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On Efficiency and Duality for Multiobjective Variational Problems

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The concept of efficiency (Pareto optimum) is used to formulate duality for multiobjective variational problems. Wolfe and Mond–Weir type duals are formulated. Under generalized (F, ρ) – convexity assumptions on the functions involved weak and strong duality theorems are proved. © 1994 Academic Press, Inc.

1. INTRODUCTION AND PRELIMINARIES

The aim of this paper is to use the concept of efficiency (Pareto optimum) to formulate some results of duality under generalized (F, ρ) – convexity assumptions for the following class of multiobjective variational problem:

$$\text{(P) Minimize } \int_a^b f(t, x(t), \dot{x}(t))\, dt = \left(\int_a^b f^1(t, x(t), \dot{x}(t))\, dt, \ldots, \int_a^b f^p(t, x(t), \dot{x}(t))\, dt\right)$$

subject to

$$x(a) = \alpha, \qquad x(b) = \beta \tag{1}$$

$$g(t, x(t), \dot{x}(t)) \leqslant 0 \tag{2}$$

$$h(t, x(t), \dot{x}(t)) = 0 \qquad \forall\, t \in [a, b], \tag{3}$$

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where $f_i = I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $i \in \{1, \ldots, p\} = P$, and

$$g = (g_1, \ldots, g_m), \qquad g_j\colon I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \qquad j = 1, \ldots, m,$$
$$h = (h_1, \ldots, h_q), \qquad h_k\colon I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \qquad k = 1, \ldots, q,$$

are assumed to be continuously differentiable functions. $I = [a, b]$ is a real interval.

Following Bector and Husain [1] and Mishra and Mukherjee [3], we consider for primal problem (P) the Wolfe type dual and Mond–Weir type dual. Notations are same as in [1] and [3]. In [2], Egudo has used the concept of efficiency (Pareto optimum) to formulate duality for multiobjective non-linear programs. Preda [4] has used the same concept under a weaker assumption, namely generalized (F, ρ) – convexity. Bector and Husain [1] have discussed duality theorems and related proper efficient solutions of the primal and dual problems for multiobjective variational problems.

Let X denote the set of all feasible solutions of (P) and

$$X := \{x \in C(I, \mathbb{R}^n)\colon x(a) = \alpha, x(b) = \beta, g(t, x, \dot{x}) \leqslant 0, \\ h(t, x, \dot{x}) = 0, \forall\, t \in I\},$$

where $C(I, \mathbb{R}^n)$ is the space of piecewise smooth functions x with norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty$.

Definition 1. A point $x \in X$ is said to be an efficient solution of (P) if for all $x \in X$

$$\int_a^b f^i(t, x^*(t), \dot{x}^*(t))\, dt \geqslant \int_a^b f^i(t, x(t), \dot{x}(t))\, dt, \qquad \forall\, i \in P$$
$$\Rightarrow \int_a^b f^i(t, x^*(t), \dot{x}^*(t))\, dt = \int_a^b f^i(t, x(t), \dot{x}(t))\, dt, \qquad \forall\, i \in P.$$

In this paper proofs for strong duality results will invoke the following:

Lemma 1. *x^0 is an efficient solution for (P) if and only if x^0 solves*

$$P_k(x^0) = \begin{cases} \int_a^b f^k(t, x(t), \dot{x}(t))\, dt \\ \text{subject to} \end{cases}$$

$$\int_a^b f^j(t, x(t), \dot{x}(t))\, dt \leqslant \int_a^b f^j(t, x^0(t), \dot{x}^0(t))\, dt$$

for all $j \neq k$,

$$g(t, x(t), \dot{x}(t)) \geqq 0, \qquad h(t, x(t), \dot{x}(t)) = 0$$

for each $k = 1, \ldots, p$.

DEFINITION 2. A functional F: $I \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is sublinear if for any $x, x^0 \in \mathbb{R}^n$, $\dot{x}, \dot{x}^0 \in \mathbb{R}^n$,

$$F(t, x, \dot{x}, x^0, \dot{x}^0; a_1 + a_2) \leqq F(t, x, \dot{x}, x^0, \dot{x}^0; a_1) + F(t, x, \dot{x}, x^0, \dot{x}^0; a_2)$$

for any $a_1, a_2 \in \mathbb{R}$,

and

$$F(t, x, \dot{x}, x^0, \dot{x}^0; \alpha a) = \alpha F(t, x, \dot{x}, x^0, \dot{x}^0; a)$$

for any $\alpha \in \mathbb{R}$, $\alpha \geqq 0$ and $a \in \mathbb{R}^n$.

