Analytic Solution of Fractional-Order Heat- and Wave-Like Equations Using Generalized n-dimensional Differential Transform Method

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In this paper, we have introduced a generalized n-dimensional differential transform method to propose a user friendly algorithm to obtain the closed form analytic solution for n-dimensional fractional heat- and wave-like equations. Three examples are given to establish the simplicity of the algorithm. In Example 5.3, we show that ten terms of the series representing the solution, even in fractional order, give a very accurate solution.

Key words: Generalized n-Dimensional Differential Transform Method; n-Dimensional Fractional Heat- and Wave-Like Equation; Caputo Fractional Derivative.
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2. Introduction

The idea of fractional-order derivatives initially arose from a letter by Leibnitz to L'Hospital in 1695. Fractional calculus has gained considerable popularity and importance during the past three decades, mainly due to its applications in numerous fields of science and engineering. One of the main advantages of using fractional-order differential equations in mathematical modelling is their non-local property. It is a well known fact that the integer-order differential operator is a local operator whereas the fractional-order differential operator is non-local in the sense that the next state of the system depends not only upon its current state but also upon all of its proceeding states.

In the last decade, many authors have made notable contributions to both theory and application of fractional differential equations in areas as diverse as finance [1 – 3], physics [4 – 7], control theory [8], and hydrology [9 – 13].

In this paper, we consider the following n-dimensional fractional heat- and wave-like equations which are the generalized form of the model in [14]:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = f_1(x_1,x_2,\ldots,x_{n-1})\frac{\partial^2 u}{\partial x_1^2} + f_2(x_1,x_2,\ldots,x_{n-1})\frac{\partial^2 u}{\partial x_2^2} + \ldots + f_{n-1}(x_1,x_2,\ldots,x_{n-1})\frac{\partial^2 u}{\partial x_{n-1}^2} + g(x_1,x_2,\ldots,x_{n-1},t), \tag{1}$$

$$0 < x_i < a_i,\ \ i = 1,2,\ldots,(n-1),\ \ 0 < \alpha \le 2,\ \ t > 0,$$

subject to the initial conditions

$$\begin{aligned} u(x_1,x_2,\ldots,x_{n-1},0) &= \Psi(x_1,x_2,\ldots,x_{n-1}),\\ u_t(x_1,x_2,\ldots,x_{n-1},0) &= \eta(x_1,x_2,\ldots,x_{n-1}), \end{aligned} \tag{2}$$

where α is a parameter describing the fractional derivative. Fractional heat-like and wave-like equations are obtained from (1) by restricting the parameter α in $(0,1]$ and $(1,2]$, respectively. The fractional wave-like equation can be used to describe different models in anomalous diffusive and sub diffusive systems, description of fractional random walk, unification of diffusion and wave propagation phenomenon [15 – 18].

Several authors [14, 19, 20] applied the Adomian decomposition method (ADM), the variational iteration method (VIM), and the homotopy analysis method (HAM) successfully to solve two- and three-dimensional fractional heat- and wave-like equations.

In 1986, a new powerful numerical technique named differential transform method (DTM), was developed by Zhao [21], to solve various scientific and engineering problems. Originally, he developed DTM to solve the electric circuit problems. DTM is based on the Taylor series expansion which constructs analytical solutions in the form of a polynomial. The traditional higher-order Taylor series method requires symbolic computations, but the DTM does not require high symbolic computations. However, the solution is obtained by DTM in the form of polynomial series through an iterative procedure. Various applications of DTM are given in [22–26]. Recently Kurnaz et al. [27] have applied DTM for solving partial differential equations. Arikoglu and Ozkol [28] developed the fractional differential transform method which is based on the classical differential transform method, on fractional power series, and on Caputo fractional derivatives. Odibat and Momani proposed the one- and two-dimensional generalized differential transform method (GDTM) to solve various ordinary/partial differential equations of integer and fractional order [29–31].

In this paper, we extend the two-dimensional GDTM [29–31] to n-dimensions and apply it to solve n-dimensional fractional heat- and wave-like equations. The accuracy and applicability of the above method is established by means of several examples.

3. Fractional Calculus

We give some basic definitions and properties of fractional calculus [32–35] that are prerequisite for further development.

Definition 2.1. A real function $f(x), x>0$, is said to be in a space $C_\mu, \mu \in \mathbb{R}$ if there exists a real number $p(<\mu)$ such that $f(x)=x^p f_1(x)$ where $f_1(x)\in C[0,\infty)$, and is said to be in the space C_m^μ if $f^{(m)} \in C_\mu, m \in \mathbb{N}$.

Definition 2.2. The Riemann–Liouville fractional integral operator of order $\alpha \geq 0$ of a function $f \in C_\mu, \mu \geq -1$ is defined as

$$J_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\,\mathrm{d}t, \quad \alpha>0,\; x>0. \tag{3}$$

For $\alpha, \beta > 0$, $a \geq 0$, and $\gamma \geq -1$, the operator J_a^α has the following properties:

$$\begin{aligned} &1.\; J_a^\alpha (x-a)^\gamma = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma+\alpha)}(x-a)^{\gamma+\alpha},\\ &2.\; J_a^\alpha J_a^\beta f(x) = J_a^{\alpha+\beta} f(x),\\ &3.\; J_a^\alpha J_a^\beta f(x) = J_a^\beta J_a^\alpha f(x). \end{aligned} \tag{4}$$

Definition 2.3. The fractional derivative of order α of a function $f(x)$ in the Caputo sense is defined as

$$D_a^\alpha f(x) = J_a^{m-\alpha} D_a^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-t)^{m-\alpha-1} f^m(t)\,\mathrm{d}t \tag{5}$$

for $m-1<\alpha\leq m,\; m\in\mathbb{N},\; x>a,\; f\in C_{-1}^m$.

The following properties of the operator D_a^α are well known:

$$D_a^\alpha (x-a)^\gamma = \begin{cases} \dfrac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)}(x-a)^{\gamma-\alpha}, & \text{for } \alpha \leq \gamma\\ 0, & \text{for } \alpha > \gamma, \end{cases} \tag{6}$$

$$D_a^\alpha J_a^\alpha f(x) = f(x), \tag{7}$$

$$J_a^\alpha D_a^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(a)\frac{(x-a)^k}{k!}, \quad x>0. \tag{8}$$

The following theorem involving generalized Taylor's formula is needed for the further development of the theory.

