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Inductive algebras for compact groups

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Abstract
Inductive algebras for a compact group are self-adjoint.

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compact group, representation, inductive algebra, Schur's lemma

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1 | INTRODUCTION

Let G be a separable locally compact group and π an irreducible unitary representation of G on a separable Hilbert space $\mathcal{H}$. Let $\mathcal{B}(\mathcal{H})$ denote the algebra of bounded operators on $\mathcal{H}$. An *inductive algebra* is a weakly closed abelian subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ that is normalized by $\pi(G)$, that is, $\pi(g)\mathcal{A}\pi(g)^{-1} = \mathcal{A}$ for each $g \in G$. If we wish to emphasize the dependence on π, we will use the term π-inductive algebra. A *maximal inductive algebra* is a maximal element of the set of inductive algebras, partially ordered by inclusion.

The identification of inductive algebras can shed light on the possible realizations of $\mathcal{H}$ as a space of sections of a homogeneous vector bundle (see e.g., [8–12]). For self-adjoint maximal inductive algebras, there is a precise result known as Mackey's imprimitivity theorem, as explained in the introduction to [9]. Inductive algebras have also found applications in operator theory (see e.g., [4, 5]).

In [6], it was shown that finite-dimensional inductive algebras for a connected group are trivial. However, the title of [6] is somewhat misleading, as finite groups can have non-trivial finite-dimensional inductive algebras.

In this note, we show that inductive algebras for a compact group are self-adjoint. This is significant because, in general, the classification of self-adjoint inductive algebras is easier than the classification of all inductive algebras. This is because the methods of spectral theory are available only in the former case. Also, unlike in the classification work cited above, we do not need to assume maximality.

In Section 2, we prove some results about subalgebras of $L^\infty(X, \mu)$, which will be used in the proof of our main theorem, but which are also of independent interest.

2 | SUBALGEBRAS OF L^∞

Theorem 1. *Let (X, μ) be a measure space. The algebra $L^\infty(X, \mu)$ is finite-dimensional if and only if all of its subalgebras are self-adjoint.*

Proof. Assume first that $L^\infty(X, \mu)$ is finite dimensional. Observe that under this hypothesis, if $f \in L^\infty(X, \mu)$, then there exists a simple function s such that $f = s$ almost everywhere (see [2, Proposition 3.4.2] and [7, Section 13.3, Corollary 6]).

Let $\mathcal{A} \subseteq L^\infty(X,\mu)$ be a subalgebra. Let $\{f_1, f_2, \dots, f_n\}$ be a basis for $\mathcal{A}$, and choose simple functions $s_1, s_2, \dots, s_n$ such that $f_j = s_j$ (a.e.), $j = 1, 2, \dots, n$. Define a map $\mathbf{s} : X \to \mathbb{C}^n$ by

$$\mathbf{s}(x) = (s_1(x), s_2(x), \dots, s_n(x)).$$

Since simple functions attain only finitely many values, $\mathbf{s}(X)$ is finite, and we may write

$$\mathbf{s}(X) \setminus \{0\} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\},$$

for some $m \in \mathbb{N}$.

Put $A_0 = \mathbf{s}^{-1}(0)$ and

$$A_k = \mathbf{s}^{-1}(\mathbf{v}_k), \qquad k = 1, 2, \dots, m.$$

Then $\{A_k\}_{k=0}^m$ are disjoint, and

$$X = \bigcup_{k=0}^m A_k.$$

Let v_{kj} denote the j-th component of the vector $\mathbf{v}_k$. Observe that if $x \in A_k$ then $s_j(x) = v_{kj}$, $j = 1, \dots, n$, $k = 1, \dots, m$, that is, each s_j is constant on each A_k. Therefore,

$$s_j \in \operatorname{span}\{\chi_{A_k}\}_{k=1}^m, \qquad j = 1, \dots, n.$$

Therefore $\mathcal{A} \subseteq \operatorname{span}\{\chi_{A_k}\}_{k=1}^m$.

Fix distinct $h, k \in \{1, \dots, m\}$. Since $\mathbf{v}_h \neq \mathbf{v}_k$, there exists $j = j(h,k)$ such that $v_{hj} \neq v_{kj}$. Since $\mathbf{v}_h \neq 0$, there exists $l = l(h)$ such that $v_{hl} \neq 0$. Observe that

$$\varphi_{hk} = (s_j - v_{kj}) s_l \in \mathcal{A}.$$

If $x \in A_k$, then

$$\begin{aligned}\varphi_{hk}(x) &= (s_j(x) - v_{kj}) s_l(x) \\ &= (v_{kj} - v_{kj}) v_{kl} \\ &= 0,\end{aligned}$$

and if $x \in A_h$, then

$$\begin{aligned}\varphi_{hk}(x) &= (s_j(x) - v_{kj}) s_l(x) \\ &= (v_{hj} - v_{kj}) s_l(x) \\ &= (v_{hj} - v_{kj}) v_{hl} \\ &\neq 0.\end{aligned}$$

Put

$$\psi_{hk} = \frac{\varphi_{hk}}{(v_{hj} - v_{kj}) v_{hl}}.$$

Then, $\psi_{hk} \in \mathcal{A}$, $\psi(x) = 1$ if $x \in A_h$ and $\psi(x) = 0$ if $x \in A_k$.