Now consider the function f: $I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, and suppose that f is with first partial derivatives at x^0, an interior point of X, $\nabla_x f(t, x^0(t), \dot{x}^0(t))$ is the gradient vector of f with respect to x at x^0, and $\nabla_{\dot{x}} f(t, x^0(t), \dot{x}^0(t))$ is the gradient vector of f with respect to $\dot{x}$ at x^0. Let $d(t, \cdot, \cdot)$ be a pseudo metric on $\mathbb{R}^n$, and $\rho \in \mathbb{R}$.

DEFINITION 3. $f(t, x(t), \dot{x}(t))$ is said to be (F, ρ) − convex if

$$\begin{aligned}
&\int_a^b \{f(t, x(t), \dot{x}(t)) - f(t, x^0(t), \dot{x}^0(t))\}\, dt \\
&\quad \geqq \int_a^b F(t, x(t), \dot{x}(t), x^0(t), \dot{x}^0(t); \nabla_x f(t, x^0(t), \dot{x}^0(t)) \\
&\quad - \frac{d}{dt}(\nabla_{\dot{x}} f(t, x^0(t), \dot{x}^0(t))))\, dt \\
&\quad + \rho \int_a^b d^2(t, x(t), x^0(t))\, dt.
\end{aligned}$$

The function f is said to be strongly F-convex, F − convex, or weakly F − convex at x^0, according to $\rho > 0$, $\rho = 0$, or $\rho < 0$.

DEFINITION 4. The function $f(t, \cdot, \cdot)$ is (F, ρ) − quasiconvex at x^0 if for all $x \in X$ such that

$$\int_a^b f(t, x(t), \dot{x}(t))\, dt \leqq \int_a^b f(t, x^0(t), \dot{x}^0(t))\, dt$$

we have

$$\int_a^b F(t, x(t), \dot{x}(t), x^0(t)\dot{x}^0(t); \nabla_x f(t, x^0(t), \dot{x}^0(t)) - \frac{d}{dt}(\nabla_{\dot{x}} f(t, x^0(t), \dot{x}^0(t)))) \, dt \leq -\rho \int_a^b d^2(t, x(t), x^0(t)) \, dt.$$

We say that $f(t, \cdot, \cdot)$ is strongly F-quasiconvex, F-quasiconvex, or weakly F-quasiconvex at x^0 according to $\rho > 0$, $\rho = 0$, or $\rho < 0$.

DEFINITION 5. The function $f(t, \cdot, \cdot)$ is (F, ρ) − pseudoconvex at x^0 if for all $x \in X$ such that

$$\int_a^b F(t, x, \dot{x}, x^0, \dot{x}^0; \nabla_x f(t, x^0(t), \dot{x}^0(t)) - \frac{d}{dt}(\nabla_{\dot{x}} f(t, x^0(t), \dot{x}^0(t))) \, dt \geq -\rho \int_a^b d^2(t, x(t), x^0(t)) \, dt$$

we have

$$\int_a^b f(t, x(t), \dot{x}(t)) \, dt \geq \int_a^b f(t, x^0(t), \dot{x}^0(t)) \, dt.$$

Function $f(t, \cdot, \cdot)$ is strongly F-pseudoconvex, F-pseudoconvex, or weakly F-pseudoconvex according to $\rho > 0$, $\rho = 0$, or $\rho < 0$.

DEFINITION 6. The function $f(t, \cdot, \cdot)$ is strictly (F, ρ) − pseudoconvex at x^0 if for all $x \in X$. $x \neq x^0$ such that

$$\int_a^b F(t, x(t), \dot{x}(t), x^0(t), \dot{x}^0(t); \nabla_x f(t, x^0(t), \dot{x}^0(t)) - \frac{d}{dt}(\nabla_{\dot{x}} f(t, x^0(t), \dot{x}^0(t)))) \, dt \geq -\rho \int_a^b d^2(t, x(t), x^0(t)) \, dt$$

and we have

$$\int_a^b f(t, x(t), \dot{x}(t)) \, dt > \int_a^b f(t, x^0(t), \dot{x}(t)) \, dt,$$

or equivalently, if

$$\int_a^b f(t, x(t), \dot{x}(t)) \, dt \leq \int_a^b f(t, x^0(t), \dot{x}(t)) \, dt$$

we have

$$\int_a^b F(t, x(t), \dot{x}(t), x^0(t), \dot{x}^0(t); \nabla_x f(t, x^0(t), \dot{x}^0(t)) - \frac{d}{dt}(\nabla_{\dot{x}} f(t, x^0(t), \dot{x}^0(t)))) \, dt < -\rho \int_a^b d^2(t, x(t), x^0(t)) \, dt.$$