Theorem 3.1. If $u(x,y)=f(x)g(y),\; f(x)=x^\lambda g(x)$, $\lambda>-1$, and $g(x)$ has the generalized power series expansion $g(x)=\sum_{n=0}^\infty a_n \cdot (x-x_0)^{n\alpha}$ with the radius of convergence $R>0$, then for $0<\alpha\leq 1,\; x\in(0,R)$,

$$D_{x_0}^\gamma D_{x_0}^\beta f(x) = D_{x_0}^{\gamma+\beta} f(x), \tag{9}$$

when either of the two conditions hold:

(a) $\beta<\lambda+1$ and γ is arbitrary or
(b) $\beta \geq \lambda+1,\; \gamma$ is arbitrary, and $a_n=0$ for $n=0,1,\ldots,m-1$, where $m-1<\beta\leq m$.

Proof is given in [31].

4. Generalized n-Dimensional Differential Transform Method

We have used the following symbolic notations for convenience:

(i) $(x_1,x_2,\ldots,x_n)\equiv(x)_n$,
(ii) $\alpha_1,\alpha_2,\ldots,\alpha_n \equiv (\alpha)_n$,
(iii) $(*,x_2,\ldots,x_n)\equiv(*,\bar{x}_n)$.

The generalized n-dimensional differential transform of a function $u(x)_n$ is defined as

$$U_{(\alpha)_n}(k)_n = \frac{1}{\prod_{i=1}^{n}\Gamma(\alpha_i k_i+1)}\left[\left(\prod_{i=1}^{n}(D_{\tilde{x}_i}^{\alpha_i})^{k_i}\right)u(x)_n\right]_{(\tilde{x})_n}. \tag{10}$$

The inversion of (10) is given by

$$u(x)_n = \sum_{k_1,k_2,\ldots,k_n=0}^{\infty}\left[U_{(\alpha)_n}(k)_n\prod_{i=1}^{n}(x_i-\tilde{x}_i)^{k_i\alpha_i}\right]. \tag{11}$$

For $\alpha=1,\forall i$ the generalized n-dimensional differential transform reduces to the classical n-dimensional differential transform. For the special case when $u(x)_n$ can be split as

$$u(x)_n = \prod_{i=1}^{n} f_i(x_i),$$
$$\text{then } u(x)_n = \prod_{i=1}^{n}\left[\sum_{k_i=0}^{\infty}F_{\alpha_i}(k_i)\cdot(x_i-\tilde{x}_i)^{k_i\alpha_i}\right], \tag{12}$$

where $F_{\alpha_i}(k_i)$ are the generalized one-dimensional differential transforms of $f_i(x_i), 1\le i\le n$. From (11) and (12) we deduce that $U_{(\alpha)_n}(k)_n=\prod_{i=1}^{n}F_{\alpha_i}(k_i)$.

Now we give some theorems outlining the different properties of $u(x)_n$ and $U_{(\alpha)_n}(k)_n$. These theorems are the n-dimensional generalisations of the corresponding theorems of [29–31].

Theorem 4.1. If $u(x)_n = v(x)_n \pm w(x)_n$, then $U_{(\alpha)_n}(k)_n = V_{(\alpha)_n}(k)_n \pm W_{(\alpha)_n}(k)_n$.

Theorem 4.2. If $u(x)_n = cv(x)_n$, then $U_{(\alpha)_n}(k)_n = cV_{(\alpha)_n}(k)_n$, where c is a scalar.

Theorem 4.3. For $u(x)_n = v(x)_n\cdot w(x)_n$, $U_{(\alpha)_n}(k)_n = \sum_{a_1=0}^{k_1}\sum_{a_2=0}^{k_2}\cdots\sum_{a_n=0}^{k_n}V_{(\alpha)_n}(a_1,\overline{k_n-a_n})\cdot W_{(\alpha)_n}(k_1-a_1,\overline{a_n})$.

Proof. The theorem is proved by using induction on n. The assertion follows trivially for $n=1$, as

$$U_{\alpha_1}(k_1) = \sum_{a_1=0}^{k_1}V_{\alpha_1}(k_1-a_1)W_{\alpha_1}(a_1). \tag{13}$$

Assuming the theorem holds for $n=m$,

$$U_{(\alpha)_m}(k)_m = \sum_{a_1=0}^{k_1}\sum_{a_2=0}^{k_2}\cdots\sum_{a_m=0}^{k_m}V_{(\alpha)_m}(a_1,\overline{k_m-a_m})W_{(\alpha)_m}(k_1-a_1,\overline{a_m}). \tag{14}$$

The inverse of above follows from (11) and is given as

$$u(x)_m = \sum_{k_1=k_2=\ldots k_m=0}^{\infty}\sum_{a_1=0}^{k_1}\sum_{a_2=0}^{k_2}\cdots\sum_{a_m=0}^{k_m}\left[V_{(\alpha)_m}(a_1,\overline{k_m-a_m})\cdot W_{(\alpha)_m}(k_1-a_1,\overline{a_m})\prod_{i=1}^{m}(x_i-\tilde{x}_i)^{k_i\alpha_i}\right]$$

$$u(x)_m = \left[\sum_{k_1=k_2=\ldots k_m=0}^{\infty}V_{(\alpha)_m}(k)_m\prod_{i=1}^{m}(x_i-\tilde{x}_i)^{k_i\alpha_i}\right]\cdot\left[\sum_{k_1=k_2=\ldots k_n=0}^{\infty}W_{(\alpha)_m}(k)_m\prod_{i=1}^{m}(x_i-\tilde{x}_i)^{k_i\alpha_i}\right], \tag{15}$$

since $u(x)_n = v(x)_n\cdot w(x)_n$. Replacing m by $m+1$, we obtain

$$u(x)_{m+1} = \left[\sum_{k_1=k_2=\ldots k_{m+1}=0}^{\infty}V_{(\alpha)_{m+1}}(k)_{m+1}\prod_{i=1}^{m+1}(x_i-\tilde{x}_i)^{k_i\alpha_i}\right]\cdot\left[\sum_{k_1=k_2=\ldots k_{m+1}=0}^{\infty}W_{(\alpha)_{m+1}}(k)_{m+1}\prod_{i=1}^{m+1}(x_i-\tilde{x}_i)^{k_i\alpha_i}\right]. \tag{16}$$

Let $V^1_{(\alpha)_m}(k)_m =$

$$\left[\sum_{k_{m+1}=0}^{\infty}V_{(\alpha)_{m+1}}(k)_{m+1}(x_{m+1}-\tilde{x}_{m+1})^{k_{m+1}\alpha_{m+1}}\right] \tag{17}$$

and $W^1_{(\alpha)_m}(k)_m =$

$$\left[\sum_{k_{m+1}=0}^{\infty}W_{(\alpha)_{m+1}}(k)_{m+1}(x_{m+1}-\tilde{x}_{m+1})^{k_{m+1}\alpha_{m+1}}\right]. \tag{18}$$

