J. Math. Anal. Appl. 527 (2023) 127419

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Regular Articles

Reverse Faber-Krahn inequality for the p-Laplacian in hyperbolic space

Mrityunjoy Ghosh [a,*], Sheela Verma [b]

[a] *Department of Mathematics, Indian Institute of Technology Madras, Chennai 600036, India*
[b] *Department of Mathematical Sciences, Indian Institute of Technology (BHU), Varanasi, India*

A R T I C L E I N F O

Article history:
Received 13 December 2022
Available online 16 May 2023
Submitted by J. Galvez

Keywords:
p-Laplacian
h-convexity
Steiner formula
Nagy's inequality
Reverse Faber-Krahn inequality
Interior parallels

A B S T R A C T

In this paper, we study the shape optimization problem for the first eigenvalue of the p-Laplace operator with the mixed Neumann-Dirichlet boundary conditions on multiply-connected domains in hyperbolic space. Precisely, we establish that among all multiply-connected domains of a given volume and prescribed $(n-1)$-th quermassintegral of the convex Dirichlet boundary (inner boundary), the concentric annular region produces the largest first eigenvalue. We also derive Nagy's type inequality for outer parallel sets of a convex domain in the hyperbolic space.

1. Introduction

The study of isoperimetric type inequalities for the eigenvalues of elliptic operators remains one of the most attracted areas in spectral theory after a famous conjecture by Lord Rayleigh stating that: *among all domains of the given volume, the ball minimizes the first eigenvalue* λ_1 *of the Dirichlet Laplacian*, i.e.,

$$\lambda_1(\Omega) \geq \lambda_1(B), \tag{1}$$

for all domains Ω such that $\mathrm{Vol}(\Omega)=\mathrm{Vol}(B)$. Here B represents the ball. This conjecture was proved by Faber [9] for planar Euclidean domains, and later Krahn [18] generalized it to higher dimensions. Inequality (1) is known as the *Rayleigh-Faber-Krahn inequality*. Similar results also hold for domains in Riemannian manifolds and for Robin boundary conditions; see [4,6,17] for instance. We refer to the monographs [14,15] for various such isoperimetric type problems.

In this article, we focus on the first eigenvalue of the p-Laplace operator with the mixed Neumann-Dirichlet boundary conditions on domains in the hyperbolic space. Let $\mathbb{H}^n$ denote the n-dimensional hyper-

* Corresponding author.
E-mail addresses: ghoshmrityunjoy22@gmail.com (M. Ghosh), sheela.mat@iitbhu.ac.in (S. Verma).

https://doi.org/10.1016/j.jmaa.2023.127419

bolic space with constant sectional curvature −1. Let $\Omega \subset \mathbb{H}^n$ be a bounded domain with $\partial\Omega = \Gamma_D \sqcup \Gamma_N$. For $1 < p < \infty$, the p-Laplace operator is defined as $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$. Here ∇ denotes the hyperbolic gradient. For $p = 2$, the p-Laplace operator coincides with the classical *Laplace-Beltrami* operator. We consider the following eigenvalue problem of the p-Laplace operator:

$$\left.\begin{aligned} -\Delta_p u &= \tau |u|^{p-2}u \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \Gamma_D, \\ \frac{\partial u}{\partial \eta} &= 0 \quad \text{on} \quad \Gamma_N, \end{aligned}\right\} \tag{$\mathscr{P}$}$$

where $\tau \in \mathbb{R}$ and η is the outward unit normal vector to Γ_N. A real number τ is said to be an eigenvalue of ($\mathscr{P}$) if there exists $\phi \in W^{1,p}_{\Gamma_D}(\Omega) \setminus \{0\}$ satisfying the following

$$\int_\Omega |\nabla\phi|^{p-2} \langle \nabla\phi, \nabla w\rangle \, \mathrm{d}V_g = \tau \int_\Omega |\phi|^{p-2}\phi w \, \mathrm{d}V_g, \quad \forall\, w \in W^{1,p}_{\Gamma_D}(\Omega),$$