Since $\chi_{A_h} = \prod_{k \neq h} \psi_{hk}$, it follows that $\chi_{A_h} \in \mathcal{A}$, $h = 1, \dots, m$. Therefore $\mathcal{A} = \operatorname{span}\{\chi_{A_k}\}_{k=1}^m$. Therefore $\mathcal{A}$ is self-adjoint.

Assume now that $L^\infty(X,\mu)$ is infinite dimensional. We claim first that X has a sequence $\{E_n\}_{n=1}^\infty$ of disjoint measurable subsets of positive measure. Indeed, there exists a real-valued function $f \in L^\infty(X,\mu)$ such that

$$-\infty < \operatorname{ess\,inf} f < \operatorname{ess\,sup} f < \infty.$$

Put

$$c = \frac{\operatorname{ess\,inf} f + \operatorname{ess\,sup} f}{2}.$$

Let

$$Y = \{x \in X \,|\, f(x) > c\}, \quad \text{and} \quad Z = \{x \in X \,|\, f(x) \leq c\}.$$

Then Y and Z are disjoint measurable sets, and by the definitions of essential supremum and essential infimum $\mu(Y) > 0$ and $\mu(Z) > 0$. Since

$$L^\infty(X) \cong L^\infty(Y) \oplus L^\infty(Z),$$

either $L^\infty(Y)$ or $L^\infty(Z)$ must be infinite dimensional, say $\dim L^\infty(Z) = \infty$. Let $E_1 = Y$. We may iterate the previous argument with Z in place of X to produce the required sequence.

Choose points $e_n \in E_n$, and let $\mathcal{A}$ consist of all $f \in L^\infty(X,\mu)$ which are constant on each E_n, $n = 1, 2, \ldots$, and

$$\lim_{m\to\infty} \frac{f(e_{2m+1}) - f(e_1)}{(1/m)} = i \lim_{m\to\infty} \frac{f(e_{2m}) - f(e_1)}{(1/m)}.$$

It is easy to check that $\mathcal{A}$ is a subalgebra of $L^\infty(X,\mu)$. Now, define $f : X \to \mathbb{C}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \in E_1, \\ \frac{1}{n} & \text{if } x \in E_n,\ n \text{ even}, \\ \frac{i}{n} & \text{if } x \in E_n,\ n > 1 \text{ odd}. \end{cases}$$

Then, $f \in \mathcal{A}$ but $\overline{f} \notin \mathcal{A}$. Therefore $\mathcal{A}$ is not self-adjoint. □

3 | COMPACT GROUPS

Theorem 2. *Let G be a compact group and π an irreducible unitary representation of G on a Hilbert space $\mathcal{H}$. If $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is a π-inductive algebra, then $\mathcal{A}$ is self-adjoint.*

Proof. By the Peter–Weyl theorem, $\mathcal{H}$ is finite dimensional.

Let $\mathcal{N}$ denote the set of nilpotent elements in $\mathcal{A}$ (the nilradical of $\mathcal{A}$). Let

$$\mathcal{K} = \{x \in \mathcal{H} \mid Tx = 0, \quad \forall T \in \mathcal{N}\}.$$

By (a trivial case of) Engel's theorem [3, Section 3.3], $\mathcal{K} \neq 0$. Observe that $\mathcal{N}$ is normalized by $\pi(G)$, so $\mathcal{K}$ is $\pi(G)$-invariant. However, since π is irreducible, it follows that $\mathcal{K} = \mathcal{H}$, whence $\mathcal{N} = 0$.

Let $\mathcal{A}^*$ denote the space of linear functionals on $\mathcal{A}$. For each $\lambda \in \mathcal{A}^*$, let

$$\mathcal{H}_\lambda = \{v \in \mathcal{H} \mid Tv = \lambda(T)v \quad \text{for all } T \in \mathcal{A}\},$$

and

$$\Lambda = \{\lambda \in \mathcal{A}^* \mid \mathcal{H}_\lambda \neq 0\}.$$

Then Λ is a finite set.