Using (14)–(18), we have

$$u(x)_{m+1} = \sum_{k_1=k_2=\ldots k_m=0}^{\infty}\sum_{a_1=0}^{k_1}\sum_{a_2=0}^{k_2}\cdots\sum_{a_m=0}^{k_m}\left[V^1_{(\alpha)_m}(a_1,\overline{k_m-a_m})\cdot W^1_{(\alpha)_m}(k_1-a_1,\overline{a_m})\prod_{i=1}^{m}(x_i-\tilde{x}_i)^{k_i\alpha_i}\right]$$
$$= \sum_{k_1=k_2=\ldots k_m=0}^{\infty}\sum_{a_1=0}^{k_1}\sum_{a_2=0}^{k_2}\cdots\sum_{a_m=0}^{k_m}\sum_{k_{m+1}=0}^{\infty}\left[V_{(\alpha)_{m+1}}\cdot(a_1,\overline{k_m-a_m},k_{m+1})(x_{m+1}-\tilde{x}_{m+1})^{k_{m+1}\alpha_{m+1}}\right.$$

$$\cdot \sum_{k_{m+1}=0}^{\infty} W_{(\alpha)_{m+1}}(k_1-a_1,\overline{a_m},k_{m+1})$$
$$\cdot (x_{m+1}-\tilde{x}_{m+1})^{k_{m+1}\alpha_{m+1}}\Bigg] \prod_{i=1}^{m}(x_i-\tilde{x}_i)^{k_i\alpha_i}. \tag{19}$$

Using (14) in (19), we have

$$u(x)_{m+1} = \sum_{k_1=k_2=\ldots k_{m+1}=0}^{\infty} \sum_{a_1=0}^{k_1}\sum_{a_2=0}^{k_2}\cdots\sum_{a_{m+1}=0}^{k_{m+1}}\Bigg[V_{(\alpha)_{m+1}}$$
$$\cdot (a_1,\overline{k_{m+1}-a_{m+1}})W_{(\alpha)_{m+1}}(k_1-a_1,\overline{a_{m+1}})$$
$$\cdot \prod_{i=1}^{m+1}(x_i-\tilde{x}_i)^{k_i\alpha_i}\Bigg]. \tag{20}$$

Substituting (13) into (20), the validity of the theorem holds for $n=m+1$, thus proving the theorem by induction.

From now onwards $0<\alpha_i\le 1$, and $i=1,2,\ldots,n$.

Theorem 4.4. For $u(x)_n = D_{\tilde{x}_i}^{\alpha_i}v(x)_n, U_{(\alpha)_n}(k)_n = \frac{\Gamma(\alpha_i(k_i+1)+1)}{\Gamma(\alpha_i k_i+1)}V_{(\alpha)_n}(k_1,k_2,\ldots,k_{i-1},k_i+1,k_{i+1},\ldots k_n)$.

Proof. From (10) we have

$$U_{(\alpha)_n}(k)_n = \frac{1}{\prod_{i=1}^{n}\Gamma(\alpha_i k_i+1)}\left[\left(\prod_{i=1}^{n}(D_{\tilde{x}_i}^{\alpha_i})^{k_i}\right)u(x)_n\right]_{(\tilde{x})_n}$$
$$= \frac{\Gamma(\alpha_i(k_i+1)+1)}{\Gamma(\alpha_i(k_i+1)+1)\prod_{i=1}^{n}\Gamma(\alpha_i k_i+1)} \cdot \left[\left(\prod_{i=1}^{n}(D_{\tilde{x}_i}^{\alpha_i})^{k_i}\right)D_{\tilde{x}_i}^{\alpha_i}v(x)_n\right]_{(\tilde{x})_n}$$
$$= \frac{\Gamma(\alpha_i(k_i+1)+1)}{\Gamma(\alpha_i k_i+1)} \cdot V_{(\alpha)_n}(k_1,k_2,\ldots,k_{i-1},k_i+1,k_{i+1},\ldots k_n).$$

Theorem 4.5. If $u(x)_n = D_{\tilde{x}_1}^{\alpha_1}D_{\tilde{x}_2}^{\alpha_2}\ldots D_{\tilde{x}_n}^{\alpha_n}v(x)_n$, then

$$U_{(\alpha)_n}(k)_n = \frac{\prod_{j=1}^{n}\Gamma(\alpha_j(k_j+1)+1)}{\prod_{j=1}^{n}\Gamma(\alpha_j k_j+1)} \cdot V_{(\alpha)_n}(k_1+1,k_2+1,\ldots,k_n+1).$$

Proof. From (10) we have

$$U_{(\alpha)_n}(k)_n = \frac{1}{\prod_{i=1}^{n}\Gamma(\alpha_i k_i+1)}\left[\left(\prod_{i=1}^{n}(D_{\tilde{x}_i}^{\alpha_i})^{k_i}\right)u(x)_n\right]_{(\tilde{x})_n}$$
$$= \frac{\prod_{j=1}^{n}\Gamma(\alpha_j(k_j+1)+1)}{\prod_{j=1}^{n}\Gamma(\alpha_j(k_j+1)+1)\prod_{i=1}^{n}\Gamma(\alpha_i k_i+1)} \cdot \left[\left(\prod_{i=1}^{n}(D_{\tilde{x}_i}^{\alpha_i})^{k_i}\right)D_{\tilde{x}_1}^{\alpha_1}D_{\tilde{x}_2}^{\alpha_2}\ldots D_{\tilde{x}_n}^{\alpha_n}v(x)_n\right]_{(\tilde{x})_n}$$
$$= \frac{\prod_{j=1}^{n}\Gamma(\alpha_j(k_j+1)+1)}{\prod_{j=1}^{n}\Gamma(\alpha_j k_j+1)} \cdot V_{(\alpha)_n}(k_1+1,k_2+1,\ldots,k_n+1).$$

Theorem 4.6. If $u(x)_n = \prod_{i=1}^{n}(x_i-\tilde{x}_i)^{m_i\alpha_i}$, then $U_{(\alpha)_n}(k)_n = \prod_{i=1}^{n}\delta(k_i-m_i)$.

Proof. From (11) we have

$$u(x)_n = \prod_{i=1}^{n}(x_i-\tilde{x}_i)^{m_i\alpha_i},$$
$$= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\cdots\sum_{k_n=0}^{\infty}\left(\prod_{i=1}^{n}(\delta(k_i-m_i)(x_i-\tilde{x}_i)^{k_i\alpha_i})\right).$$

So, applying the inverse differential transform (10), we get $U_{(\alpha)_n}(k)_n = \prod_{i=1}^{n}\delta(k_i-m_i)$.