where $\mathrm{d}V_g$ is the volume element induced by the hyperbolic metric g and $W^{1,p}_{\Gamma_D}(\Omega)$ is the space of all Sobolev functions that vanishes on Γ_D, i.e.,

$$W^{1,p}_{\Gamma_D}(\Omega) = \{u \in W^{1,p}(\Omega) : u|_{\Gamma_D} = 0\}.$$

It is well known that ($\mathscr{P}$) admits a least positive eigenvalue $\tau_1(\Omega)$ (cf. [13]) whose variational characterization is given by

$$\tau_1(\Omega) = \inf_{u \in W^{1,p}_{\Gamma_D}(\Omega)\setminus\{0\}} \left\{ \frac{\int_\Omega |\nabla u|^p \mathrm{d}V_g}{\int_\Omega |u|^p \mathrm{d}V_g} \right\} \tag{2}$$

and $\tau_1(\Omega)$ is simple.

Let $W_{n-1}(C)$ denote the $(n-1)$-th quermassintegral (see Section 2.1 for precise definition) of a convex domain C. In this article, we choose the following types of domains:

$$\left.\begin{aligned} \Omega = \Omega_N \setminus \overline{\Omega_D},\ &\text{where } \Omega_D, \Omega_N \text{ are two smooth, bounded domains in} \\ &\mathbb{H}^n \text{ such that } \Omega_D \text{ is simply connected and } \overline{\Omega_D} \subset \Omega_N. \\ A_\Omega = B_R \setminus \overline{B_r},\ &\text{where } B_R, B_r \text{ are two concentric open geodesic balls} \\ &\text{of radius } R, r\ (0 < r < R) \text{ respectively in } \mathbb{H}^n \text{ such that} \\ &|\Omega| = |A_\Omega| \text{ and } W_{n-1}(B_r) = W_{n-1}(\Omega_D). \end{aligned}\right\} \tag{$\mathscr{D}$}$$

Assume that $\Gamma_D := \partial\Omega_D$ and $\Gamma_N := \partial\Omega_N$. Here Γ_D and Γ_N, respectively, represent the Dirichlet and Neumann boundary, i.e., we consider the inner Dirichlet-outer Neumann boundary condition for ($\mathscr{P}$).

Now we state some existing isoperimetric bounds of $\tau_1(\Omega)$ for domains in the Euclidean space. Suppose Ω and A_Ω are domains in $\mathbb{R}^n$ as defined in ($\mathscr{D}$). For $\Omega \subset \mathbb{R}^2$, Hersch [16] studied problem ($\mathscr{P}$) for the classical Laplace operator and proved that A_Ω maximizes the first eigenvalue of ($\mathscr{P}$), i.e.,

$$\tau_1(\Omega) \leq \tau_1(A_\Omega).$$

The above inequality is known as the *reverse Faber-Krahn* inequality for the mixed eigenvalue problem. Note that in the planar case, the quermassintegral constraint, imposed on the Dirichlet boundary, reduces to the perimeter constraint (see Section 2.1). In [1, Theorem 1.2], Anoop and Ashok extended Hersch's result for the p-Laplacian and to the higher dimensions under the assumptions that Ω_D is a ball. Later, in [8, Theorem 1.1], the authors extended this result to the case when Ω_D is convex. The proof given by Hersch [16] is based on the *"method of interior parallels"* for planar domains. Hersch's idea was to construct a test function whose level sets are the parallel sets to the Dirichlet boundary. The key step for applying this method is the Nagy's inequality [26] for outer parallel sets of a planar domain, which is as follows:

Let $K \subset \mathbb{R}^2$ be a bounded, simply connected domain and $\delta > 0$. Let K_δ denote the set of all points in $\mathbb{R}^2$ that are at a distance (Euclidean) at most δ from K. Suppose $K^\#$ is an open ball in $\mathbb{R}^2$ of same perimeter as K, i.e., $P(K) = P(K^\#)$. Then Sz. Nagy [26] proved that

$$P(K_\delta) \leq P(K^\#_\delta). \tag{3}$$

In [1], the authors derive an analogue of the above inequality for multiply connected domains in higher dimensions under the assumption that Ω_D is a ball. However, a rigorous version of Nagy's type inequality (3) for convex domains in $\mathbb{R}^n$ $(n \geq 3)$ has been proved in [2, Corollary 3.4]. It is worth mentioning that Nagy's type inequality has its own importance as it can be applied to obtain several bounds for the first eigenvalue of Laplacian and torsional rigidity; see [21,22], for instance. To the best of our knowledge, the analogue of Hersch's result and Nagy's type inequality for the outer parallel sets are not available in the hyperbolic space. Indeed, the reverse Faber-Krahn inequality is not available even for the Laplacian ($p = 2$) in the hyperbolic space.

A Similar reverse type isoperimetric inequality is also proved for the first Robin eigenvalue of the Laplacian on an exterior of a convex set by Krejčiřík and Lotoreichik [19]. They established that among all exterior of planar convex sets with fixed perimeter or area, the first Robin eigenvalue is maximized by the exterior of a disk. Later, they extended this result (without convexity constraint) to the class of exterior of a compact set in [20].

The main objective of this article is to prove Hersch's result in the hyperbolic space $\mathbb{H}^n$. Moreover, we establish a hyperbolic version of Nagy's inequality (3). To state the main results, we need the following definitions.

Definition 1.1 *(Outer parallel set).* Let $K \subset \mathbb{H}^n$ and $\delta > 0$. Then the *Outer parallel body* of K at a distance $\delta > 0$ is defined as

$$K_\delta = \{x \in \mathbb{H}^n : d_{\mathbb{H}}(x, K) \leq \delta\},$$

where $d_{\mathbb{H}}$ is the hyperbolic distance function. The boundary ∂K_δ is called as the *Outer parallel set* of K at a distance δ.

Next, we recall the definition of *h-convex* (or *horoconvex*) domains in the hyperbolic space; cf. [11, Section 2]. For more details, see Section 2.2.

Definition 1.2 *(h-convex domain).* A domain $K \subset \mathbb{H}^n$ is said to be *h-convex* if all the principal curvatures of ∂K are bounded below by 1, i.e., if κ_i, $1 \leq i \leq n-1$, are the principal curvatures of ∂K, then $\kappa_i \geq 1$, $\forall\, 1 \leq i \leq n-1$.

Let $P(A) := |\partial A|$ denote the perimeter of a set $A \subset \mathbb{H}^n$. Now we state the first main result of this article which is an analogue of Sz. Nagy's inequality for outer parallel sets of a domain in the hyperbolic space.

Theorem 1.3 *(Nagy's inequality). Let $K \subset \mathbb{H}^n$ be a smooth, bounded, convex domain and $\delta > 0$. Let K^* be an open geodesic ball in $\mathbb{H}^n$ such that $W_{n-1}(K) = W_{n-1}(K^*)$. Then the followings hold:*

(i) *If* $n = 2$*, then* $P(K_\delta) \leq P(K_\delta^*)$*. Further, equality holds if and only if* K *is a geodesic ball.*
(ii) *If* $n \geq 3$ *and* K *is h-convex, then* $P(K_\delta) \leq P(K_\delta^*)$*. Further, equality holds if and only if* K *is a geodesic ball.*

The main ingredients to prove Theorem 1.3 are (i) the Steiner formula for convex domains in $\mathbb{H}^n$, and (ii) classical hyperbolic isoperimetric inequality (for $n = 2$) and a version of Alexandrov-Fenchel inequality involving the quermassintegrals due to Wang and Xia [28] (for $n \geq 3$). First, we express the perimeter of outer parallel sets of a convex domain in terms of a polynomial in δ using the Steiner formula. Then we derive an isoperimetric type inequality between $W_i(K)$ and $W_i(K^*)$, which gives the desired result upon substituting in the Steiner formula. At this point, it is necessary to mention that for $n = 2$, we are able to get Nagy's type estimate for the convex domains, thanks to the classical hyperbolic isoperimetric inequality that holds for any domain. However, for $n \geq 3$, we need a stronger assumption than convexity, called h-convexity. This assumption is necessary to apply a class of Alexandrov-Fenchel inequalities (Proposition 2.8) which are not available for convex domains in the hyperbolic space. The extension of Theorem 1.3 for general domains in the hyperbolic space seems a challenging open problem.

