θ	=	time	[s]
ρ	=	density	[kg/m³]
τ	=	delay time	[s]
ν	=	kinematic viscosity	[m²/s]

<Subscripts>

j	=	1, 2 . . . , 20; j-th part of cross-section
kl	=	kl-th part of cross-section
p	=	particle
T	=	tube
W	=	liquid

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DEVELOPING AND FULLY DEVELOPED TURBULENT FLOW THROUGH ANNULI

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Turbulent flow velocity profiles for the inner and outer flow regions of annuli are proposed. Theoretical expressions for friction factors and entrance region pressure drops are developed and are further extended to the limiting cases of pipes and parallel plates. Experimental data are found to be in good agreement with theoretical results.

Introduction

Study of turbulent flow through an annulus primarily requires a knowledge of the velocity profile, which has been a subject of discussion for the last three decades. A number of investigators[1–7,11] have presented extensive work on the subject, but the discrepancy between experimental and theoretical results is still to be explained.

Lundgren *et al.*[8] presented a method for estimation of entrance loss in laminar flow through a channel of arbitrary cross-section. The investigation made by Okiishi and Serovy[10] shows the dependence of flow development on the geometry of the entrance. Thorpe[12,13] made extensive review of the previous investigations on turbulent flow through annuli and showed that modified logarithmic velocity profiles provide a simple, accurate relationship for annular friction coefficient. The purpose of the present investigation is to propose an appropriate turbulent flow velocity profile and relationships for friction factor in an annulus. A method to predict the entrance region pressure drop in an annulus, which is an important consideration in the design of heat exchangers, is also proposed.

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1. Theoretical Consideration

1.1 Fully developed flow

The equation for turbulent flow velocity profile in a pipe for $Re<10^5$ derived from the one-seventh power law

$$u^+=8.74(y^+)^{1/7}$$

is assumed to be valid for the outer region of the annulus also. This may be written as

$$u_2^+=8.74(y_2^+)^{1/7} \tag{1}$$

By simple logic a similar equation with different constants will describe the velocity profile in the inner region. Thus the equation proposed is

$$u_1^+=B(y_1^+)^b \tag{2}$$

The parameter b becomes a function of Reynolds number and the expression is too complicated for engineering applications. For simplicity it is assumed that $b=1/7$ as in the outer region.

For the inner and outer flow regions, the momentum balance yields, respectively,

$$\tau_{w1}=(r_1/2)\{(\lambda^2/k^2)-1\}\Delta P/L$$

and (3)

$$\tau_{w2}=(r_2/2)(1-\lambda^2)\Delta P/L$$

Defining friction velocities as

$$u_1^*=(\tau_{w1}/\rho)^{1/2}$$

and (4)

$$u_2^*=(\tau_{w2}/\rho)^{1/2}$$

the ratio u_2^*/u_1^* may be obtained as

$$u_2^*/u_1^*=\left[\frac{k(1-\lambda^2)}{(\lambda^2-k^2)}\right]^{1/2} \tag{5}$$

Imposing the condition $u_1=u_2$ at $r=r_m$, dividing Eq. (1) by Eq. (2) and substituting the value of u_2^*/u_1^* from Eq. (5), the expression for the constant B may be obtained as

$$B=8-74\left[\frac{1-\lambda}{\lambda-k}\right]^{1/7}\left[\frac{1-\lambda^2}{\lambda^2-k^2}k\right]^{4/7} \tag{6}$$

Therefore, the velocity profile in the inner region takes the following form:

$$u_1^+=8.74\left[\frac{1-\lambda}{\lambda-k}\right]^{1/7}\left[\frac{(1-\lambda^2)}{(\lambda^2-k)}k\right]^{4/7}[(y_1^+)]^{1/7} \tag{7}$$

For determination of the radius of maximum velocity the following widely accepted Kays and Leung[6] equation is used

$$(\lambda-k)/(1-\lambda)=(k)^{0.343} \tag{8}$$

The shear stress variation in the radial direction is not linear in the case of an annulus. The stress is more on the inner wall and less on the outer wall. Hence the friction factors on the two walls are related to the Reynolds number in corresponding flow regions. Friction factors and Reynolds numbers are defined as