Theorem 4.7. Let $u(x)_n = \prod_{i=1}^{n} f_i(x_i), f_i(x_i) = x_i^{\lambda}h_i(x_i)$, $\lambda>-1, h_i(x_i)$ has the generalized Taylor series expansion $h_i(x_i) = \sum_{n=0}^{\infty}a_n(x_i-\tilde{x}_i)^{n\alpha_i}$, and either of the two conditions hold:

(a) $\beta<\lambda+1$ and γ is arbitrary or
(b) $\beta\ge\lambda+1$, γ is arbitrary, and $a_n=0$ for $n=0,1,\ldots,m-1$, where $m-1<\beta\le m$.

Then the generalized n-dimensional differential transform (10) becomes

$$U_{(\alpha)_n}(k)_n = \frac{1}{\prod_{j=1}^{n}\Gamma(\alpha_j k_j+1)} \cdot \left[\left(\prod_{\substack{j=1\\ j\neq i}}^{n}(D_{\tilde{x}_j}^{\alpha_j})^{k_j}\right)(D_{\tilde{x}_i}^{\alpha_i k_i})u(x)_n\right]_{(\tilde{x})_n}.$$

Proof. The proof follows immediately from the fact that $D_{\tilde{x}_i}^{\gamma_1}D_{\tilde{x}_i}^{\gamma_2}f_i(x_i) = D_{\tilde{x}_i}^{\gamma_1+\gamma_2}f_i(x_i)$ under the conditions given in Theorem 3.1.

In Theorems 4.8–4.10, the functions $f_i(x_i)$ satisfy the conditions given in Theorem 3.1.

Theorem 4.8. Let $u(x)_n = D^{\gamma}_{\tilde{x}_i} v(x)_n, m-1<\gamma\le m, v(x)_n=\prod_{i=1}^{n} f_i(x_i)$, then

$$U_{(\alpha)_n}(k)_n = \frac{\Gamma(\alpha_i k_i+\gamma+1)}{\Gamma(\alpha_i k_i+1)} \cdot V_{(\alpha)_n}(k_1,k_2,\ldots,k_{i-1},k_i+\gamma/\alpha_i,k_{i+1},\ldots k_n).$$

Proof. From (10) we have

$$U_{(\alpha)_n}(k)_n = \frac{1}{\prod_{i=1}^{n}\Gamma(\alpha_i k_i+1)}\left[\left(\prod_{i=1}^{n}(D^{\alpha_i}_{\tilde{x}_i})^{k_i}\right)u(x)_n\right]_{(\tilde{x})_n}$$
$$= \frac{\Gamma(\alpha_i k_i+\gamma+1)}{\Gamma(\alpha_i k_i+\gamma+1)\prod_{i=1}^{n}\Gamma(\alpha_i k_i+1)} \cdot \left[\left(\prod_{i=1}^{n}(D^{\alpha_i}_{\tilde{x}_i})^{k_i}\right)D^{\gamma}_{\tilde{x}_i}v(x)_n\right]_{(\tilde{x})_n}$$
$$= \frac{\Gamma(\alpha_i k_i+\gamma+1)}{\Gamma(\alpha_i k_i+1)} \cdot V_{(\alpha)_n}(k_1,k_2,\ldots,k_{i-1},k_i+\gamma/\alpha_i,k_{i+1},\ldots k_n).$$

Theorem 4.9. If $u(x)_n=\prod_{i=1}^{n} f_i(x_i)$, then $U_{(\alpha)_n}(k)_n = \frac{1}{\prod_{i=1}^{n}\Gamma(\alpha_i k_i+1)}\left[\left(\prod_{i=1}^{n} D^{\alpha_i k_i}_{\tilde{x}_i}\right)u(x)_n\right]_{(\tilde{x})_n}$.

Theorem 4.10. Let $u(x)_n = D^{\gamma_1}_{\tilde{x}_1}D^{\gamma_2}_{\tilde{x}_2}\ldots D^{\gamma_n}_{\tilde{x}_n}v(x)_n, m_i-1<\gamma_i\le m_i, v(x)_n=\prod_{i=1}^{n} f_i(x_i)$, then

$$U_{(\alpha)_n}(k)_n = \frac{\prod_{i=1}^{n}\Gamma(\alpha_i k_i+\gamma_i+1)}{\prod_{i=1}^{n}\Gamma(\alpha_i k_i+1)} \cdot \left[V_{(\alpha)_n}(k_1+\gamma_1/\alpha_1,k_2+\gamma_2/\alpha_2,\ldots,k_n+\gamma_n/\alpha_n)\right].$$

Proof. From (10) we have

$$U_{(\alpha)_n}(k)_n = \frac{1}{\prod_{i=1}^{n}\Gamma(\alpha_i k_i+1)}\left[\left(\prod_{i=1}^{n}(D^{\alpha_i}_{\tilde{x}_i})^{k_i}\right)u(x)_n\right]_{(\tilde{x})_n}$$
$$= \frac{\prod_{i=1}^{n}\Gamma(\alpha_i k_i+\gamma_i+1)}{\prod_{i=1}^{n}\Gamma(\alpha_i k_i+\gamma_i+1)\prod_{i=1}^{n}\Gamma(\alpha_i k_i+1)} \cdot \left[\left(\prod_{i=1}^{n}(D^{\alpha_i}_{\tilde{x}_i})^{k_i}\right)D^{\gamma_1}_{\tilde{x}_1}D^{\gamma_2}_{\tilde{x}_2}\ldots D^{\gamma_n}_{\tilde{x}_n}v(x)_n\right]_{(\tilde{x})_n}$$
$$= \frac{\prod_{i=1}^{n}\Gamma(\alpha_i k_i+\gamma_i+1)}{\prod_{i=1}^{n}\Gamma(\alpha_i k_i+1)}[V_{(\alpha)_n}(k_1+\gamma_1/\alpha_1,k_2+\gamma_2/\alpha_2,\ldots,k_n+\gamma_n/\alpha_n)].$$

5. Numerical Examples

Let $x=(x_1,x_2,\ldots,x_n)\in\mathbb{R}^{n-1}$, $\tilde{\alpha}=(\alpha_1,\alpha_2,\ldots,\alpha_{n-1})\in(0,1]^{n-1}$ and $k=(k_1,k_2,\ldots,k_n)\in\mathbb{N}_0^{n-1}$, where $\mathbb{N}_0^{n-1}=\mathbb{N}\cup\{0\}$. We use the following standard notations: $x_1^{\alpha_1 k_1},x_2^{\alpha_2 k_2},\ldots,x_{n-1}^{\alpha_{n-1}k_{n-1}}=x^{\tilde{\alpha}k}$. In the following examples $u(x,t)$ denotes the exact solution of the problem under consideration and is given as

$$u(x,t)=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\ldots\sum_{k_{n-1}=0}^{\infty}\sum_{h=0}^{\infty}U_{(\alpha)_{n-1},\beta}(k,h)x^{\tilde{\alpha}k}t^{h\beta}. \tag{21}$$

Further, we define the error by

$$E_m^{\alpha}=|u(x,t)-\tilde{u}_m(x,t)|\,, \tag{22}$$

where $\tilde{u}_m(x,t)$ is the approximate solution containing m terms obtained by truncating the solution series (21).