Then by applying the Nagy's inequality (Theorem 1.3), we prove the reverse Faber-Krahn inequality for domains in the hyperbolic space. More precisely, we obtain the following result.

Theorem 1.4 *(Reverse Faber-Krahn inequality). Let* Ω, A_Ω *be as defined in* ($\mathscr{D}$) *and* τ_1 *be the first eigenvalue of* ($\mathscr{P}$)*. Assume that* Ω_D *is convex for* $n = 2$ *and* Ω_D *is h-convex for* $n \geq 3$*. Then*

$$\tau_1(\Omega) \leq \tau_1(A_\Omega).$$

Moreover, equality occurs only when $\Omega = A_\Omega$*.*

To prove Theorem 1.4, we apply the method of interior parallels in the hyperbolic space with the help of Nagy's type inequality (Theorem 1.3) for outer parallel sets. Namely, we produce a test function on Ω using the first eigenfunction of A_Ω that remains constant on the outer parallel sets to Ω_D. Indeed, the construction of the test function on Ω is done in such a way that its gradient norm coincides with the first eigenfunction of A_Ω, whereas its p-norm increases. We would like to mention that the analogue of Theorem 1.4 for the case when Ω_D is a non-convex domain remains completely open (see Section 4).

The rest of this article is organized as follows. In Section 2, we discuss a few geometric tools related to the convex domains in the hyperbolic space and mention some facts about the h-convexity. The proofs of Theorem 1.3 and Theorem 1.4 are given in Section 3. Finally, in Section 4, we mention some open problems related to Nagy's type inequality and reverse Faber-Krahn inequality.

2. Preliminaries

In this section, we first discuss the notion of quermassintegrals (or mixed volumes) for a convex domain in $\mathbb{H}^n$. Then we state a few well known facts about the h-convex domains. We complete this section by providing some isoperimetric inequalities in the hyperbolic space, which will be used in later sections. Throughout the article, we denote the boundary of a set $A \subset \mathbb{H}^n$ by ∂A. Also, $P(A)$ stands for the perimeter of A, i.e., $P(A) = |\partial A|$.

2.1. Quermassintegrals & curvature integrals

Let $K \subset \mathbb{H}^n$ be a bounded, convex domain. Then the *Quermassintegrals* $W_j(K)$, for $1 \leq j \leq n-1$, is defined (cf. [23,28]) as

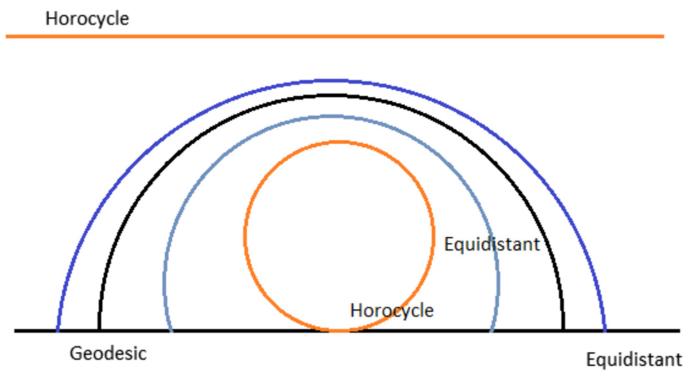

Fig. 1. Equidistants in the hyperbolic plane.

$$W_j(K) = \frac{(n-j)\omega_{j-1}\cdots\omega_0}{n\omega_{n-2}\cdots\omega_{n-j-1}} \int\limits_{\mathcal{L}_j} \chi(L_j \cap K)\mathrm{d}L_j, \tag{4}$$

where L_j is a j-dimensional totally geodesic subspace, $\mathcal{L}_j$ is the space of all totally geodesic subspaces of dimension j, $\mathrm{d}L_j$ is the natural measure on $\mathcal{L}_j$, ω_i denotes the i-dimensional Hausdorff measure of the i-dimensional unit sphere and χ is the characteristic function acting as $\chi(A) = 1$, if $A \neq \emptyset$ and $\chi(A) = 0$, if $A = \emptyset$. As a convention, we assume $W_0(K) = \mathrm{Vol}(K)$ and $W_n(K) = \frac{\omega_{n-1}}{n}$. Also we observe that $W_1(K) = \frac{P(K)}{n}$; cf. [23].