$$f_1=2\tau_{w1}/U_1^2,\quad Re_1=D_{e1}U_1/\nu \tag{9}$$

$$f_2=2\tau_{w2}/U_2^2,\quad R_{e2}=D_{e2}U_2/\nu \tag{10}$$

where

$$D_{e1}=2r_1[(\lambda^2/k^2)-1]$$
$$D_{e2}=2r_2(1-\lambda^2) \tag{11}$$

Average velocities U_2 and U_1 can be obtained by using the velocity profile Eqs. (1) and (7) for outer and inner regions respectively. Thus

$$U_2=[7u_m/4(1+\lambda)][\lambda+(7/15)(1-\lambda)] \tag{12}$$

and

$$U_1=[7u_m/4(\lambda+k)][\lambda-(7/15)(\lambda-k)] \tag{13}$$

The friction coefficient on the outer wall may be obtained using the outer velocity profile as

$$\frac{U_2}{U_2^*}=\left\{8.74(u_2^*/\nu)^{1/7}\int_{r_m}^{r_2}2\pi r(r_2-r)^{1/7}dr\right\}\Big/\pi(r_2^2-r_m^2) \tag{14}$$

Equations (4), (10) and (14) are combined to give

$$f_2=\phi_2(k)Re_2^{-0.25} \tag{15}$$

where

$$\phi_2(k)=0.0201(1+\lambda)^2/[\lambda+(7/15)(1-\lambda)]^{7/4} \tag{16}$$

For a tube, $\lambda\to 0$ and Eq. (16) reduces to

$$f_2=0.0765(Re_2)^{-0.25} \tag{17}$$

For parallel plates, $\lambda\to 1$ and

$$f_2=0.0804(Re_2)^{-0.25} \tag{18}$$

Similarly, inner wall friction coefficient

$$f_1=\phi_1(k)Re_1^{-0.25} \tag{19}$$

where

$$\phi_1(k)=0.0201\left[\frac{\lambda-k}{(1-\lambda)k}\right]^{0.25}\left[\frac{(\lambda^2-k^2)}{(1-\lambda^2)k}\right]\times\left[\frac{(\lambda+k)^2}{[\lambda-(7/15)(\lambda-k)]^{1.75}}\right]$$

which reduces to Eq. (18) for parallel plates.

1.2 Entrance region flow

Total pressure drop in a conduit, including the entrance region, is given by

$$\frac{(\Delta P)}{(1/2)U^2\rho}=\frac{(\Delta P)_e}{(1/2)U^2\rho}+\frac{(\Delta P)_f}{(1/2)U^2\rho} \tag{20}$$

Assuming the entire developing boundary layer to be turbulent, the above equation can be written in the following dimensionless forms for the inner and outer regions respectively:

$$\Delta P_1^+=4\phi_1(k)L_1^++C_1 \tag{21}$$

and

$$\Delta P_2^+=4\phi_2(k)L_2^++C_2 \tag{22}$$

where

$$\Delta P_1^+=2\Delta P/\rho U_1^2,\quad \Delta P_2^+=2\Delta P/\rho U_2^2$$
$$L_1^+=(L/D_{e1})(Re_1)^{0.25},\quad L_2^+=(L/D_{e2})(Re_2)^{0.25} \tag{23}$$

The linearisation technique proposed by Lundgren *et al.*[8] for determination of the entrance loss coefficient in a conduit of arbitrary cross-section by a knowledge of the fully developed velocity profile is found to be more practical. The loss coefficient is given by

$$C=(2/A)\iint^{A}[(u/U)^3-(u/U)^2]dA \tag{24}$$

where u is point velocity, U is average velocity in fully developed flow condition. The first term is a kinetic energy correction and the second is a momentum correction. Although the expression was developed for laminar flow, it can also be extended to turbulent flow.

Due to difference in velocity profiles in the two flow regions, different loss coefficients and entrance lengths are expected for the inner and outer regions. For the outer region of the annulus the velocity profile may be used in Eq. (24) to give

$$C_2=(2/A_2)\int_0^{2\pi}\int_{r_m}^{r_2}\left[M_2^3\left(\frac{r_2-r}{r_2-r_m}\right)^{3/7}-M_2^2\left(\frac{r_2-r}{r_2-r_m}\right)^{2/7}\right]rdrd\theta \tag{25}$$

where M_2 is the ratio of the fully developed maximum velocity to the average velocity in the outer region.

$$M_2=4(1+\lambda)/7[\lambda+(7/15)(1-\lambda)] \tag{26}$$

On integration, Eq. (25) gives

$$C_2=\frac{4}{(1+\lambda)}\left[\frac{7}{10}M_2^3\left\{\lambda+\frac{7}{17}(1-\lambda)\right\}-\frac{7}{9}M_2^2\left\{\lambda+\frac{7}{16}(1-\lambda)\right\}\right] \tag{27}$$

For a tube, $\lambda\to 0$, $M_2=60/49$, and Eq. (27) gives $C_2=0.0759$. For parallel plates, $\lambda\to 1$, $M_2=8/7$, and $C_2=0.058$.