2. Wolfe Type Duality

In the present section we prove weak and strong duality relations between (P) and the Wolfe type dual [1]

$$\text{(WD) Maximize } \left(\int_a^b \{f^1(t, u(t), \dot{u}(t)) + y^T g(t, u(t), \dot{u}(t)) + z^T h(t, u(t), \dot{u}(t))\} \, dt, \ldots, \int_a^b \{f^p(t, u(t), \dot{u}(t) + y^T g(t, u(t), \dot{u}(t) + z^T h(t, u(t), \dot{u}(t))\} \, dt \right)$$

subject to

$$x(a) = \alpha, \qquad x(b) = \beta \tag{4}$$

$$\sum_{i=1}^p \tau_i f_x^i(t, u(t), \dot{u}(t)) + y^T(t) \, g_x(t, u(t), \dot{u}(t)) + z^T(t) \, h_x(t, u(t), \dot{u}(t)) = D\left[\sum_{i=1}^p \tau_i f_{\dot{x}}^i(t, u(t), \dot{u}(t)) + y^T(t) \, g_{\dot{x}}(t, u(t), \dot{u}(t)) + z^T h_{\dot{x}}(t, u(t), \dot{u}(t)) \right]$$

$$t \in I \tag{5}$$

$$y \geqslant 0 \tag{6}$$

$$\tau_i \geqslant 0, \qquad \sum_{i=1}^p \tau_i = 1. \tag{7}$$

THEOREM 1 (Weak Duality). *Assume that, for all feasible x for* (P) *and all feasible* (u, τ, y, z) *for* (WE),

(a) $\tau^T f(t, \cdot, \cdot) + y^T g(t, \cdot, \cdot) + Z^T h(t, \cdot, \cdot)$ *is F-convex at u. Further, if either*

(b) $\tau_i > 0 \; \forall \, i \in \{1, \ldots, p\}$, *or*

(c) $\tau^T f(t, \cdot, \cdot) + y^T g(t, \cdot, \cdot) + Z^T h(t, \cdot, \cdot)$ *is strictly F-convex at u, then*

$$\int_a^b f^i(t, x(t), \dot{x}(t))\, dt \leq \int_a^b \{f^i(t, u(t), \dot{u}(t)) + y^T(t)\, g(t, u(t), \dot{u}(t)) + Z(t)^T h(t, u(t), \dot{u}(t))\}\, dt \qquad \forall\, i \in \{1, .., p\} \tag{8}$$

and

$$\int_a^a f^j(t, x(t), \dot{x}(t))\, dt < \int_a^b \{f^j(t, u(t), \dot{u}(t)) + y(t)^T g(t, u(t), \dot{u}(t)) + Z(t)^T h(t, u(t), \dot{u}(t))\}\, dt \tag{9}$$

for some $j \{1, \ldots, p\}$ *cannot hold.*

Proof. Suppose, contrary to the result, that (8) and (9) hold. Then since x is feasible for (P) and $y \geqslant 0$, (8) and(9) imply

$$\int_a^b \{f^i(t, x(t), \dot{x}(t)) + y(t)^T g(t, x(t), \dot{x}(t)) + Z(t)^T h(t, x(t), \dot{x}(t))\}\, dt \leq \int_a^b \{f^i(t, u(t), \dot{u}(t)) + y^T g(t, u(t), \dot{u}(t)) + Z^T h(t, u(t), \dot{u}(t))\}\, dt \tag{10}$$

for all $i \in P$

and

$$\int_a^b \{f^j(t, x(t), \dot{x}(t)) + y^T g(t, x(t), \dot{x}(t)) + Z^T h(t, x(t), \dot{x}(t))\}\, dt < \int_a^b \{f^j(t, u(t), \dot{u}(t)) + y^T g(t, u(t), \dot{u}(t)) + Z^T h(t, u(t), \dot{u}(t))\}\, dt,$$

for some $j \in P$, (11)

respectively. Now hypothesis (b) and $\sum_{i=1}^p \tau_i = 1$ imply

$$\int_a^b \{\tau^T f(t, x(t), \dot{x}(t)) + y^T g(t, x(t), \dot{x}(t)) + Z^T h(t, x(t), \dot{x}(t))\}\, dt < \int_a^b \{\tau^T f(t, u(t), \dot{u}(t)) + y^T g(t, u(t), \dot{u}(t)) + Z^T h(t, u(t), \dot{u}(t))\}\, dt. \tag{12}$$