Let $\kappa_1, \kappa_2, \ldots, \kappa_{n-1}$ be the principal curvatures of ∂K and H_j, for $0 \leq j \leq n-1$, denote the normalized elementary symmetric functions of principal curvatures of ∂K. Then the *Curvature integrals* are defined by

$$V_{n-j-1}(K) = \int\limits_{\partial K} H_j \mathrm{d}S, \text{ for } j = 0, 1, \ldots, n-1, \tag{5}$$

where $\mathrm{d}S$ is the volume element on ∂K induced from $\mathbb{H}^n$. Now by [25, Proposition 7], curvature integrals and quermassintegrals are related by the following formula:

$$V_{n-j-1}(K) = n\left(W_{j+1}(K) + \frac{j}{n-j+1}W_{j-1}(K)\right), \text{ for } \ 0 \leq j \leq n-1. \tag{6}$$

2.2. Horoconvexity (h-convexity) in the hyperbolic space

We first define h-convexity in the hyperbolic plane via λ-geodesics and state some of its properties. Then we give the definition of h-convexity in higher dimensions. For more details, see [12].

Definition 2.1 *(Equidistants or λ-geodesics).* The curves which are equidistant to geodesics are called *Equidistants*. A λ-geodesic is an equidistant that meets the infinity line with angle α such that $|\cos\alpha| = \lambda$.

Remark 2.2. For $\lambda = 0(\alpha = 90°)$, equidistants are geodesics and for $\lambda = 1$, they are horocycles. The geodesic curvature of a λ geodesic is $\pm\lambda$. Some equidistants in the hyperbolic plane have been drawn in Fig. 1.

The following lemma shows the relation between positions of different λ geodesics.

Lemma 2.3. *Given any two points p and q in the hyperbolic plane and $0 < \lambda \leq 1$, there are exactly two λ-geodesics passing through them. These λ-geodesics are symmetric with respect to the geodesic passing through p and q and lie in the region bounded by the two horocycles passing through these points.*

Now we define λ-convexity of a set in the hyperbolic plane, h-convexity is the particular case of this.

Definition 2.4. For given $\lambda \in [0,1]$, a set Ω in the hyperbolic plane is said to be λ-convex if for every $p, q \in \Omega$, the λ-geodesics joining them lie inside Ω. 1-convex sets are also called h-convex sets.

Lemma 2.5. *A compact domain Ω with C^2-boundary is λ-convex if and only if the geodesic curvature k_g of the boundary satisfies $k_g \geq \lambda$ ($k_g \leq -\lambda$, in case of opposite orientation).*

Remark 2.6. If a domain is λ_0-convex then it is λ-convex for all $\lambda \leq \lambda_0$. In particular, every h-convex set is convex but converse is not true. For example, consider convex polygon.

In higher dimensions, h-convexity can be defined in the similar way. We define h-convexity of domains in n-dimensional hyperbolic space $\mathbb{H}^n$ in terms of horospheres.

A horoball is the limit of a sequence of increasing balls sharing a tangent hyperplane and its point of tangency. A horosphere is the boundary of a horoball.

Definition 2.7. A domain $\Omega \subset \mathbb{H}^n$ is said to be horoconvex (or h-convex) if, for every point $p \in \partial\Omega$, there exists a horosphere passing through the point p such that the domain Ω lies entirely in the horoball bounded by the horosphere.