Since $\mathcal{A}$ is abelian, and $\mathcal{N} = 0$, the Jordan–Chevalley decomposition [3, Section 4.2] implies that

$$\mathcal{H} = \bigoplus_{\lambda\in\Lambda} \mathcal{H}_\lambda. \tag{1}$$

Let $\langle\cdot,\cdot\rangle$ denote the inner product of $\mathcal{H}$. There exists an inner product $\langle\cdot,\cdot\rangle_1$ on $\mathcal{H}$ such that $\mathcal{H}_\lambda$ and $\mathcal{H}_\mu$ are orthogonal with respect to $\langle\cdot,\cdot\rangle_1$ if $\lambda \neq \mu$. Let σ denote the Haar probability measure on the compact group G. By Schur's lemma (see [1]), there exists a constant c such that

$$\langle v, w\rangle = c \int_G \langle \pi(g)v, \pi(g)w\rangle_1 \, d\sigma.$$

If $g \in G$ and $\lambda \in \mathcal{A}^*$, define $g\lambda : \mathcal{A} \to \mathbb{C}$ by

$$g\lambda(T) = \lambda(\pi(g)^{-1}T\pi(g)), \quad T \in \mathcal{A}.$$

This defines an action of G on $\mathcal{A}^*$, which preserves Λ.

Note that for any $g \in G$, $\lambda \in \mathcal{A}^*$ and $v \in \mathcal{H}_\lambda$, $\pi(g)v \in \mathcal{H}_{g\lambda}$. Also, $\lambda \neq \mu$ implies $g\lambda \neq g\mu$. Therefore, if $\lambda \neq \mu$, $v \in \mathcal{H}_\lambda$ and $w \in \mathcal{H}_\mu$, then

$$\begin{aligned}\langle v, w\rangle &= c \int_G \langle \pi(g)v, \pi(g)w\rangle_1 \, d\mu \\ &= 0.\end{aligned}$$

Therefore $\mathcal{H}_\lambda$ and $\mathcal{H}_\mu$ are orthogonal with respect to $\langle\cdot,\cdot\rangle$ if $\lambda \neq \mu$.

Observe that if $\lambda \in \Lambda$, then λ is multiplicative. Indeed, if $\lambda \in \Lambda$, then there exists $v \in \mathcal{H}_\lambda \setminus \{0\}$. Therefore, if $T_1, T_2 \in \mathcal{A}$, then

$$\lambda(T_1T_2)v = T_1T_2v = T_1(\lambda(T_2)v) = \lambda(T_2)T_1v = \lambda(T_2)\lambda(T_1)v.$$

Since $v \neq 0$, it follows that $\lambda(T_1T_2) = \lambda(T_1)\lambda(T_2)$.

It follows that the map $\mathcal{G} : \mathcal{A} \to L^\infty(\Lambda)$ (with respect to counting measure) defined by

$$[\mathcal{G}(T)](\lambda) = \lambda(T).$$

is an algebra homomorphism.

Since Λ is finite, $L^\infty(\Lambda)$ is finite dimensional, and so $\mathcal{G}(\mathcal{A})$ is self-adjoint by Theorem 1.

Let $T \in \mathcal{A}$. Then there exists $T_1 \in \mathcal{A}$ such that $\mathcal{G}(T_1) = \overline{\mathcal{G}(T)}$, that is, $\lambda(T_1) = \overline{\lambda(T)}$ for all $\lambda \in \Lambda$. We claim that $T_1 = T^*$, that is, that

$$\langle Tv, w\rangle = \langle v, T_1w\rangle \quad \text{for all } v, w \in \mathcal{H}.$$

By Equation (1), it suffices to check this assuming that $v \in \mathcal{H}_\lambda$ and $w \in \mathcal{H}_\mu$ for $\lambda, \mu \in \Lambda$.

If $\lambda = \mu$, then

$$\begin{aligned}\langle Tv, w\rangle &= \langle \lambda(T)v, w\rangle \\ &= \langle v, \overline{\lambda(T)}w\rangle \\ &= \langle v, T_1 w\rangle.\end{aligned}$$

If $\lambda \neq \mu$, then $\langle v, w\rangle = 0$, and so

$$\begin{aligned}\langle Tv, w\rangle &= \langle \lambda(T)v, w\rangle \\ &= 0, \quad \text{and} \\ \langle v, T_1 w\rangle &= \langle v, \mu(T_1)w\rangle \\ &= 0.\end{aligned}$$

□

Corollary 3. *Let G be a finite group and π an irreducible unitary representation of G on a Hilbert space $\mathcal{H}$. If $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is a π-inductive algebra, then $\mathcal{A}$ is self-adjoint.*

In view of Raghavan's theorem [6], it might appear that Corollary 3 may be used whenever Theorem 2 is applicable. However, that is not the case. Indeed, if $G = \mathrm{O}(2)$, the group of orthogonal 2×2 matrices, then G is compact and not abelian, but its group of components is abelian. If π is an irreducible representation of G of dimension greater than one, then Theorem 2 implies that all π-inductive algebras are self-adjoint, but Corollary 3 is not applicable.

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