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GENERALIZATION OF LEADER'S FIXED POINT PRINCIPLE

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In *J. Math. Anal. Appl.* 61 (1977), 466-474, Leader has given a fixed point principle for an operator $f : X \to X$, where X is a metric-space, based on conditional uniform equivalence of orbits. We generalize this principle for two mappings f_1 and f_2 to give common fixed point results in two different ways. Further we derive an f-generalized fixed point theorem for two commuting mappings.

Introduction

Let (f_l, X, d) , $l = 1, 2$, be two operators where $f_l : X \to X$ and X is a metric space. Let $F \cong f_2 f_1$ be the composite map of f_1 and f_2 . Then we define the generalized orbit of a point $x \in X$ as the sequence of iterates x , $f_1(x)$, $f_2 f_1(x) \cong F(x)$, $f_1 F(x)$, $F^2(x)$, $f_1 F^2(x)$, $F^3(x)$, ... The limit p of a generalized convergent orbit must be fixed under certain weak conditions (for example, f_1, f_2 have closed graphs or $d(x, f_l(x))$, $l = 1, 2$, is lower semicontinuous), that is, $p = f_l(p)$, $l = 1, 2$. We call a fixed point p a generalized fixed

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point if it is the limit of every generalized orbit in X . A necessary condition for a fixed point is the equivalence of all generalized orbits. Now we give our first result as follows.

THEOREM 1. *Let* (f_l, X, d) , $l = 1, 2$, *be two operators on a metric space* (X, d) . *Let* $\{x_n\}$ *and* $\{y_n\}$ *be two sequences generated by generalized orbits of* x_0 *and* y_0 *respectively with the applications of maps* f_1 *and* f_2 *in the following way:*

$$x_1 = f_1(x_0), x_2 = f_2(x_1), \ldots ,$$

and in general

$$x_{2n+1} = f_1(x_{2n}), x_{2n+2} = f_2(x_{2n+1}) , \quad n = 0, 1, 2, \ldots ,$$

and

$$y_1 = f_2(y_0), y_2 = f_1(y_1), \ldots ,$$

and in general

$$y_{2n+1} = f_2(y_{2n}), y_{2n+2} = f_1(y_{2n+1}) , \quad n = 0, 1, 2, \ldots .$$

Let ε_i , $i \in N$, *be positive real numbers defined by*

(1) $$\varepsilon_n = \sup\{d(x_i, y_i) : i \geq n, d(x_0, y_0) \leq c\} .$$

If $(m/2)(\varepsilon_n+\varepsilon_{n+1}) + \frac{1}{2}(\varepsilon_n-\varepsilon_{n+1}) + \varepsilon_m \leq c$ *and* $d(x, f_l(x)) \leq c$, $l = 1, 2$, *then, for all* $i \geq n$ *and all* j *in* N ,

(2) $$d(x_i, x_{i+j}) \leq \frac{m}{2} (\varepsilon_n+\varepsilon_{n+1}) + \frac{1}{2}(\varepsilon_n-\varepsilon_{n+1}) + \varepsilon_m .$$

Further if

(3) $$d(x_n, y_n) \to 0$$

uniformly for all x_0, y_0 *in* X *with* $d(x_0, y_0) \leq c$ *then*

(4) *the sequence* $\{x_n\}$ *is uniformly Cauchy.*

If the graphs of both (f_l, X, d) , $l = 1, 2$, *are complete and* (3) *holds then* $d(x, f_l(x)) \leq c$, $l = 1, 2$, *implies that* f_1 *and* f_2 *have a* *common fixed point.*

Proof. By the induction process on k we shall prove (2) for $j \le km$ for all k in N under the given condition that, for a given m, n,

$$\frac{m}{2}\left(\varepsilon_n+\varepsilon_{n+1}\right) + \tfrac{1}{2}\left(\varepsilon_n-\varepsilon_{n+1}\right) + \varepsilon_m \le c$$

and

$$d\left(x, f_l(x)\right) \le c\ , \quad l = 1, 2 \quad \text{for all} \quad x \in X\ .$$

