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A Mathematical Model of Consumers' Buying Behaviour Based On Multiresolution Analysis

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Abstract

In this paper we have presented a generalized mathematical model of consumers' buying behaviour. This model provides better insights and perceptions that can be used to take many important managerial decisions for any product to improve the buying behaviour of consumers towards that product. In this paper we have proved that consumers' buying behaviour is a $L^2(\mathbb{R})$ function. Such functions can take two values. 1 (if the buying behaviour is satisfied) or 0 (if the buying behaviour is not satisfied). Through multiresolution analysis (MRA), we have proved that all the factors affecting consumers' buying behaviour are the subspaces of $L^2(\mathbb{R})$. We have also proved that the satisfaction of consumers' buying behaviour is convex with respect to all the factors that affect it. We have given a relationship among all the factors influencing consumers' buying behaviour.We have provided a way by which the overall inclination of buying behaviour of any consumer or his inclination towards any particular product can be investigated.

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Keywords: Consumers' buying behaviour; $L^2(\mathbb{R})$ functions; Multiresolution analysis (MRA); Subspaces; Convexity property.

1. Introduction

Consumers' buying behaviour is one of the most important areas of study in managament science. Most of the researchers [1], [2], [3], [4], [5] have elucidated their works based on either theory or survey. Apart from some pertinent models for consumers' buying behaviour by different authors that exist, our efforts and motivation are to develop a more generalized mathematical model for consumers' buying behaviour.

From [6],[7], the factors that influence consumers buying behaviour are cultural, social, personal and psychological. Among them, cultural factors influence the buying behaviour most.

Cultural factors include subcultures which in turn include nationalities, religions, racial groups and geographic regions.

Social factors include family, group, social roles and status.

Personal factors include age of buyer, stages in the life cycle, occupation, economic circumstances, lifestyle, personality and self concept.

psychological factors include motivation, perception, attitudes, learning and beliefs.

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2. Litreature review

Denni arli et al. studied about the effect of religion on consumers' decision making [8]. This is the first cross cultural study which compares moral ideology and ethical beliefs of religious and nonreligious consumers . Authors found that religious consumers incline more towards idealism than that of relativism. They have stronger ethical beliefs comparative to negative consumer ethical behaviours in the case of non religious consumers. Findings of this study also suggest that effect of religion gets overshadowed by cultural differences in the cases of recycling and software piracy/buying counterfeit. In this study authors did not provide any concrete or proper quantitative way as how to overcome this situation. Xinxin chen et al. investigated about the effects of customers' personality characteristics on CEB (customer engagement behaviours) [9]. Authors did not say anything about the effect of rest of the three buying behaviour governing factors on CEB. Andrew duffy discussed about three different types of trust and found that when individuals get plenty of data on internet, their buying behaviour is affected by trust in self [10]. Without loss of generality, we can say that in this study author has focussed only on the personal factors. The author does not say anything about if there is any effect of such situation on rest of the three buying behaviour governing factors. David s. Ackerman et al. studied about the consumers' reactions towards used goods [11]. Authors found that consumers' reactions are generally negative towards used goods. In this study authors have not provided any concrete quantitative way to reduce the negative emotions of consumers' towards used goods. Danny tengti kao et al. studied about the effects of ad metaphor and goal orientation on consumers' buying behaviour in the case of brand commitment and competitor brands [12]. Authors did not provide any way to optimize the effects of findings of their study in order to make the ads regarding a particular product more promising.

Pascual fernándeza et al. studied about the discrete location problems for entering firms which face competition with other established firms in a market where customers are spatially separated [13]. The formulation and resolution of such problems are decided by customers' behaviour. The findings of this study might have been more effective if the problem taken at hand had been more generalized in terms of all the four buying behaviour governing factors.

From the literature review, it can be clearly seen that the findings of the above studies do not reflect the all around aspects of consumers' buying behaviour. In other words, most of the studies are centered about any one of the four buying behaviour governing factors and unable to predict or say if there are any effects of the study concerned on rest of the three buying behaviour governing factors. Therefore to get better insights and perceptions from the existing studies about any one of the four buying behaviour governing factors on the rest of the three buying behaviour governing factors, there is an obvious need of a new model of consumers' buying behaviour which can predict or say if there is any effect on rest of the three buying behaviour governing factors by the study concerned.

3. Methodology

For the development of a new model of consumers' buying behaviour which covers all the above mentioned shortcomings, we mainly need the following aspects.

Without loss of generality we assume that any consumer always has two choices for any product. Either he/she buys the product or he/she does not buy the product.