Entrance loss coefficient in the inner region is estimated by using the fully developed velocity profile for the inner region

$$C_1=\frac{4}{(\lambda+k)}\left[\frac{7}{10}M_1^3\left\{\lambda-\frac{7}{17}(\lambda-k)\right\}-\frac{7}{9}M_1^2\left\{\lambda-\frac{7}{16}(\lambda-k)\right\}\right] \tag{28}$$

where M_1 is the ratio of maximum velocity to average velocity in the inner region

$$M_1=4(\lambda+k)/7[\lambda-(7/15)(\lambda-k)] \tag{29}$$

For flow between parallel plates both Eqs. (27) and (28) give the same single value of the loss coefficients, $C_1=0.058$, by imposing the conditions $\lambda\to 1$ and $k\to 1$. For a very thin wire, forming an annulus, $k\to 0$ and Eq. (28) gives $C_1=0.01688$.

2. Experimental

Flow of water through annuli of aspect ratio $k=0.424$, 0.1073, 0.0485 having the same outer pipe diameter 1.85 cm and with an inner rod or wire of various diameters was studied. To satisfy the requirement of a hydrodynamically smooth conduit, a brass tube with a well-stretched vertical rod or wire of mild steel was preferred. The entrance was round-edged. Copper tappings of 0.4 cm diameter fixed at various distances (0, 2.5, 5.1, 7.6, 10.1, 12.7, 20.3, 28.0, 38.1, 48.2, 61.0, 73.6 cm) from the entrance were connected to mercury and carbon tetrachloride manometers for pressure gradient measurement.

Axial pressure gradients at different flow rates were recorded for water in turbulent flow through the annuli. For fully developed flow data the tappings beyond 70 equivalent diameter distance from the entrance of the annulus were employed.

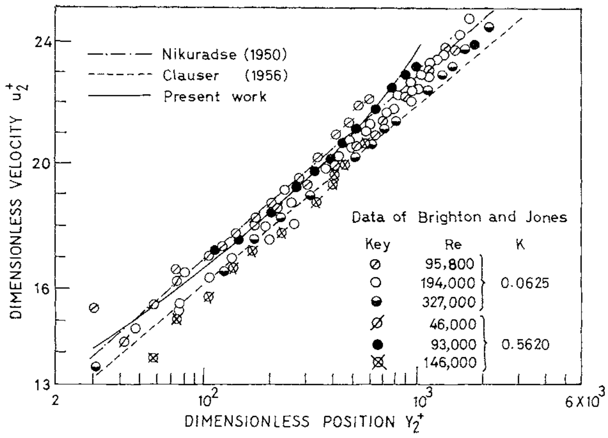

Fig. 1 Velocity profile in outer region of annulus

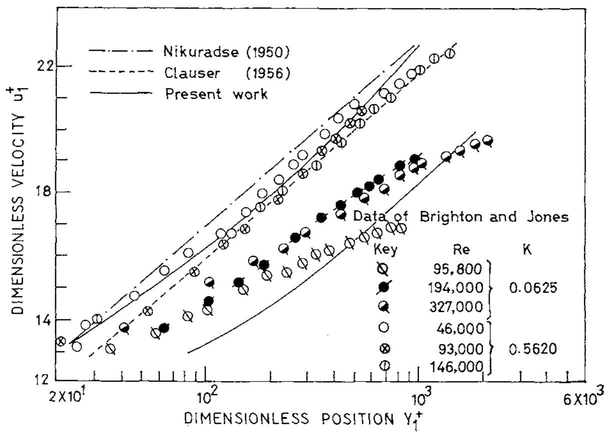

Fig. 2 Velocity profile in inner region of annulus

3. Results and Discussion

The proposed velocity profiles are compared with experimental data of Brighton and Jones[2] in **Figs. 1** and **2** along with the well-known tube flow correlations. In Fig. 1, where u_2^+ is plotted against y_2^+, good agreement is seen between the present work and experimental data of Brighton and Jones, which are at $Re<10^5$ and $y_2^+>1000$. Figure 2 presents a plot u_1^+ vs.