Let $x_i = (x)_i$, that is, i iterations on initial point x by f_1 and f_2 accordingly. Let $k = 1$, that is, $j \le m$; then (1) implies for all $i \ge n$ and $j \le m$, m even, that

$$\begin{aligned}(5)\quad d\left(x_i, x_{i+j}\right) &\le d\left(x_i, x_{i+1}\right) + d\left(x_{i+1}, x_{i+2}\right) + \ldots + d\left(x_{i+j-1}, x_{i+j}\right)\\ &\le d\left(\left(x_0\right)_i, \left(x_1\right)_i\right) + d\left(\left(x_0\right)_{i+1}, \left(x_1\right)_{i+1}\right) + d\left(\left(x_2\right)_i, \left(x_3\right)_i\right)\\ &\qquad + d\left(\left(x_2\right)_{i+1}, \left(x_3\right)_{i+1}\right) + \ldots + d\left(\left(x_{j-2}\right)_{i+1}, \left(x_{j-1}\right)_{i+1}\right)\\ &\le \varepsilon_n + \varepsilon_{n+1} + \varepsilon_n + \varepsilon_{n+1} + \ldots + \varepsilon_n + \varepsilon_{n+1}\\ &\le \frac{j}{2}\left(\varepsilon_n+\varepsilon_{n+1}\right) \le \frac{m}{2}\left(\varepsilon_n+\varepsilon_{n+1}\right)\ .\end{aligned}$$

When m is odd we get

$$\begin{aligned}d\left(x_i, x_{i+j}\right) &\le d\left(x_i, x_{i+1}\right) + \ldots + d\left(x_{i+j-1}, x_{i+j}\right)\\ &\le d\left(\left(x_0\right)_i, \left(x_1\right)_i\right) + d\left(\left(x_0\right)_{i+1}, \left(x_1\right)_{i+1}\right)\\ &\qquad + \ldots + d\left(\left(x_{j-3}\right)_{i+1}, \left(x_{j-2}\right)_{i+1}\right) + d\left(\left(x_{j-1}\right)_i, \left(x_j\right)_i\right)\\ &\le \varepsilon_n + \varepsilon_{n+1} + \varepsilon_n + \ldots + \varepsilon_{n+1} + \varepsilon_n\\ &= \left(\frac{j+1}{2}\right)\varepsilon_n + \left(\frac{j-1}{2}\right)\varepsilon_{n+1}\\ &\le \frac{m}{2}\left(\varepsilon_n+\varepsilon_{n+1}\right) + \tfrac{1}{2}\left(\varepsilon_n-\varepsilon_{n+1}\right)\ .\end{aligned}$$

Thus

$$(6)\qquad d\left(x_i, x_{i+j}\right) \le \frac{m}{2}\left(\varepsilon_n+\varepsilon_{n+1}\right) + \tfrac{1}{2}\left(\varepsilon_n-\varepsilon_{n+1}\right)$$

for all $i \ge n$ and $j \le m$, independent of m even or odd, that is, (2) holds for all $j \le m$. Now we suppose that (2) holds for all $j \le km$ and prove it for $j \le (k+1)m$. Consider $km < j \le (k+1)m$. Then $0 < j-m \le km$

and so our induction process gives

$$d(x_i, x_{i+j-m}) \le \frac{m}{2}(\varepsilon_n+\varepsilon_{n+1}) + \tfrac{1}{2}(\varepsilon_n-\varepsilon_{n+1}) + \varepsilon_m \le c$$

for all $i \ge n$. Now iterating m times by f_1 and f_2 accordingly the above we get

(7) $$d(x_{i+m}, x_{i+j}) \in \varepsilon_m \quad \text{for all} \quad i \ge n ;$$

here $x_0 = x_i$ and $y_0 = x_{i+j-m}$.

Therefore from (6) with $j = m$ and (7) we get

$$d(x_i, x_{i+j}) \le d(x_i, x_{i+m}) + d(x_{i+m}, x_{i+j})$$

$$\le \frac{m}{2}(\varepsilon_n+\varepsilon_{n+1}) + \tfrac{1}{2}(\varepsilon_n-\varepsilon_{n+1}) + \varepsilon_m ,$$

for all $i \ge n$. Thus (2) is true for all $j \le (k+1)m$ and hence for all j in N.

Now from (1) and (3) we have $\varepsilon_n \downarrow 0$. Therefore for a given $0 < \varepsilon < c$, we can choose m so large that $\varepsilon_m < \varepsilon$. Further we take n so large that

(8) $$(\varepsilon_n+\varepsilon_{n+1}) + m^{-1}(\varepsilon_n-\varepsilon_{n+1}) < 2m^{-1}(\varepsilon-\varepsilon_m) .$$

(For this we choose n large enough so that $\varepsilon_n < (\varepsilon-\varepsilon_m)/m$ and this gives

$$\left(1+\frac{1}{m}\right)\varepsilon_n + \left(1-\frac{1}{m}\right)\varepsilon_n < 2\,\frac{(\varepsilon-\varepsilon_m)}{m}$$

or

$$\left(1+\frac{1}{m}\right)\varepsilon_n + \left(1-\frac{1}{m}\right)\varepsilon_{n+1} < 2\,\frac{(\varepsilon-\varepsilon_m)}{m}$$

which consequently satisfies (8)). Thus we have

$$\frac{m}{2}(\varepsilon_n+\varepsilon_{n+1}) + \tfrac{1}{2}(\varepsilon_n-\varepsilon_{n+1}) + \varepsilon_m < \varepsilon < c .$$

So (2) holds for all j in N and hence $d(x_i, x_{i+j}) < \varepsilon$ for all $i \ge n$ and j in N and all x with $d(x, f_l(x)) \le c$, which proves (4).