Example 5.1. Consider the following n-dimensional heat-like equation:

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}}=\gamma\sum_{i=1}^{n-1}\frac{\partial^2 u}{\partial x_i^2},\quad 0<x_i<c_i, \quad i=1,2,\ldots,(n-1),\ 0<\alpha\le 1,\ t>0, \tag{23}$$

subject to the initial condition

$$u(x,0)=\prod_{i=1}^{n-1}\sin x_i\,, \tag{24}$$

having $u(x,t)=\mathrm{e}^{-(n-1)\gamma t}\prod_{i=1}^{n-1}\sin x_i$ as the exact solution for $\alpha=1$.

Taking $\alpha_1=1,\alpha_2=1,\ldots,\alpha_{n-1}=1,\beta=\alpha$, and applying the generalized n-dimensional transform to both sides of (23) and (24), we get

$$\frac{\Gamma(\alpha(h+1)+1)}{\Gamma(\alpha h+1)}U_{1,1,\ldots,1,\alpha}(k,h+1)=$$
$$\gamma[(k_1+1)(k_1+2)U_{1,1,\ldots,1,\alpha}(k_1+2,k_2,\ldots,k_{n-1},h)$$
$$+(k_2+1)(k_2+2)U_{1,1,\ldots,1,\alpha}(k_1,k_2+2,k_3,\ldots,k_{n-1},h)$$

$$+\ldots+(k_{n-1}+1)(k_{n-1}+2)U_{1,1,\ldots,1,\alpha}\cdot(k_1,k_2,\ldots,k_{n-2},k_{n-1}+2,h)] \quad (25)$$

and

$$U_{1,1,\ldots,1,\alpha}(k,0)=\frac{(-1)^{(k_1+k_2+\ldots+k_{n-1}-(n-1))/2}}{k_1!k_2!\ldots k_n!}, \quad (26)$$

respectively.

Substituting $h=0,1,2,3,\ldots$ in the recurrence relation (25) and using (26), we obtain the different components of $U_{1,1,\ldots,1,\alpha}(k,h)$ as follows:

$$U_{1,1,\ldots,1,\alpha}(k,h)=\frac{(-(n-1)\gamma)^h}{\Gamma(\alpha h+1)}\cdot\frac{(-1)^{(k_1+k_2+\ldots+k_{n-1}-(n-1))/2}}{k_1!k_2!\ldots k_n!}.$$

The solution $u(x,t)$ of (23) is given as

$$u(x,t)=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\ldots\sum_{k_{n-1}=0}^{\infty}\sum_{h=0}^{\infty}U_{1,1,\ldots,1,\alpha}(k,h)x^k t^{h\alpha} = E_\alpha(-(n-1)\gamma t^\alpha)\prod_{i=1}^{n-1}\sin x_i, \quad (27)$$

where $E_\alpha(z)$ is the Mittag–Leffler function defined by $E_\alpha(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)}, \alpha>0, z\in C$.

For $\alpha=1$, the solution (27) of the fractional-order partial differential equation (PDE) reduces to the exact solution of the integer-order PDE

$$u(x,t)=\prod_{i=1}^{n-1}\sin x_i\sum_{h=0}^{\infty}\frac{(-(n-1)\gamma t)^h}{h!} = \mathrm{e}^{-(n-1)\gamma t}\prod_{i=1}^{n-1}\sin x_i.$$

Taking $n=3, c_1=c_2=2\pi$, and $\gamma=1$ in (23) and (27), we obtain the analytical solution of $\frac{\partial^\alpha u}{\partial t^\alpha}=\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}$, $0<x_1,x_2<2\pi$, $0<\alpha\le 1$, $t>0$, as $u(x,t)=\sin x_1\sin x_2 E_\alpha(-2t^\alpha)$, [14, 19].

Similarly, Example 1 in [27] follows as a special case of our general solution (27) by substituting $n=4$ and $\alpha=1$.

Example 5.2. Next, we apply our algorithm to the following n-dimensional heat-like equation:

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\prod_{i=1}^{n-1}x_i^4+\frac{1}{12(n-1)}\sum_{i=1}^{n-1}\left[x_i^2\frac{\partial^2 u}{\partial x_i^2}\right], \quad (28)$$

$0<x_i<1,\ i=1,2,\ldots,(n-1),\ 0<\alpha\le 1,\ t>0,$ subject to the initial condition

$$u(x,0)=0, \quad (29)$$

having $u(x,t)=\prod_{i=1}^{n-1}x_i^4(\mathrm{e}^t-1)$ as the exact solution for $\alpha=1$.

Taking $\alpha_1=1,\alpha_2=1,\ldots,\alpha_{n-1}=1,\beta=\alpha$, and applying the generalized n-dimensional transform to both sides of (28) and (29), we obtain

$$\begin{aligned}
&\frac{\Gamma(\alpha(h+1)+1)}{\Gamma(\alpha h+1)}U_{1,1,\ldots,1,\alpha}(k,h+1)=\\
&\prod_{i=1}^{n-1}\delta(k_i-4)\delta(h)\\
&+\frac{1}{12(n-1)}\sum_{a_1=0}^{k_1}\sum_{a_2=0}^{k_2}\ldots\sum_{a_{n-1}=0}^{k_{n-1}}\sum_{b=0}^{h}\Bigg[\Bigg\{(k_1-a_1+1)\\
&\cdot(k_1-a_1+2)\delta(a_1-2)\prod_{j=2}^{n-1}\delta(k_j-a_j)\delta(h-b)\\
&\cdot U_{1,1,\ldots,1,\alpha}(k_1-a_1+2,a_2,a_3,\ldots,a_{n-1},b)\Bigg\}\\
&+\Bigg\{(a_2+1)(a_2+2)\delta(a_1)\delta(k_2-a_2-2)\\
&\cdot\prod_{j=3}^{n-1}\delta(k_j-a_j)\delta(h-b)U_{1,1,\ldots,1,\alpha}\\
&\cdot(k_1-a_1,a_2+2,a_3,\ldots,a_{n-1},b)\Bigg\}\\
&+\Bigg\{(a_3+1)(a_3+2)\delta(a_1)\delta(k_3-a_3-2)\\
&\cdot\prod_{\substack{j=2\\ j\neq 3}}^{n-1}\delta(k_j-a_j)\delta(h-b)U_{1,1,\ldots,1,\alpha}\\
&\cdot(k_1-a_1,a_2,a_3+2,a_4,\ldots,a_{n-1},b)\Bigg\}+\ldots\\
&+\Bigg\{(a_{n-1}+1)(a_{n-1}+2)\delta(a_1)\delta(k_{n-1}-a_{n-1}-2)\\
&\cdot\prod_{j=2}^{n-2}\delta(k_j-a_j)\delta(h-b)U_{1,1,\ldots,1,\alpha}\\
&\cdot(k_1-a_1,a_2,a_3,\ldots,a_{n-2},a_{n-1}+2,b)\Bigg\}\Bigg]
\end{aligned} \quad (30)$$