According to (9), (12), and by sublinearity of F, we have

$$\int_a^b F(t, x, \dot{x}, u, \dot{u}; ([\nabla_x f(t, u, \dot{u})]^T\tau + ([\nabla_x g(t, u, \dot{u})]^T y + [\nabla_x h(t, u, \dot{u})]^T Z) - \frac{d}{dt}([\nabla_{\dot{x}} f(t, u, \dot{u})]^T\tau + [\nabla_{\dot{x}} g(t, u, \dot{u})]^T y + [\nabla_{\dot{x}} h(t, u, \dot{u})]^T Z))\, dt < 0, \tag{13}$$

which contradicts (5), because

$$\int_a^b F(t, x, \dot{x}, u, \dot{u}; 0)\, dt = 0.$$

When the hypothesis (c) holds, since $\tau_i \geqslant 0$, $i \in P$, and $\Sigma_{i=1}^{p} = 1$, (10) and (11) imply

$$\int_a^b \{\tau^T f(t, x(t), \dot{x}(t)) + y(t)^T g(t, x(t), \dot{x}(t)) + Z(t)^T h(t, x(t), \dot{x}(t))\}\, dt < \int_a^b \{\tau^T f(t, u(t), \dot{u}(t)) + y^T(t) g(t, u(t), \dot{u}(t)) + Z(t)^T h(t, u(t), \dot{u}(t))\}\, dt,$$

and then again we reach (13). Hence the proof is complete.

THEOREM 1′ (Weak Duality). *Assume that for all feasible x for* (P) *and all feasible (u, τ, y, z) for* (WD), (a) f_i, $i \in P$, g_j, $j = 1, \ldots, m$, h_k, $-h_k$, $k = 1, \ldots, q$ *are F-convex. Further, if either* (b) *or* (c) *from Theorem* 1 *is satisfied, then* (8) *and* (9) *cannot hold.*

Proof. See Preda [4] and the proof of Theorem 1, above.

Now we give weak duality results under (F, ρ) − convexity.

THEOREM 2 (Weak Duality). *Assume that for all feasible x for* (P) *and all feasible (u, τ, y, z) for* (WD), (a) $\tau^T f(t, \cdot, \cdot) + y^T g(t, \cdot, \cdot) + Z^T h(t, \cdot, \cdot)$ *is (F, ρ) − convex at u. Further, if either* (b) $\tau_i > 0$, *for all $i \in P$ and $\rho \geqslant 0$, or* (c), *given any metric $d(t, \cdot, \cdot)$* on $\mathbb{R}^n$ *and preset $\rho > 0$, we have the following, i.e.,* (8) *and* (9) *cannot hold.*

Proof. We suppose contrary to the result that (8) and (9) hold. Because x and (u, τ, y, z) are feasible solutions for (P) and (WD), respectively, in the case (b), we find

$$\int_a^b \{\tau^T f(t, x(t), \dot{x}(t)) + y^T g(t, x(t), \dot{x}(t)) + Z^T h(t, x(t), \dot{x}(t))\}\, dt < \int_a^b \{\tau^T f(t, u(t), \dot{u}(t)) + y^T g(t, u(t), \dot{u}(t)) + Z^T h(t, u(t), \dot{u}(t))\}\, dt. \tag{15}$$

Now, from (15) and (9), we obtain

$$\int_a^b F(t, x, \dot{x}, u, \dot{u}; ([\nabla_x f(t, u, \dot{u})]^T\tau + [\nabla_x g(t, u, \dot{u})]^T y + [\nabla_x h(t, u, \dot{u})]^T z) - \frac{d}{dt}([\nabla_{\dot{x}} f(t, u, u)]^T\tau + [\nabla_{\dot{x}} g(t, u, \dot{u})]^T y + [\nabla_{\dot{x}} h(t, u, \dot{u})]^T z))\, dt < -\rho \int_a^b d^2(t, x, u)\, dt \tag{16}$$

and then, from (5) and sublinearity of F, this implies

$$\rho \int_a^b d^2(t, x, u)\, dt < 0,$$

which is a contradiction to the fact that $\rho \geqslant 0$.

When we have (c) from (8) and (9) and $\tau \geqslant 0$, we obtain

$$\int_a^b \{\tau^T f(t, x(t), \dot{x}(t)) + y(t)^T g(t, x(t), \dot{x}(t)) + Z(t)^T h(t, x(t), \dot{x}(t))\}\, dt$$
$$\leqslant \int_a^b \{\tau^T f(t, u(t), \dot{u}(t)) + y(t)^T g(t, u(t), \dot{u}(t)) + Z(t)^T h(t, u(t), \dot{u}(t))\}\, dt$$

and then, by (a), we find (13) with equality as well.