Let $f(x)$ be a function which gives the value of any consumer's buying behaviour for a particular product. Let x denotes the product that a consumer can buy and $x \in \mathbb{Z}^+$. For every x we get a value of $f(x)$.

Thus we get two possibilities of the function

Let there be two sets A and A'. The set A contains those items which a consumer buys and set A' contains those items which a consumer does not buy. Clearly set A' is the complement of set A

The function $f(x)$ can be shown as

(i) $f : \mathbb{Z}^+ \longrightarrow A$; when a consumer buys something.

(ii) $f : \mathbb{Z}^+ \longrightarrow A'$; When a consumer does not buy anything.

From our above mentioned assumption we say that, for any consumer, the function $f(x)$ has two values with respect to any particular product 1 (if the consumer buys the product) and 0 (if the consumer does not buy the product).

$f(x)$ can be written as

$$f(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \in A' \end{cases}$$

Since the function $f(x)$ has two values for any consumer and for any product thus we have to deal with the squared value of $f(x)$ i.e. $[f(x)]^2$.

Theorem 1. *The satisfaction of consumers' buying behaviour is convex to all the four spaces of psychological, personal,social and cultural factors.*

Proof of theorem 1

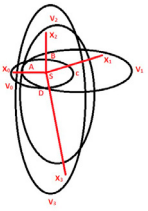

Figure 1: Space representation of buying behaviour governing factors

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Where V_0, V_1, V_2, V_3 are the spaces of psychological, personal, social and cultural factors respectively and S denotes the point of satisfaction of consumers' buying behaviour.
This theorem can be proved from figure 1.

Theorem 2. *The consumers' buying behaviour function* $f(x) \in L^2(\mathbb{R})$

Proof of theorem 2

The proof of this theorem is obvious.

Theorem 3. *Consumers' buying behaviour forms multiresolution analysis.*

Proof of theorem 3

The proof of this theorem is quite rigorous and based on the methodology given in [14].
From [15], the set of orthonormal wave packets(which also form wavelets) are given as

$W^{\pm}(t) = \sqrt{2^{-k}}\Psi^{\pm}(2^{-k}t - \frac{2\pi}{\varepsilon_0}l)$ where $l = 0, \pm1, \pm2, \pm3, ...$

$W_{00}^{\pm}$ is the mother wavelet, defined by

$$W_{00}^{\pm} \equiv \Psi^{\pm}(t) = \sqrt{\frac{\varepsilon_0}{2\pi}} e^{\pm 3it\varepsilon_0/2} \frac{sint\varepsilon_0/2}{t\varepsilon_0/2}$$

±signs are for positive and negative frequency parts respectively.

It is to be noted that the resulting set of orthonormal wavelets get decomposed into different classes. For fixed k, the wavelets belong to the same class. These wavelets are time translated relative to each other ($l = 0, \pm1, \pm2, \pm3, ...$).

In our case, l is the value of brands of any certain type of product. Thus we confine ourselves to only positive values of l. Without loss of generality we assume $l \neq 0$. We take a small positive value for ε_0 ($\varepsilon_0 = 0.2$). Though the choice of ε_0 is completely arbitrary in every case, we keep the value of ε_0 constant throughout our study to have the similar shapes of orthonormal wavelet packets for every product. We take a fixed value of k for a certain type of product, that will remain constant for different brands of the same product. The values of k will change only for different types of products. We also take some arbitrary values of t for every case. Now we can plot graphs between W^+ and t for different brands of the same product keeping k constant. We can also plot the graphs between W^+ and t for different brands of different products or for different products by varying the value of k. Since in our case, the negative frequency parts are insignificant, we restrict ourselves to only positive frequency parts. As an example we are giving here the plots of two different products (say product A and product B) (Fig 2). We also assume that both the products have four different brands brand 1, brand 2, brand 3 and brand 4 i .e. $l = 1, 2, 3, 4$ in both the cases.

We see from the plots that the peak values of all the four brands of product A are same and the peak values of all the four brands of product B are also same. These peak values always allow us to study the overall consumers' buying behaviour between two different products or among four different brands of two different products. These peak values always indicate about the inclination of consumers towards a particular product also. The same procedure can be repeated for a number of times, for a number of products and/or for their different brands which will finally lead to get an idea about the inclination of overall consumers' buying behaviour and/or to get an idea about the inclination of consumers' buying behaviour towards a particular product.