and $U_{1,1,\ldots,1,\alpha}(k,0)=0$, respectively. (31)

Substituting $h=0,1,2,3,\ldots$ in the recurrence relation (30) and using (31), the different components of

$U_{1,1,\dots,1,\alpha}(k,h)$ are obtained as

$$U_{1,1,\dots,1,\alpha}(k,h) = \begin{cases} \dfrac{1}{\Gamma(\alpha h+1)}, & k_1 = k_2 = \dots \\ & \quad = k_{n-1} = 4 \\ 0, & \text{otherwise.} \end{cases}$$

Thus the solution $u(x,t)$ of (28) is given by

$$\begin{aligned} u(x,t) &= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\dots\sum_{k_{n-1}=0}^{\infty}\sum_{h=0}^{\infty} U_{1,1,\dots,1,\alpha}(k,h)x^k t^{h\alpha} \\ &= (E_\alpha(t^\alpha)-1)\prod_{i=1}^{n-1} x_i^4. \end{aligned} \tag{32}$$

For $\alpha = 1$, the solution (32) reduces to

$$u(x,t) = x^4\left(\sum_{h=0}^{\infty}\frac{1}{\Gamma(h+1)}t^h - 1\right) = (\mathrm{e}^t-1)\prod_{i=1}^{n-1}x_i^4,$$

which is the solution of the integer-order PDE.

The differential equation (28) and its solution (32) become

$$\frac{\partial^\alpha u}{\partial t^\alpha} = x_1^4x_2^4x_3^4 + \frac{1}{36}\left[x_1^2\frac{\partial^2 u}{\partial x_1^2} + x_2^2\frac{\partial^2 u}{\partial x_2^2} + x_3^2\frac{\partial^2 u}{\partial x_3^2}\right],$$
$$0 < x_1,x_2,x_3 < 1,\ \ 0<\alpha\le 1,\ \ t>0,$$

and $u(x_1,x_2,x_3,t) = x_1^4x_2^4x_3^4(E_\alpha(t^\alpha)-1)$, respectively, for $n=4$, which is the same as the solution obtained by other methods [14, 19].

Example 5.3. Now, we consider the following n-dimensional wave-like equation with initial conditions:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \sum_{i=1}^{n-1}x_i^2 + \frac{1}{2}\sum_{i=1}^{n-1}\left[x_i^2\frac{\partial^2 u}{\partial x_i^2}\right], \tag{33}$$
$$0<x_i<1,\ \ i=1,2,\dots,(n-1),\ \ 1<\alpha\le 2,\ \ t>0,$$

$$u(x,0) = 0,\ \ u_t(x,0) = \sum_{i=1}^{n-2}x_i^2 - x_{n-1}^2, \tag{34}$$

having $u(x,t) = \left(\sum_{i=1}^{n-2}x_i^2\right)(\mathrm{e}^t-1) + x_{n-1}^2(\mathrm{e}^{-t}-1)$ as the exact solution for $\alpha = 2$.

We solve (33) for various values of α.

(a) $\alpha = 2$

Taking $\alpha_i = 1, 1\le i\le n-1$, $\beta = 1$, applying the generalized n-dimensional transform to both sides of (33)–(34), and using theorem (17), we get

$$\begin{aligned} &\frac{\Gamma(h+3)}{\Gamma(h+1)}U_{1,1,\dots,1,1}(k,h+2) = \\ &\sum_{i=1}^{n-1}\left[\delta(k_i-2)\prod_{\substack{j=1\\ j\neq i}}^{n-1}\delta(k_j)\delta(h)\right] \\ &+\frac{1}{2}\sum_{a_1=0}^{k_1}\sum_{a_2=0}^{k_2}\dots\sum_{a_{n-1}=0}^{k_{n-1}}\sum_{b=0}^{h}\Bigg[\Bigg\{(k_1-a_1+1) \\ &\cdot(k_1-a_1+2)\delta(a_1-2)\prod_{j=2}^{n-1}\delta(k_j-a_j)\delta(h-b) \\ &\cdot U_{1,1,\dots,1,1}(k_1-a_1+2,a_2,a_3,\dots,a_{n-1},b)\Bigg\} \\ &+\Bigg\{(a_2+1)(a_2+2)\delta(a_1)\delta(k_2-a_2-2) \\ &\cdot\prod_{j=3}^{n-1}\delta(k_j-a_j)\delta(h-b)U_{1,1,\dots,1,1} \\ &\cdot(k_1-a_1,a_2+2,a_3,\dots,a_{n-1},b)\Bigg\} \\ &+\Bigg\{(a_3+1)(a_3+2)\delta(a_1)\delta(k_3-a_3-2) \\ &\cdot\prod_{\substack{j=2\\ j\neq 3}}^{n-1}\delta(k_j-a_j)\delta(h-b)U_{1,1,\dots,1,1} \\ &\cdot(k_1-a_1,a_2,a_3+2,a_4,\dots,a_{n-1},b)\Bigg\}+\dots \\ &+\Bigg\{(a_{n-1}+1)(a_{n-1}+2)\delta(a_1)\delta(k_{n-1}-a_{n-1}-2) \\ &\cdot\prod_{j=2}^{n-2}\delta(k_j-a_j)\delta(h-b)U_{1,1,\dots,1,1} \\ &\cdot(k_1-a_1,a_2,a_3,\dots,a_{n-2},a_{n-1}+2,b)\Bigg\}\Bigg], \end{aligned} \tag{35}$$

$$\begin{aligned} &U_{1,1,\dots,1,1}(k,0) = 0, \\ &U_{1,1,\dots,1,1}(k,1) = V_1(k,1)+V_2(k,1)+\dots+V_{n-1}(k,1), \end{aligned} \tag{36}$$

where

$$V_j(k,1) = \begin{cases} 1, & k_j=2,\ k_i=0,\ i\neq j \\ 0, & \text{otherwise,} \end{cases} \quad \begin{matrix} 1\le i\le n-1, \\ 1\le j\le n-2, \end{matrix}$$

$$V_{n-1}(k,1) = \begin{cases} -1, & k_1=k_2=\dots=k_{n-2}=0, k_{n-1}=2, \\ 0, & \text{otherwise.} \end{cases}$$