But this contradicts (5) again. Hence the proof is complete.

THEOREM 2′ (Weak Duality). *Assume that for all feasible x for* (P) *and all feasible (u, τ, y, z) for* (WD)

(a) *f_i is (F, ρ_{1i}) − convex, $i \in P$;*

(b) *g_j is (F, ρ_{2j}) − convex, $j = 1, \ldots, m$;*

(c) *h_k is (F, ρ_{3k}) − convex, $-h_k$ is (F, ρ_{4k}) convex with $\rho_{3k} + \rho_{4k} \geqslant 0$, $1 \leqslant k \leqslant q$.*

Also, if either

(d) *$\tau_i > 0$, for all $i \in P$ and $\sum_{i=1}^{p} \rho_{1i}\tau_i + \sum_{j=1}^{m} \rho_{2j} y_j + \sum_{k=1}^{q} \rho_{3k} Z_k \geqslant 0$*

or

(e) *$\sum_{i=1}^{p} \rho_{1i}\tau_i + \sum_{j=1}^{m} \rho_{2j} y_j + \sum_{k=1}^{q} \rho_{3k} Z_k > 0$*

and $d(t, \cdot, \cdot)$ is a metric on $\mathbb{R}^n$, then (8) *and* (9) *cannot hold.*

Proof. See Preda [4] and Theorem 2, above.

COROLLARY 1. *Let* (u^0, τ^0, y^0, Z^0) *be a feasible solution for* (WD) *such that*

$$\int_a^b y^{0^T} g(t, u^0, \dot{u}^0)\, dt = 0, \qquad \int_a^b Z^{0^T} h(t, u^0, \dot{u}^0)\, dt = 0$$

and assume that u^0 *is feasible for* (P). *If weak duality (any of Theorems* 1, 1′, 2, *or* 2′) *holds between* (P) *and* (WD), *then* u^0 *is efficient for* (D) *and* (u^0, τ^0, y^0, Z^0) *is efficient for* (WD).

Proof. Suppose that u^0 is not efficient for (P); then there exists a feasible x for (P) such that, for some $i \in P$,

$$\int_a^b f^i(t, x(t), \dot{x}(t))\, dt < \int_a^b f^i(t, u^0(t), \dot{u}^0(t))\, dt \tag{17}$$

and

$$\int_a^b f^j(t, x(t), \dot{x}(t))\, dt \leq \int_a^b f^j(t, u^0(t), \dot{u}^0(t))\, dt \tag{18}$$

for all $j \in P$.

By hypothesis

$$\int_a^b \{y^{0^T} g(t, u^0, \dot{u}^0) + Z^{0^T} h(t, u^0, \dot{u}^0)\}\, dt = 0,$$

so (17) and (18) can be written as

$$\int_a^b f^i(t, x(t), \dot{x}(t))\, dt < \int_a^b f^i(t, u^0(t), \dot{u}^0 u) + y^{0^T} g(t, u^0(t), \dot{u}(t)) + Z^{0^T} h(t, u^0(t), \dot{u}(t))\}\, dt \qquad \text{for some } i \in P$$

and

$$\int_a^b f^j(t, x(t), \dot{x}(t))\, dt \leq \int_a^b \{f^j(t, u^0(t), \dot{u}^0(t)) + y^0(t)^T g(t, u^0(t), \dot{u}^0(t)) + Z^0(t)^T h(t, u^0(t), \dot{u}^0(t))\}\, dt \qquad \forall j \in P,$$

respectively; and since (u^0, τ^0, y^0, Z^0) is feasible for (WD) and x is feasible for (P), these inequalities contradict weak duality (Theorems 1, 1′, 2, or 2′).

Also suppose that (u^0, τ^0, y^0, Z^0) is not efficient for (WD). Then there exists a feasible (u, τ, y, z) for (WD) such that for some $i \in P$

$$\int_a^b \{f^i(t, u(t), \dot{u}(t)) + y^T g(t, u(t), \dot{u}(t)) + Z^T h(t, u(t), \dot{u}(t)\}\, dt$$
$$> \int_a^b \{f^i(t, u^0(t), \dot{u}^0(t)) + y^{0^T} g(t, u^0(t), \dot{u}^0(t)) + Z^{0^T} h(t, u^0(t), \dot{u}^0(t))\}\, dt \tag{19}$$

and

$$\int_a^b \{f^j(t, u(t), \dot{u}(t)) + y^T g(t, u(t), \dot{u}(t)) + Z^T h(t, u(t), \dot{u}(t)\}\, dt$$
$$\geqslant \int_a^b \{f^j(t, u^0(t), \dot{u}^0(t)) + y^T g(t, u^0(t), \dot{u}^0(t)) + Z^T h(t, u^0(t), \dot{u}^0(t))\}\, dt \tag{20}$$

for all $j \in P$;

and since

$$\int_a^b \{y^{0^T} g(t, u^0(t), \dot{u}^0(t)) + h(t, u^0(t), \dot{u}^0(t))\}\, dt = 0,$$