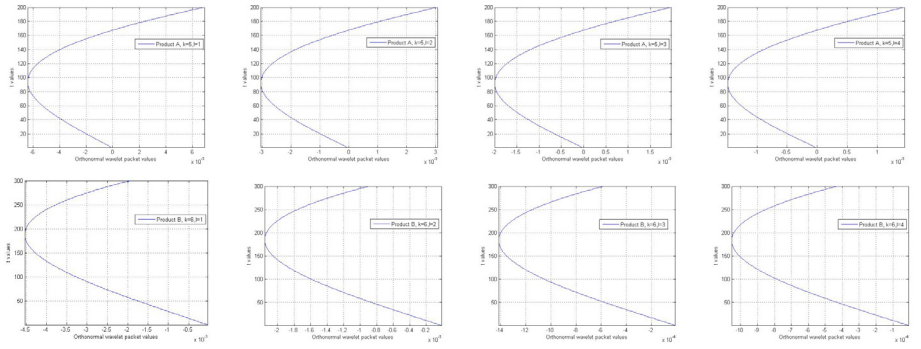

Figure 2: Orthonormal wave packet representation of four different brands of two different products

From [$Theorem1$], satisfaction of consumers' buying behaviour is convex [16] to the spaces of psychological, personal, social and cultural factors. This property of convexity can be used to optimize the sales of any product by drawing two straight line between the two least affecting buying behaviour governing factors and two most affecting buying behaviour governing factors for that product.These two straight lines along with the X axis will form a triangular region. The area of this triangular region can be calculated. This experiment can be repeated a large number of times and a new value of area can be calculated every time. Assuming there are n sets of straight lines and n is sufficiently large, the expectation values of area can be calculated as $E(x) = \sum_{i=1}^{n} A_i p_i$, where A_i is the area obtained due to i^{th} set of lines. We take each $p_i = \frac{1}{n}$. Let N be the total number of the product sold. Now the difference $D = E(x) - N$ can be calculated. If $D \geq 0$, the change of policy for the increase in sales of product at hand may not be required. If $D < 0$, the change of policy for the increase in sales of product at hand may be required.

From [Theorem 2 & [17],Example 3.3.6 p 100], we know that the space $L^2(\mathbb{R})$ is Hilbert space. We consider the inner product is Euclidean [18]. For any two such products where buying one product facilitates the consumer to buy another product, let x_1, x_2, x_3, x_4 are respectively the cultural, social, personal and psychological factors for the first product and y_1, y_2, y_3, y_4 are respectively the cultural, social, personal and psychological factors for the second product.
The Euclidean inner product for the two products can be defined as

$$\langle x, y\rangle = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4.$$

If any of the factors of x_i or y_i is zero i.e. not satisfied, it can be easily inferrred that the product will not be purchsed as a whole. Where the product as a whole means the first product along with the second product. Further, since cultural factors influence the buying behaviour most, if any one of the x_1 or y_1 is zero, the inner product will be zero and the product as a whole will not be purchased. In other words we can say that if the Euclidean inner product for any two products where buying one product facilitates the consumer to buy another product is zero, the buying behaviour for two such products will be orthogonal to each other.
From[$Theorem3$], we know that consumers' buying behaviour forms MRA. Since cultural factors exert deepest and broadest influence on consumers' buying behaviour therefore, without loss of generality, in terms of percentage, we can assume that the cultural factors affect the buying behaviour at least by 51%. In terms of probability, we can say that the cultural factors affect the buying behaviour at least with the probability 0.51. From the property (iv) of multiresolution analysis [14], we can now have the probabilites of rest of the three buying behaviour governing factors along with the total probability by which the consumers' buying behaviour is affected as

Table 1: Probabilities of Buying Behaviour Governing Factors

p_{cul}	p_{soc}	p_{per}	p_{psy}	p_{total}
0.51	0.255	0.1275	0.06375	0.95625

where p_{cul}, p_{soc}, p_{per}, p_{psy} and p_{total} represent probability of cultural factors, probability of social factors, probability of personal factors, probability of psychological factors and new total probability respectively. Due to MRA we have,

$$p_{soc} = \frac{p_{cul}}{2} \tag{1}$$

$$p_{per} = \frac{p_{soc}}{2} \tag{2}$$

$$p_{psy} = \frac{p_{per}}{2} \tag{3}$$

$$p_{total} = p_{cul} + p_{soc} + p_{per} + p_{psy} \tag{4}$$

Here we would like to mention clearly that we give equal importance to all the subfactors of any of the four buying behaviour governing factors. Which clearly means that all the subfactors of any of the four buying behaviour governing factors have equal probabilities with respect to that factor. Mathematically this can be expressed as

$$p_{sub} = \frac{p_x}{n}.$$

Where p_{sub} is the probability of any subfactor associated with any of the buying behaviour governing factors. p_x can be any one of the p_{cul}, p_{soc}, p_{per}, p_{psy} at a time. n is the total number of subfactors for the associated buying behaviour governing factor.