Substituting $h=0,1,2,3,\ldots$ in the recurrence relation (35) and using (36), we get different components of $U_{1,1,\ldots,1,1}(k,h)$ as follows:

$$U_{1,1,\ldots,1,1}(k,h)=V_1(k,h)+V_2(k,h)+\ldots+V_{n-1}(k,h),$$

where

$$V_j(k,h)=\begin{cases}\dfrac{1}{h!}, & k_j=2,\ k_i=0,\ i\neq j \quad 1\le i\le n-1,\\ & 1\le j\le n-2,\\ 0, & \text{otherwise},\end{cases}$$

$$V_{n-1}(k,h)=\begin{cases}\dfrac{(-1)^h}{h!}, & k_1=k_2=\ldots=k_{n-2}=0,\\ & k_{n-1}=2,\\ 0, & \text{otherwise}.\end{cases}$$

Hence the solution of (33) is given by

$$u(x,t)=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\cdots\sum_{k_{n-1}=0}^{\infty}\sum_{h=0}^{\infty}U_{1,1,\ldots,1,1}(k,h) \cdot x^k t^h=\left(\sum_{i=1}^{n-2}x_i^2\right)(\mathrm{e}^t-1)+x_{n-1}^2(\mathrm{e}^{-t}-1), \tag{37}$$

which is the exact solution.

(b) $\alpha=1.5$

Taking $\alpha_i=1, 1\le i\le n-1, \beta=0.5$, and following the same procedure as in case (a), the different components of $U_{1,1,\ldots,1,0.5}(k,h)$ can be computed as follows:

For $h=3n+1, n\in\mathbb{N}\cup\{0\}$, we have $U_{1,1,\ldots,1,0.5}(k,h)=0$.

For $h=3n+2, n\in\mathbb{N}\cup\{0\}, U_{1,1,\ldots,1,0.5}(k,h)=V_1(k,h)+V_2(k,h)+\ldots+V_{n-1}(k,h)$, where

$$V_j(k,h)=\begin{cases}\dfrac{1}{\Gamma(\frac{h}{2}+1)}, & k_j=2,\ k_i=0,\ i\neq j\\ & 1\le i\le n-1,\\ & 1\le j\le n-2,\\ 0, & \text{otherwise},\end{cases}$$

$$V_{n-1}(k,h)=\begin{cases}-\dfrac{1}{\Gamma(\frac{h}{2}+1)}, & k_1=k_2=\ldots=k_{n-2}=0,\\ & k_{n-1}=2,\\ 0, & \text{otherwise}.\end{cases}$$

For $h=3n+3, n\in\mathbb{N}\cup\{0\}$, we have $U_{1,1,\ldots,1,0.5}(k,h)=V_1(k,h)+V_2(k,h)+\ldots+V_{n-1}(k,h)$, where

$$V_j(k,h)=\begin{cases}\dfrac{1}{\Gamma(\frac{h}{2}+1)}, & k_j=2,\ k_i=0,\ i\neq j,\\ & 1\le i\le n-1,\\ & 1\le j\le n-2,\\ 0, & \text{otherwise}.\end{cases}$$

Hence the solution of (33) is given by

$$\begin{aligned}u(x,t)&=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\cdots\sum_{k_{n-1}=0}^{\infty}\sum_{h=0}^{\infty}U_{1,1,\ldots,1,0.5}(k,h)x^k t^{0.5h}\\ &=(x_1^2+x_2^2+\ldots+x_{n-1}^2)\left(\sum_{h=0}^{\infty}\frac{1}{\Gamma(\frac{3h}{2}+\frac{5}{2})}t^{(3h/2)+3/2}\right)\\ &+(x_1^2+x_2^2+\ldots+x_{n-2}^2-x_{n-1}^2)\\ &\cdot\left(\sum_{h=0}^{\infty}\frac{1}{\Gamma(\frac{3h}{2}+2)}t^{(3h/2)+1}\right)\\ &=(x_1^2+x_2^2+\ldots+x_{n-1}^2)t^{3/2}E_{3/2,5/2}(t^{3/2})\\ &+(x_1^2+x_2^2+\ldots+x_{n-2}^2-x_{n-1}^2)\,t\,E_{3/2,2}(t^{3/2}),\end{aligned} \tag{38}$$

where $E_{\alpha,\beta}(z)$ is the two parameter Mittag–Leffler function defined by $E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)}$.

(c) $\alpha=1.5$

Taking $\alpha_i=1, 1\le i\le n-1, \beta=0.25$, and following the same procedure as in case (a), the different components of $U_{1,1,\ldots,1,0.25}(k,h)$ can be computed as follows:

$$U_{1,1,\ldots,1,0.25}(k,0)=0.$$

For $h=5n+1, 5n+2, 5n+3, n\in\mathbb{N}\cup\{0\}$, we have $U_{1,1,\ldots,1,0.25}(k,h)=0$.

For $h=5n+4,\ n\in\mathbb{N}\cup\{0\}$, we have $U_{1,1,\ldots,1,0.25}(k,h)=V_1(k,h)+V_2(k,h)+\ldots+V_{n-1}(k,h)$, where

$$V_j(k,h)=\begin{cases}\dfrac{1}{\Gamma(\frac{h}{2}+1)}, & k_j=2,\ k_i=0,\ i\neq j\\ & 1\le i\le n-1,\\ & 1\le j\le n-2,\\ 0, & \text{otherwise},\end{cases}$$

$$V_{n-1}(k,h)=\begin{cases}-\dfrac{1}{\Gamma(\frac{h}{2}+1)}, & k_1=k_2=\ldots=k_{n-2}=0,\\ & k_{n-1}=2,\\ 0, & \text{otherwise}.\end{cases}$$

For $h=5n+5,\ n\in\mathbb{N}\cup\{0\}$, we have $U_{1,1,\ldots,1,0.25}(k,h)=V_1(k,h)+V_2(k,h)+\ldots+V_{n-1}(k,h)$, where

$$V_j(k,h)=\begin{cases}\dfrac{1}{\Gamma(\frac{h}{2}+1)}, & k_j=2,\ k_i=0,\ i\neq j,\\ & 1\le i\le n-1,\\ & 1\le j\le n-2,\\ 0, & \text{otherwise}.\end{cases}$$