Further let the graphs of (f_l, X, d), $l = 1, 2$, be complete and (3) hold, then for all x with $d(x, f_l(x)) \le c$, $l = 1, 2$, (4) gives that $\{f_1(x_{2n})\}$ and $\{f_2(x_{2n+1})\}$ are Cauchy and hence by graph-completeness we have

$$f_1(x_{2n}) \to f_1(t) \text{ with } x_{2n} \to t$$

and

$$f_2(x_{2n+1}) \to f_2(t) \text{ with } x_{2n+1} \to t, \quad t \in X .$$

Therefore $f_1(t) = t$ and $f_2(t) = t$, that is, $f_1(t) = f_2(t) = t$ which completes the whole proof.

REMARK. In case $f_1 = f_2 = f$ we get Theorem 1 of Leader [1] as a corollary. (In that case ε_{n+1} reduces to ε_n.)

In what follows we give another generalization of Leader's fixed point principle by considering a composite map.

THEOREM 2. *Let* (f_l, X, d), $l = 1, 2$, *be two operators on a metric space* (X, d). *Let, for any* $x_0 \in X$, $\{x_n\}$ *be a sequence as defined in Theorem* 1. *Now let* $\{x_{2n}\}$ *be a sequence generated by the composite map* $F \cong f_2 f_1$ *in the following way with* x_0 *as the starting point:*

$$x_2 = F(x_0), \quad x_4 = F(x_2) = F^2(x_0) \ldots$$

and in general, $x_{2n+2} = F(x_{2n}) = F^{n+1}(x_0)$, $n = 0, 1, 2, \ldots$. *Now we define real numbers* ε_{2i}, $i \in N$, *for some* $c > 0$ *as*

(9) $$\varepsilon_{2n} = \sup\left\{d\left(F^i x_0, F^i y_0\right) : i \ge n, \; d(x_0, y_0) \le c\right\}$$

where $\{y_{2n}\}$ *is a sequence generated by repeated application of* F *on* y_0. *If* $m\varepsilon_{2n} + \varepsilon_{2m} \le c$ *and* $d(x, F(x)) \le c$, *then*

(10) $$d(x_{2i}, x_{2i+2j}) \le m\varepsilon_{2n} + \varepsilon_{2m} \text{ for all } i \ge n$$

and all j *in* N . *Hence if*

(11) $$d(x_{2n}, y_{2n}) \to 0$$

uniformly for all x_0, y_0 *in* X *with* $d(x_0, y_0) \le c$, *then*

(12) *the sequence* $\{x_{2n}\}$ *is uniformly Cauchy.*

Again if the graph of (F, X, d) *is complete and* (11) *holds then* $d(x, F(x)) \le c$ *implies that* F *has a fixed point. Further if*

(13) $$d(F(x), F(y)) < d(x, y) ,$$

then F *has a unique fixed point. Now if* $f_1f_2 = f_2f_1$ *then the fixed point of* F *becomes the unique common fixed point of* f_1 *and* f_2 .

The proof of (10), (12) and that F has a fixed point follows the same line as that of Theorem 1 of Leader [1]. Further, condition (13) guarantees the uniqueness of the fixed point of F which, on combination with the commutativity of f_1 and f_2 , gives the existence of the unique common fixed point (same as that of F) of both the maps and hence the result.