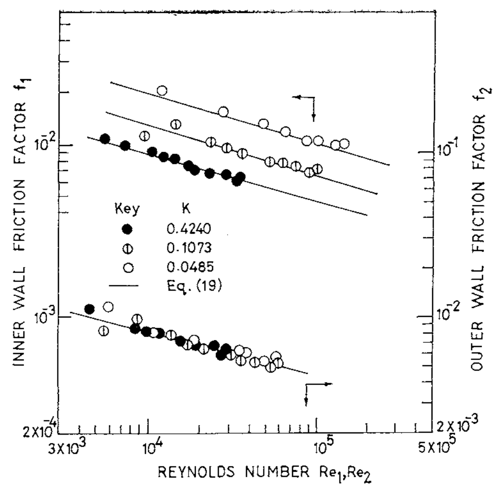

Fig. 3 Predicted and experimental friction factor with Reynolds number based on inner and outer flow region of annulus

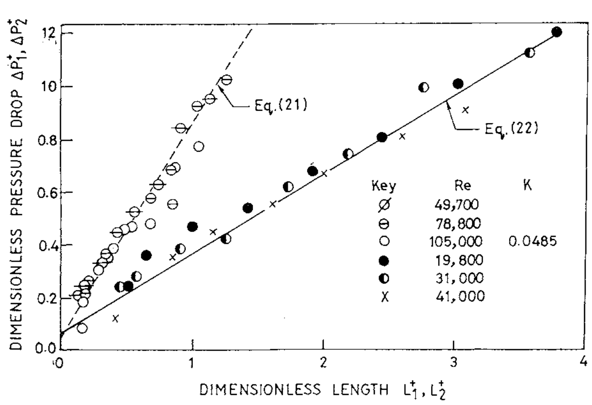

Fig. 4 Predicted and experimental dimensionless pressure drop with dimensionless length based on inner and outer regions of annulus

y_1^+ for the inner region. It is seen that the present expression agrees well with experimental data at Re=46,000 and Re=93,000, for k=0.562. The deviation of the data from the present line is less as compared to the deviation from the equation of Nikuradse and Clauser[3].

Equations (15), (17) and (18) indicate that the relationship of friction factor and Reynolds number for a tube, the outer region of an annulus and a parallel plate is almost the same. The value of $\phi_2(k)$ varies from 0.0765 to 0.0804, which can well be approximated by the constant of the Blasius equation: $f=0.079\,(Re)^{-0.25}$. Applying a simple mass balance and using the average velocity ratio from Eqs. (12) and (13) the friction factors and Reynolds numbers were calculated. In **Fig. 3** calculated values of f_2 with Re_2 are displayed and are compared with the Blasius friction equation. The data for different annuli (k= 0.4240, 0.1073, 0.0485) fall around the theoretical line and show good agreement between theoretical and experimental results. Equation (19) shows a variation in friction factor f_1 with aspect ratio. The value of f_1 equals f_2 for parallel plates and increases with decreasing aspect ratio. The experimental data for three annuli of aspect ratio, k=0.4240, 0.1073 and 0.0485 are compared with corresponding theoretical equations and are seen to be in good agreement.

Thorpe[12] has presented a logarithmic equation for the friction factor in conventional form. He has considered a mean friction factor and a Reynolds number based on total cross-sectional area. After having confirmed that the friction factor Reynolds number relationships, Eqs. (15) and (19) for outer and inner regions respectively, are in excellent agreement with the experimental data, it is now possible to express the friction factor and Reynolds number relationship based on total area.

Comparing Eqs. (12) and (13) it is seen that U_2 almost equals U_1. Therefore the total average velocity U in the annulus can be taken as approximately equal to U_2 or U_1.

Defining the mean shear stress and the mean friction factor as

$$\tau_{wm}=(\tau_{w1}r_1+\tau_{w2}r_2)/(r_1+r_2)$$
$$=f\rho U^2/2=(D_e/4)(\Delta P/L) \tag{30}$$

Equation (3) gives

$$D_{e2}/D_e=\tau_{w2}/\tau_{wm}$$
$$=(r_2^2-r_m^2)/r_2(r_2-r_1) \tag{31}$$

Substitution of f_2 and Re_2 in terms of f and Re and subsequent arrangement of terms gives the following required relationship based on total area:

$$f=\phi(k)Re^{-0.25}$$

where

$$\phi(k)=0.0201\frac{(1+\lambda)(1-k)^{5/4}}{(1-\lambda)[(7/15)+(8/15)\lambda]^{7/4}} \tag{32}$$

The entrance loss coefficient for the outer region of the annulus is a weak function of the aspect ratio and varies from 0.0759 for pipe to 0.058 for parallel plate. The loss coefficient for the inner region, however, changes much with aspect ratio, being minimum for $\lambda=1$ and increasing with decreasing aspect ratio. The curve for dimensionless pressure drop against dimensionless length is a straight line with slopes $4\phi_2(k)$ and $4\phi_1(k)$ and intercepts C_2 and C_1 for outer and inner regions respectively, as drawn in **Fig. 4**. The present experimental data at different Reynolds numbers are seen to be in agreement with the theoretical expression.