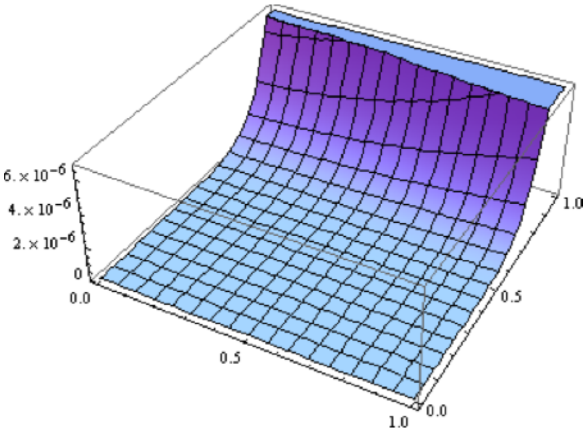

Fig. 1 (colour online). Error $E_{10}^{1.25}$ at $n=4$, $\alpha=1.25$.

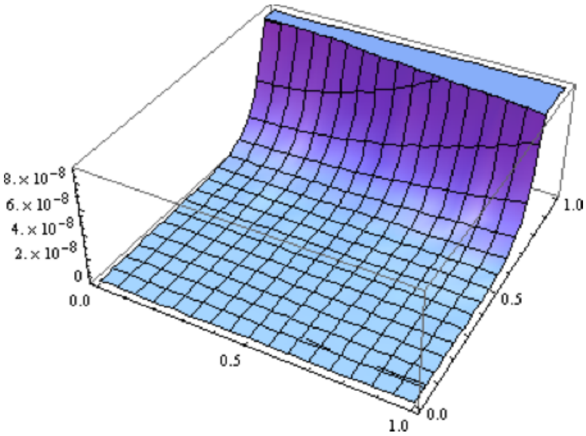

Fig. 2 (colour online). Error $E_{10}^{1.5}$ at $n=4$, $\alpha=1.5$.

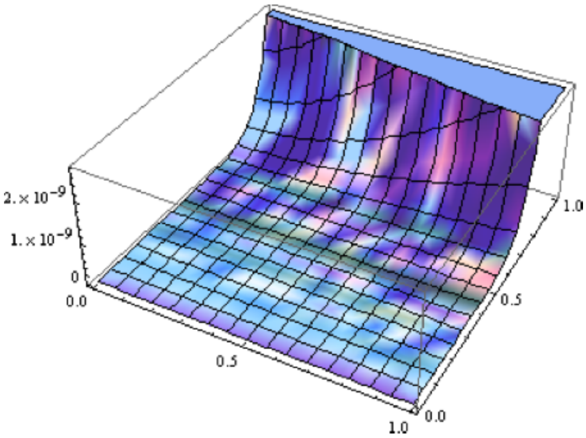

Fig. 3 (colour online). Error E_{10}^{2} at $n=4$, $\alpha=2$.

Hence the solution of (33) is given by

$$u(x,t)=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\cdots\sum_{k_{n-1}=0}^{\infty}\sum_{h=0}^{\infty}U_{1,1,\ldots,1,0.25}(k,h)x^k t^{0.25h}$$
$$=(x_1^2+x_2^2+\ldots+x_{n-1}^2)\left(\sum_{h=0}^{\infty}\frac{1}{\Gamma(\frac{5h}{4}+\frac{9}{4})}t^{(5h/4)+5/4}\right)$$
$$+(x_1^2+x_2^2+\ldots+x_{n-2}^2-x_{n-1}^2)$$
$$\cdot\left(\sum_{h=0}^{\infty}\frac{1}{\Gamma(\frac{5h}{4}+2)}t^{(5h/4)+1}\right)$$
$$=(x_1^2+x_2^2+\ldots+x_{n-1}^2)t^{5/4}E_{5/4,9/4}(t^{5/4})$$
$$+(x_1^2+x_2^2+\ldots+x_{n-2}^2-x_{n-1}^2)\,t\,E_{5/4,2}(t^{5/4}). \tag{39}$$

Taking $n=4$, the differential equation (33) reduces to

$$\frac{\partial^\alpha u}{\partial t^\alpha}=x_1^2+x_2^2+x_3^2+\frac{1}{2}\left[x_1^2\frac{\partial^2 u}{\partial x_1^2}+x_2^2\frac{\partial^2 u}{\partial x_2^2}+x_3^2\frac{\partial^2 u}{\partial x_3^2}\right],$$
$$0<x_1,x_2,x_3<1,\ \ 1<\alpha\le 2,\ \ t>0, \tag{40}$$

and its closed form solutions at various values of α becomes

$$u(x_1,x_2,x_3,t)=x_1^2(\mathrm{e}^t-1)+x_2^2(\mathrm{e}^t-1)+x_3^2(\mathrm{e}^{-t}-1),\quad(\alpha=2)$$

$$u(x_1,x_2,x_3,t)=(x_1^2+x_2^2+x_3^2)t^{3/2}E_{3/2,5/2}(t^{3/2})$$
$$+(x_1^2+x_2^2-x_3^2)\,t\,E_{3/2,2}(t^{3/2}),\qquad(\alpha=1.5)$$

$$u(x_1,x_2,x_3,t)=(x_1^2+x_2^2+x_3^2)t^{5/4}E_{5/4,9/4}(t^{5/4})$$
$$+(x_1^2+x_2^2-x_3^2)\,t\,E_{5/4,2}(t^{5/4}),\qquad(\alpha=1.25)$$

which are the same as the solutions obtained by other methods [14, 19].

Though the solution series in each example converges to the exact closed form analytic solution of the problem, we show that only few terms of the solution series (21) are required to give a quite accurate solution. Let E_m^α denote the absolute error between the exact solution and the first m contributing terms of the solution series (21) as defined in (22). Figures 1, 2, and 3, associated with Example 5.3, show that the errors are appreciably small for $m=10$ and $\alpha=1.25, 1.5$, and 2, respectively. The error is monotonically decreasing as $\alpha\to 2$.

6. Conclusion

We have extended the theory of one and two-dimensional generalized differential transform method to n-dimensions to propose a user friendly algorithm to obtain closed form analytic solutions for n-dimensional fractional heat- and wave-like equations. Though the solution series in each example converges to the exact closed form analytic solution of the problem, we show that only few terms of the solution series (21) are required to give quite accurate solution. In Example 5.3, we show that ten terms of the series representation of the solution, even in fractional order, gives a very accurate solution.

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