Fig 3 shows the plots between the four buying behaviour governing factors and their associated probabilities, its smoothened version and exponential distribution with mean $\frac{1}{0.51}$ where Cul,Soc,Per and Psy represent cultural

factors, social factors, personal factors and psychological factors respectively.

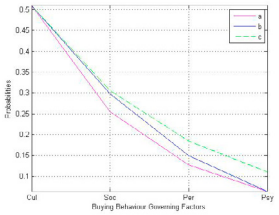

Figure 3: (a): Plot of Buying Behaviour Governing Factors With Respect to The Associated Probabilities (b): Smoothened Plot of Buying Behaviour Governing Factors With Respect to The Associated Probabilities (c): Plot of Exponential distribution With mean $\frac{1}{0.51}$.

It is evident from the plot that the consumers' buying behaviour follows nearly exponential distribution with mean $\frac{1}{0.51}$ [19]. It is observed that the mean squared error(MSE) in the case of the smoothened plot of buying behaviour governing factors with respect to the associated probabilities to that of exponential distribution with mean $\frac{1}{0.51}$ (i.e. MSE between the case (b) & case (c) of Fig. 3) is 0.00087292. It is also found that MSE in the case of the plot of buying behaviour governing factors with respect to the associated probabilities to that of exponential distribution with mean $\frac{1}{0.51}$ (i.e. MSE between the case (a) & case (c) of Fig. 3) is 0.002.
For any product, to take the total probability value of Table 1 closer to 1, the manufacturer of that product can consult the extensive body of work in economics and marketing on consumers' buying behaviour or can take important decisions on its own regarding the buying behaviour governing factors. Further, for better understanding of the advantages of our model, let us assume two different sets of cases. In the first set of cases we try to calculate the change of probabilities of the remaining three buying behaviour governing factors along with the change in total probability when any one of the four buying behaviour governing factors is incremented by 0.1%. In the second set of cases we try to calculate the change in probabilities of the remaining three buying behaviour governing factors along with the change in total probability when any one of the four buying behaviour factors is decremented by 0.1%. These two situations are depicted in table 2 and table 3 respectively.

Table 2: Increase in Buying Behaviour Governing Factors By 0.1 %

Increase by 0.1 %	$p_{cul_{new}}$	$p_{soc_{new}}$	$p_{per_{new}}$	$p_{psy_{new}}$	$p_{total_{new}}$
Cul (↑)	a_1	b_1	c_1	d_1	T_1
Soc (↑)	a_2	b_2	c_2	d_2	T_2
Per (↑)	a_3	b_3	c_3	d_3	T_3
Psy (↑)	a_4	b_4	c_4	d_4	T_4

where $a_1 = 0.511$, $b_1 = 0.2555$, $c_1 = 0.12775$, $d_1 = 0.063875$, $T_1 = 0.958125$.
$a_2 = 0.512$, $b_2 = 0.256$, $c_2 = 0.128$, $d_2 = 0.064$, $T_2 = 0.96$.
$a_3 = 0.514$, $b_3 = 0.257$, $c_3 = 0.1285$, $d_3 = 0.06425$, $T_3 = 0.96375$.
$a_4 = 0.518$, $b_4 = 0.259$, $c_4 = 0.1295$, $d_4 = 0.06475$, $T_4 = 0.97125$.

Table 3: Decrease in Buying Behaviour Governing factors by 0.1 %

Decrease by 0.1 %	$p_{cul_{new}}$	$p_{soc_{new}}$	$p_{per_{new}}$	$p_{psy_{new}}$	$p_{total_{new}}$
Cul(↓)	a_5	b_5	c_5	d_5	T_5
Soc (↓)	a_6	b_6	c_6	d_6	T_6
Per (↓)	a_7	b_7	c_7	d_7	T_7
Psy (↓)	a_8	b_8	c_8	d_8	T_8

where $a_5 = 0.509$, $b_5 = 0.2545$, $c_5 = 0.12725$, $d_5 = 0.063625$, $T_5 = 0.954375$.
$a_6 = 0.508$, $b_6 = 0.254$, $c_6 = 0.127$, $d_6 = 0.0635$, $T_6 = 0.9525$.
$a_7 = 0.506$, $b_7 = 0.253$, $c_7 = 0.1265$, $d_7 = 0.06325$, $T_7 = 0.94875$.
$a_8 = 0.502$, $b_8 = 0.251$, $c_8 = 0.1255$, $d_8 = 0.06275$, $T_8 = 0.94125$.