THEOREM 3. *Let* (X, d) *be a metric space. Let* f *and* g *be two mappings of* X *into* X *with* f *continuous. Let* f *and* g *commute with each other with* $g(X) \subset f(X)$. *For some* $x_0 \in X$ *we define a sequence* $\{y_n\}$ *as follows:*

$$y_1 = f(x_1) = g(x_0) , \quad y_2 = f(x_2) = g(x_1) , \ \ldots$$

and

$$y_n = f(x_n) = g(x_{n-1}) , \quad n = 1, 2, \ldots .$$

Similarly, for some $p_0 \in X$, *let us have a sequence* $\{z_n\}$, *that is,* $z_n = f(p_n) = g(p_{n-1})$, $n = 1, 2, \ldots$. *For some* $c > 0$, *define*

(14) $$\varepsilon_n = \sup\{d(y_i, z_i) : i \ge n-1, d(g(x_0), g(p_0)) \le c\} .$$

If $m\varepsilon_n + \varepsilon_m \le c$ *and* $d(f(x), g(x)) \le c$ *then*

(15) $$d(y_i, y_{i+j}) \le m\varepsilon_n + \varepsilon_m \quad \textit{for all} \quad i \ge n - 1$$

and all j *in* N . *Hence if*

(16) $$d(y_n, z_n) \to 0$$

uniformly for all $x_0, p_0 \in X$ *with* $d(g(x_0), g(p_0)) \le c$ *then the sequence* $\{y_n\}$ *is uniformly Cauchy, and further if* g *satisfies*

(17) $$d(g(x), g(y)) \le d(f(x), f(y)) \text{ for all } x, y \in X ,$$

then g *has an* f*-generalized fixed point.*

Before giving the proof of the theorem we give the definition of an f-generalized fixed point.

DEFINITION. Let $f, g : (X, d) \to (X, d)$ be two maps with $g(X) \subset f(X)$. If there exists a point $t \in X$ such that $g(t) = f(t)$ then we say that g has an f-generalized fixed point.

Proof. The proof of (15) and that $\{y_n\}$ is uniformly Cauchy goes in a similar fashion as that of Theorem 1 of Leader [*1*], so we omit it.

Let $y_n \to t$. Since f is continuous we have $f(y_n) \to f(t)$. Now (17) gives that $d(g(y_n), g(t)) \le d(f(y_n), f(t))$, which in the limiting case implies $g(y_n) \to g(t)$.

Thus $f(y_n) = f(g(x_{n-1})) = g(f(x_{n-1})) = g(y_{n-1}) \to g(t)$. Therefore $f(t) = g(t)$. Hence g has an f-generalized fixed point t . In case f is an identity map we get an ordinary fixed point.

THEOREM 4. *Let* (f_l, X, d) , $l = 1, 2$, *be two operators with graphs of* f_l *both complete. Let for some* $c > 0$, (X, d) *be weakly* c*-chained and* (3) *hold. Then* f_1, f_2 *have a common fixed point.*

Proof. Let $\{x_n\}$ and $\{y_n\}$ be two sequences as defined in Theorem 1 with initial points x_0 and y_0 respectively. Since (X, d) is weakly c-chained (see Leader [*1*], Theorem 2), we have for any $x, y \in X$ a finite sequence $\langle x^0, x^1, \ldots, x^m \rangle$ with $x^0 = x$, $x^m = y$ and $d(x^i, x^{i+1}) \le c$ for $i = 0, 1, \ldots, m-1$. Then in x_n, y_n applying triangle inequality we have

$$d(x_n, y_n) = d\left(x_n^0, x_n^m\right)$$
$$\leq d\left(x_n^0, x_n^1\right) + d\left(x_n^1, x_n^2\right) + \ldots + d\left(x_n^{m-1}, x_n^m\right) ,$$

where $d\left(x_n^i, x_n^{i+1}\right) \leq c$, for $i = 0, 1, \ldots, m-1$.

By using similar arguments as in proving (5) or (6) in the above and then applying (3) we get $d(x_n, y_n) \to 0$, as $n \to \infty$ for all $x, y \in X$ which further implies that all generalized orbits are equivalent.

In case $y = f_1(x_0)$, we get

$$d(x_n, f_1(x_n)) \to 0 , \quad n \text{ even},$$

and

$$d(x_n, f_2(x_n)) \to 0 , \quad n \text{ odd}.$$

Therefore for n large enough we have $d(x_n, f_l(x_n)) < c$, $l = 1, 2$, accordingly with n even or n odd. Hence Theorem 1 gives the result.

THEOREM 5. *Let* (F, X, d) *, where* $F = f_2 f_1$ *, be a composite operator as defined in Theorem* 2 *with the graph of* (F, X, d) *complete. Let for some* $c > 0$ *,* (X, d) *be weakly* c*-chained and* (11) *hold. Then* F *has a fixed point.*

Proof. The proof of the above theorem is analogous to that of the above theorem, so we omit it.

References

[1] S. Leader, "Fixed points for operators on metric spaces with conditional uniform equivalence of orbits", *J. Math. Anal. Appl.* 61 (1977), 466-474.

[2] A. Meir and E. Keeler, "A theorem on contraction mappings", *J. Math. Anal. Appl.* 28 (1969), 326-329.

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