(19) and (20) reduce to

$$\int_a^b \{f^i(t, u(t), \dot{u}(t)) + y^T g(t, u(t), \dot{u}(t)) + Z^T h(t, u(t), \dot{u}(t))\}\, dt$$
$$> \int_a^b f^i(t, u^0(t), \dot{u}^0(t))\, dt, \qquad \text{for some } i \in P$$

and

$$\int_a^b \{f^j(t, u(t), \dot{u}(t)) + y^T g(t, u(t), \dot{u}(t)) + Z^T h(t, u(t), \dot{u}(t))\}\, dt$$
$$\geqslant \int_a^b \{f^j(t, u^0(t), \dot{u}^0(t)) + dt, \qquad \text{for all } j \in P,$$

respectively. Since u^0 is feasible for (P), these inequalities contradict weak duality (Theorems 1, 1′, 2, or 2′).

Therefore u^0 and (u^0, τ^0, y^0, t^0) are efficient for their respective programs.

THEOREM 3 (Strong Duality). *Let x^0 be a feasible solution for* (P) *and assume that*

(i) *x^0 is an efficient solution;*

(ii) *for at least one i, $i \in P$, x^0 satisfies a constraint qualification for problem $P_i(x^0)$.*

Then there exist $\tau^0 \in \mathbb{R}^p$, $y^0 \in \mathbb{R}^m$, $Z^0 \in \mathbb{R}^q$ such that (x^0, τ^0, y^0, Z^0) is feasible for (WD) *and*

$$\int_a^b \{y^0(t)^T g(t, x^0(t), \dot{x}^0(t)) + Z^0(t)^T h(t, x^0(t), \dot{x}^0(t))\}\, dt = 0.$$

Further, if weak duality (Theorems 1, 1′, 2, *or* 2′) *also holds between* (P) *and* (WD) *then (x^0, τ^0, y^0, Z^0) is efficient for* (WD).

Proof. Similar to the proof of Theorem 3 of Egudo [2] and Corollary 1 above.

3. Mond–Weir Type Duality

In this section, we establish various duality theorems for the Mond–Weir dual given below:

$$\text{(MD) Maximize} \quad \left(\int_a^b f^1(t, u(t), \dot{u}(t))\, dt, \ldots., \int_a^b f^p(t, u(t), \dot{u}(t))\, dt\right)$$

subject to

$$x(a) = \alpha, \qquad x(b) = \beta \tag{21}$$

$$\begin{aligned}\sum_{i=1}^p \tau_i f_x^i(t, u(t), \dot{u}(t)) &+ y(t)^T g_x(t, u(t), \dot{u}(t)) + Z(t)^T h_x(t, u(t), \dot{u}(t)) \\ &= D\left[\sum_{i=1}^p \tau_i f_{\dot{x}}^i(t, u(t), \dot{u}(t)) + y(t)^T g_{\dot{x}}(t, u(t), \dot{u}(t)) \right. \\ &\quad \left. + Z(t)^T h_{\dot{x}}(t, u(t), \dot{u}(t))\right], \qquad t \in I\end{aligned} \tag{22}$$

$$y(t)^T g(t, u(t), \dot{u}(t)) \geqslant 0 \tag{23}$$

$$Z(t)^T h(t, \mathrm{u}(t), \dot{u}(t)) = 0 \tag{24}$$

$$\tau_e^T = 1 \tag{25}$$

$$y \geqslant 0, \qquad \tau \geqslant 0, \tag{26}$$

where $e = (1, \ldots, 1)^T \in \mathbb{R}^p$.

The weak duality results are given under conditions of generalized (F, ρ) $-$ convexity and (F, ρ) $-$ convexity.

