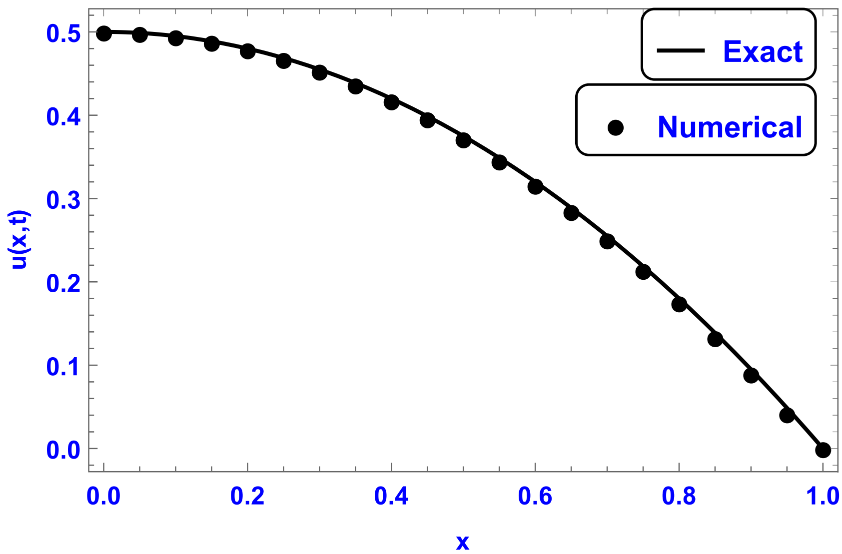

Fig. 2. Plots of $u(x,\ t)$ for $M = 20,\ \alpha = 1,\ W = 20,\ \Delta t = 0.00001$ and $r = 3.2$ in case of numerical and exact solution.

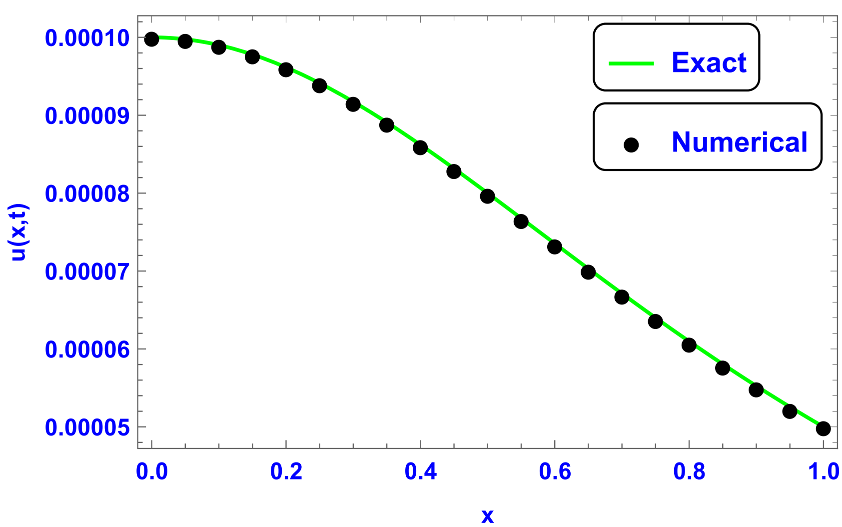

Fig. 3. Plots of $u(x,\ t)$ for $M = 20,\ \alpha = 1,\ W = 20,\ \Delta t = 0.00001$ and $r = 3.2$ in case of numerical and exact solution.

Table 3
variations of absolute error for different value of x at $\Delta t = 0.00001$.

x ↓	$\Delta t = 0.00001$
0.2	1.5×10^{-7}
0.4	3.1×10^{-7}
0.6	4.3×10^{-7}
0.8	5.3×10^{-7}
1	6.7×10^{-7}

The Eq. (51) with initial and boundary conditions

$$u(x,0) = 0, u(0,t) = \sin t, u(1,t) = \frac{\sin t}{2}, \tag{52}$$

gives the exact solution $u(x,t) = \frac{\sin t}{1+x^2}$ with suitable force function $f(x,\ t)$.