Where $p_{cul_{new}}$,$p_{soc_{new}}$, $p_{per_{new}}$, $p_{psy_{new}}$, $p_{total_{new}}$ represents new probabilities of cultural factors, social factors, personal factors, psychological factors and total probability respectively. For the first row of the table(2), if we increase the value of the p_{cul} by 0.1%, the new value of p_{cul} turns out to be $p_{cul_{new}} = p_{cul} + 0.1\% = a_1 = 0.511$. Where the value of p_{cul} can be obtained from table 1. Now the values of $p_{soc_{new}} = b_1$, $p_{per_{new}} = c_1$, $p_{psy_{new}} = d_1$ and $p_{total_{new}} = T_1$ can be obtained from the equations (2) − (5). Similar discussions can be applied for rest of the rows of table(2) and table(3) respectively.
Cul, Soc, Per and Psy represent cultural factors, social factors, personal factors and psychological factors respectively. (↑) and (↓) show the increase and decrease in the factors concerned respectively.
By performing similar experiments on different buying behaviour governing factors for any product at hand, our model can be used to optimize the chance of success for that product. The best part about our model is that the model works fine at the designing phase of the product that is, when the manufacturing of the product has not yet been started. For any product at hand, whether such changes in any one of the four buying behaviour governing factors for that product is possible or not, can be decided by the manufacturer of the product or it can be determined by the extensive body of work in economics and marketing on consumers' buying behaviour.

4. Importance of MRA on consumers' buying behaviour

The main importance of MRA on consumers' buying behaviour is that all the four buying behaviour governing factors viz cultural,social,personal and psychological are no more abstract concepts rather they are relative concepts. All the four buying behaviour governing factors are related to one another in some mathematical sense. Due to this, for any product any kind of change in any one of the four buying behaviour governing factors affects rest of the three buying behaviour governing factors also.

5. Comparison of our model with the previous models

The mathematical model of consumer behaviour proposed by [20] was mainly based on the advertising stimulation. This model related effort of advertising to change in attitude and consumer purchase. Naturally this model works fine only in the case of advertising stimulation. Our model helps taking important decisions right at the planning phase of any product when the manufacturing of the product has not yet been started. Such decisions can greatly affect the design of the product which in turn affects the sales of the product once the product enters the market. This is clearly a very big advantage for the manufacturer of the product.
[21] contrasts seven models of increasing complexity and it helps to conclude that model complexity is a must to provide a norm for omissions of simpler models. Model complexity is also important because of the requirement for the result oriented development of market simulators. Our model does not talk about different levels of complexity. Our model can predict the probability of success of any product when it is just in the design phase and neither manufacturing of the product has been started nor it is even introduced into the market. Our model can also indicate the ways by which the design of the product can be changed so as to optimize the sales.
In a nutshell we can say that our model is the generalization of the above mentioned models and it can be used at the design phase of any product. Once the product is formed and introduced to the market, the above mentioned models can be used to get further better results regarding the product at hand.

6. Conclusion

Most of the researchers have performed their studies on any one of the four buying behaviour governing factors. Such studies hardly talk about if there are any effects of the performed study on rest of the three buying behaviour governing factors. In this paper we have shown using MRA that for any product if cultural factor gets

affected by $x\%$, the social factor, personal factor and psychological factor would be affected by $\frac{x}{2}\%$, $\frac{x}{4}\%$ and $\frac{x}{8}\%$ respectively. For any product if social factors gets affected by $y\%$, due to MRA the cultural factors, personal factors and psychological factors are affected by $2x\%$, $\frac{x}{2}\%$, $\frac{x}{4}\%$ and $\frac{x}{8}\%$ respectively. For any product, similar discussions follow for investigating the effect of rest of the two buying behaviour governing factors separately on the remaining three buying behaviour governing factors. In almost all the previous studies about consumers' buying behaviour it was hardly possible to get any idea about more than one buying behaviour governing factors by performing study about just any one of the buying behaviour governing factors. Further, by convexity of the satisfaction of consumers' buying behaviour with respect to all the four spaces of cultural, social, personal and psychological factors, we have given a way by which the sales of any product can be optimized in the most promising direction. Almost in all the previous studies there were very few concrete quantitative way by which the optimization of sales could be taken in the most promising direction. Using orthonormal wavelet packets, an idea about any consumers' buying behaviour for any specific product or his overall buying inclination can be obtained. Almost in all the prior studies there were hardly any methods by which the overall inclination of the consumer or his inclination towards a particular product could have been studied. We have also shown that consumers' buying behaviour nearly follow exponential distribution. This formulation can be used to draw important statistical inferences. Prior to our paper there was hardly any study performed in this direction.

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