THEOREM 4 (Weak Duality). *Assume that for all feasible* x *for* (P) *and all feasible* (u, τ, y, Z) *for* (MD)

(a) $y^T g(t, \cdot, \cdot) + Z^T h(t, \cdot, \cdot)$ *is* (F, ρ) $-$ *quasiconvex at* u, *and also if any of the following holds*

(b) $\tau_i > 0 \ \forall\, i \in P$, *and* f_i *is* (F, ρ_{1i}) $-$ *pseudoconvex at* u *for any* $i \in P$, *with* $\rho + \sum_{i=1}^{p} \rho_{1i}\tau_i \geqslant 0$;

(c) $\tau_i > 0 \ \forall\, i \in P$ *and* $\tau^T f(t, \cdot, \cdot)$ *is* (F, ρ') $-$ *pseudoconvex at* u, *with* $\rho + \rho' \geqslant 0$;

(d) $\tau^T f(t, \cdot, \cdot)$ *is strictly* (F, ρ') *pseudoconvex at* u, *with* $\rho + \rho' > 0$, *then*

$$\int_a^b f^i(t, x(t), \dot{x}(t))\, dt \leqslant \int_a^b f^i(t, u(t), \dot{u}(t))\, dt \qquad \forall\, i \in P \tag{27}$$

and

$$\int_a^b f^j(t, x(t), \dot{x}(t))\, dt < \int_a^b f^j(t, u(t), \dot{u}(t))\, dt \tag{28}$$

for some $j \in P$ *cannot hold.*

Proof. Let x be an arbitrary feasible solution of (P) and (u, τ, y, z) be an arbitrary feasible solution of (MD). Then in view of $y \geqslant 0$ we have that

$$\int_a^b y(t)^T g(t, x(t), \dot{x}(t))\, dt \leqslant \int_a^b y(t)^T g(t, u(t), \dot{u}(t))\, dt$$

and

$$\int_a^b Z(t)^T h(t, x(t), \dot{x}(t))\, dt = \int_a^b Z(t)^T h(t, u(t), \dot{u}(t))\, dt.$$

Hence

$$\int_a^b \{y(t)^T g(t, x(t), \dot{x}(t)) + Z(t)^T h(t, x(t), \dot{x}(t))\}\, dt$$
$$\leqslant \int_a^b \{y(t)^T g(t, u(t), \dot{u}(t)) + Z(t)^T h(t, u(t), \dot{u}(t))\}\, dt$$

and since

$$y(t)^T g(t, \cdot, \cdot) + Z(t)^T h(t, \cdot, \cdot) \text{ is}$$

(F, ρ) − quasiconvex at u, this implies

$$\int_a^b F(t, x(t), \dot{x}(t), u(t), \dot{u}(t); ([\nabla_x g(t, u(t), \dot{u}(t))]^T y + [\nabla_x h(t, u(t), \dot{u}(t))]^T Z) - \frac{d}{dt}([\nabla_{\dot{x}} g(t, u(t), \dot{u}(t))]^T y + [\nabla_{\dot{x}} h(t, u(t), \dot{u}(t))]^T z))\, dt \leqslant -\rho \int_a^b d^2(t, x(t), u(t))\, dt. \tag{29}$$

From (29) feasibility of (u, τ, y, Z) and sublinearity of F we obtain

$$\int_a^b F(t, x(t), \dot{x}(t), u(t), \dot{u}(t); [\nabla_x f(t, u(t), \dot{u}(t))]^T \tau - \frac{d}{dt}([\nabla_{\dot{x}} f(t, u(t), \dot{u}(t))]^T \tau))\, dt \geqslant \int_a^b d^2(t, x(t), u(t))\, dt. \tag{30}$$

On the other hand, suppose, contrary to the result of the theorem, that (27) and (28) hold. If we have the hypothesis (b), then (27), (28), and (F, ρ_{1i}) − pseudoconvexity of f_i, $i \in P$ imply

$$\int_a^b F(t, x(t), \dot{x}(t), u(t), \dot{u}(t); \nabla_x f^i(t, u(t), \dot{u}(t)) - D(\nabla_{\dot{x}} f^i(t, u(t), \dot{u}(t))))\, dt \leqslant -\rho_{1i} \int_a^b d^2(t, x(t), u(t))\, dt, \forall\, i \in P \tag{31}$$

and

$$\int_a^b F(t, x(t), \dot{x}(t), u(t), \dot{u}(t); \nabla_x f^i(t, u(t), \dot{u}(t)) - D(\nabla_{\dot{x}} f^i(t, u(t), \dot{u}(t))))\, dt < -\rho_{1i} \int_a^b d^2(t, x(t), u(t))\, dt, \tag{31}$$

for some $j \in P$.