We plot the graph of exact and numerical solution with $N = 10, M = 20, \Delta t = 0.00001$ which is depicted by Fig. 3. The absolute error for various M and Δt is presented in Table 3 which clearly predict that our numerical results are in complete agreement with the existing results.

Example 4. If we take $\kappa(t,x) = e^{-t}$, $a = 1$, $\alpha = 0.9$ and $b = 1$ then our model (27) is reduced to

$$\begin{aligned} {}_{0}^{ABC}D_t^{0.9}u(t,x) = \frac{\partial^2 u(t,x)}{\partial x^2} + u(t,x)(1 - u(t,x)) \\ + u(t,x)\int_0^t e^{-(t-s)}u(s,x)ds + f(x,t). \end{aligned} \tag{53}$$

The Eq. (53) with initial and boundary conditions

$$\begin{aligned} u(x,0) &= 0, \\ u(0,t) &= 0, \\ u(1,t) &= t, \end{aligned} \tag{54}$$

gives the exact solution $u(x,t) = xt$.

The graph of numerical and exact solution for $N = 10, M = 20, \Delta t = 0.00001$ is shown in Fig. 4 and the absolute error is shown in Table 4. The results clearly predict that our numerical results are in complete agreement with the existing results.

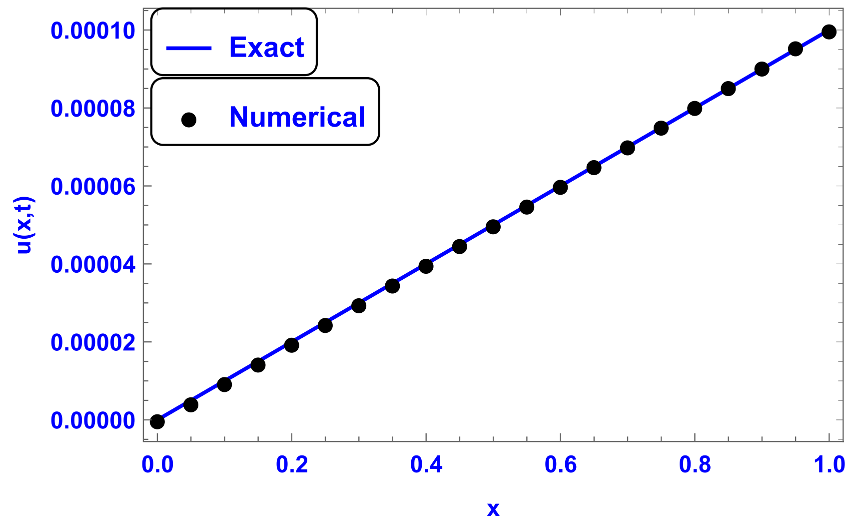

Fig. 4. Plots of $u(x, t)$ for $M = 20$, $\alpha = 0.9$, $W = 20$, $\Delta t = 0.00001$ and $r = 3.2$ in case of numerical and exact solution.

Table 4
variations of absolute error for different value of x at $\Delta t = 0.00001$.

x ↓	$\Delta t = 0.00001$
0.2	1.2×10^{-6}
0.4	5.8×10^{-7}
0.6	6.7×10^{-8}
0.8	4.7×10^{-7}
1	1.7×10^{-6}

5. Conclusion

In this article, we developed a difference scheme for discretization of time fractional derivative of Atangana–Baleanu type with the help of Taylor series and quasi wavelet method for discretization of spatial derivative and unknown function. This difference scheme in combination with quasi-wavelet numerical method is developed for solving time fractional with ABC derivative nonlinear reaction-diffusion and integro reaction-diffusion equation. In the knowledge of author presented method is first time used with this ABC fractional derivative. We easily conclude that quasi wavelet method has good accuracy and also valid for time fractional integro-reaction diffusion equation. It also has good ability to analyze the local characteristic of functions.

Declaration of Competing Interest

None

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