Prediction of the loss coefficient C and the entrance length, based on the total area of the annulus, will be more practical. In this case Eq. (20) may be written as

$$\Delta P^+=4\phi(k)L^++C \tag{33}$$

where

$$\Delta P^{+}=2\Delta P/\rho U^{2}$$
$$L^{+}=(L/De)(Re)^{0.25}$$

Using the respective velocity profiles in the inner and outer regions the evaluation of the integral in Eq. (24) for total area gives the loss coefficient C as

$$C=\frac{4}{1+k}\left[\frac{7}{10}M^{3}\left\{\lambda+\frac{7}{17}(1+k-2\lambda)\right\}-\frac{7}{9}M^{2}\left\{+\frac{7}{16}(1+k-2\lambda)\right\}\right] \quad (34)$$

where

$$M=u_m/U$$
$$=4(1+k)/7\{\lambda+(7/15)(1+k-2\lambda)\} \quad (35)$$

A relationship between the pressure drop and the maximum velocity in the entrance region was given by McComs[9] as

$$\Delta P^{+}=M^{2}-1 \quad (36)$$

Mathematical manipulation of Eqs. (33) and (36) yields a relation for the entrance length at which the pressure gradient becomes constant:

$$L_e/D_e=M^{2}-1-C)\{Re^{0.25}/4\phi(k)\} \quad (37)$$

This expression provides a method to estimate the entrance length of an annulus, a tube and a parallel plate channel. Equation (37) reduces to

$$L_e/D_e=1.41(Re)^{0.25} \quad (38)$$

for $k=0$, which was derived by J. T. Davies[5] for a circular smooth pipe.

Conclusions

The theoretical analysis of developing and fully developed turbulent flow of Newtonian fluid through a concentric annular passage, supported by experimental data, results in the conclusion that the flow characteristics in the two flow regions of the annulus are quite different and hence a separate analysis of these two regions should be made. The stress in the inner region is higher than that in the outer region, which may be explained by a higher value of velocity fluctuation in the inner region and a smaller wall area. Also, the energy loss per unit volume is higher in the inner region due to higher turbulence. The expression developed following the linearisation technique of Lundgren *et al.*[8] for the entrance region is extended to turbulent flow through the annulus and the theoretical expressions are in good agreement with experimental data.

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Nomenclature

A	= area of cross-section	[m²]
B, b	= constants	[—]
C	= entrance loss coefficient	[—]
D_e	= equivalent diameter	[m]
f	= friction factor	[—]
k	= aspect ratio (r_1/r_2)	[—]
L	= total length of the conduit	[m]
L^+	= dimensionless length	[—]
L_e	= entrance length	[m]
M	= ratio of maximum velocity to average velocity	[—]
P	= pressure	[N/m²]
ΔP	= axial pressure gradient	[Pa/m]
ΔP^+	= dimensionless pressure drop	[—]
Re	= Reynolds number	[—]
r	= radius	[m]
r_m	= radius of maximum velocity	[m]
r_1	= outer radius of inner pipe of annulus	[m]
r_2	= inner radius of outer pipe of annulus	[m]
U	= average velocity	[m/s]
u	= velocity	[m/s]
u_m	= maximum velocity	[m/s]
u^*	= friction velocity	[m/s]
u^+	= dimensionless velocity	[—]
y^+	= modified Reynolds number	[—]
τ_w	= shear stress at wall	[Kg/m·s²]
θ	= angular coordinate	[—]
λ	= (r_m/r_2)	[—]
ϕ	= function of k	[—]
μ	= viscosity coefficient	[Kg/m·s]
ρ	= density of fluid	[Kg/m³]
ν	= kinematic viscosity	[m²/s]

〈Subscripts〉

1	= inner flow region of an annulus
2	= outer flow region of an annulus
e	= entrance
f	= friction

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