Because $\tau_i > 0$, $\forall\, i \in P$, from (31), (32), and the sublinearity of F we have

$$\int_a^b F(t, x(t), \dot{x}(t), u(t), \dot{u}(t); [\nabla_x f(t, u(t), \dot{u}(t))]^T\tau - D[\nabla_{\dot{x}} f(t, u(t), \dot{u}(t))]^T\tau)\, dt < -\left(\sum_{i=1}^{p} \rho_{1i}\tau_i\right) \int_a^b d^2(t, x(t), u(t))\, dt, \tag{33}$$

which is a contradiction to (30), because

$$\rho + \sum_{i=1}^{p} \rho_{1i}\tau_i \geqslant 0.$$

When the hypothesis (c) holds, from (27) and (28) we obtain

$$\int_a^b \tau^T f(t, x(t), \dot{x}(t))\, dt < \int_a^b \tau^T f(t, u(t), \dot{u}(t))\, dt$$

and then we have a contradiction to (30). Finally, in the last case, if the hypothesis (d) holds, from (27) and (28) we have

$$\int_a^b \tau^T f(t, x(t), \dot{x}(t))\, dt \leqslant \int_a^b \tau^T f(t, u(t), \dot{u}(t))\, dt$$

and the strictly (F, ρ) − pseudoconvexity of $\tau^T f(t, \cdot, \cdot)$ at u implies again a contradiction to (30). Hence the proof is complete.

THEOREM 5 (Weak Duality). *Assuming that for all feasible x for* (P) *and all feasible (u, τ, y, Z) for* (MD)

(a) *$f_i(t, \cdot, \cdot)$ is (F, ρ_{1i}) − convex, $i \in P$;*

(b) *$g_j(t, \cdot, \cdot)$ is (F, ρ_{2j}) convex, $j = 1, \ldots, m$;*

(c) *$h_k(t, \cdot, \cdot)$ is (F, ρ_{3k}) − convex; $-h_k(t, \cdot, \cdot)$ is (F, ρ_{4k}) − convex with $\rho_{3k} + \rho_{4j} \geqslant 0$ for all $k = 1, \ldots, q$.*

Further, if either (d) *or* (c) *of Theorem* 2′ *holds, then* (27) *and* (28) *cannot hold.*

Proof. Similar to the proofs of Theorems 2′ and 4.

The following Theorem is weak duality under (F, ρ) − convexity.

THEOREM 5′ (Weak Duality). *Assume that for all feasible x for* (P) *and all feasible (u, τ, y, z) for* (MD), (a) *and either* (b) *or* (c) *of Theorem* 2 *holds then* (27) *and* (28) *cannot hold.*

Proof. Similar to the proof of Theorem 2.

Now we give the following result, which is very similar to Corollary 1 in Section 2.

COROLLARY 2. *Let* (u^0, τ^0, y^0, Z^0) *be a feasible solution for* (MD) *and assume that* u^0 *is feasible for* (P). *If weak duality* (*Theorems* 4, 5, *or* 5′) *holds between* (P) *and* (MD), *then* u^0 *is efficient for* (P) *and* (u^0, τ^0, y^0, Z^0) *is efficient for* (MD).

Proof. Suppose that u^0 is not efficient for (P); then there exists a feasible x for (P) such that (27) and (28) hold. But (u^0, τ^0, y^0, Z^0) is feasible for (MD), hence the result of weak duality (Theorems 4, 5, or 5′) is contradicted. Therefore, u^0 must be efficient for (P). Similarly, assuming that (u^0, τ^0, y^0, Z^0) is not efficient for (MD) leads us to a contradiction and hence (u^0, τ^0, y^0, Z^0) is efficient for (MD).

THEOREM 6 (Strong Duality). *Let* x^0 *be a feasible solution for* (P) *and assume that*

(a) x^0 *is efficient;*
(b) x^0 *satisfies a constraint qualification for* $P_i(x^0)$ *for at least one* $i \in P$.

Then there exist $\tau^0 \in \mathbb{R}^p$, $y^0 \in \mathbb{R}^m$, $Z^0 \in \mathbb{R}^q$ *such that* (x^0, τ^0, y^0, Z^0) *is feasible for* (MD).

Further, if weak duality (*Theorems* 4, 5, *or* 5′) *also holds, then* (u^0, τ^0, y^0, Z^0) *is efficient for* (MD).

Proof. See Preda [4] and Egudo [2].

REFERENCES

1. C. R. BECTOR AND I. HUSAIN, Duality for multiobjective variational problems, *J. Math. Anal. Appl.* **166** (1992), 214–229.
2. R. R. EGUDO, Efficiency and generalized convex duality for multiobjective programs, *J. Math. Anal. Appl.* **138** (1989), 84–94.
3. S. K. MISHRA, AND R. N. MUKHERJEE, Duality for multiobjective fractional variational problems, *J. Math. Anal. Appl.*, in press.
4. V. PREDA, On efficiency and duality for multiobjective programs, *J. Math. Anal. Appl.* **166** (1992), 365–377.