2.3. Steiner formula & Alexandrov–Fenchel inequality

Let $K \subset \mathbb{H}^n$ be a smooth, bounded, convex domain and $\delta > 0$. Then by Steiner formula (cf. [23, Chapter 18, Section 4]), the volume of K_δ is given by

$$\mathrm{Vol}(K_\delta) := |K_\delta| = \mathrm{Vol}(K) + \sum_{j=0}^{n-1} \binom{n}{j} V_j(K) \int_0^\delta \cosh^j(t)\sinh^{n-j-1}(t)\mathrm{d}t. \tag{7}$$

Therefore, the perimeter $P(K_\delta) := |\partial K_\delta|$ of K_δ has the following expansion:

$$P(K_\delta) = \frac{d}{d\delta}(\mathrm{Vol}(K_\delta)) = \sum_{j=0}^{n-1} \binom{n}{j} V_j(K) \left(\frac{d}{d\delta} \int_0^\delta \cosh^j(t)\sinh^{n-j-1}(t)\mathrm{d}t \right). \tag{8}$$

Now using (6), we restate (8) in terms of the quermassintegrals of K:

$$\begin{aligned} P(K_\delta) = \sum_{j=0}^{n-2} n\binom{n}{j} \left\{ W_{n-j}(K) + \frac{n-j-1}{j+2} W_{n-j-2}(K) \right\} \left(\frac{d}{d\delta} \int_0^\delta \cosh^j(t)\sinh^{n-j-1}(t)\mathrm{d}t \right) \\ + n^2 W_1(K) \left(\frac{d}{d\delta} \int_0^\delta \cosh^{n-1}(t)\mathrm{d}t \right). \end{aligned} \tag{9}$$

Hyperbolic Isoperimetric inequality: Let γ be a closed curve in $\mathbb{H}^2$ and K be the domain enclosed by γ. Then the hyperbolic isoperimetric inequality (cf. [24]) states that

$$P(K)^2 \geq 4\pi|K| + |K|^2, \tag{10}$$

where $|K|$ is the area of K and $P(K)$ is the length of γ. Furthermore, equality occurs if and only if γ is a circle.

Next, we state an isoperimetric inequality between the quermassintegrals of an h-convex domain in hyperbolic space obtained by Wang and Xia [28, Theorem 1.1].

Proposition 2.8 *(Alexandrov–Fenchel inequality in $\mathbb{H}^n$). Let $K \subset \mathbb{H}^n$ be a smooth, h-convex, bounded domain. Also let $f_m : [0,\infty) \longmapsto \mathbb{R}_+$ be defined by $f_m(r) = W_m(B_r)$, where B_r is the geodesic ball of radius r. Then for $0 \le i < j \le n-1$,*

$$W_j(K) \ge f_j \circ f_i^{-1}(W_i(K)).$$

The equality occurs if and only if K is a geodesic ball.

3. Main results

In this section, we first give a proof of Theorem 1.3.

Proof of Theorem 1.3. (i) Since $n = 2$, we have $P(K) = P(K^*)$, i.e., $W_1(K) = W_1(K^*)$. Therefore, by isoperimetric inequality (10), we have $|K| \le |K^*|$, i.e., $W_0(K) \le W_0(K^*)$. Hence from Steiner formula (9), we get

$$\begin{aligned} P(K_\delta) &\le 2\Big\{W_2(K^*) + \frac{1}{2}W_0(K^*)\Big\}\left(\frac{d}{d\delta}\int_0^\delta \sinh(t)\mathrm{d}t\right) + 4W_1(K^*)\left(\frac{d}{d\delta}\int_0^\delta \cosh(t)\mathrm{d}t\right) \\ &= P(K^*_\delta). \end{aligned}$$

The equality case follows immediately from the isoperimetric inequality (10).

(ii) Given that

$$W_{n-1}(K) = W_{n-1}(K^*). \tag{11}$$

Let $0 \le j < n-1$. Since K is h-convex, applying Proposition 2.8 for j and $n-1$ and using (11), we get

$$f_{n-1} \circ f_j^{-1}(W_j(K)) \le W_{n-1}(K) = W_{n-1}(K^*) = f_{n-1} \circ f_j^{-1}(W_j(K^*)).$$

Thus

$$f_{n-1} \circ f_j^{-1}(W_j(K)) \le f_{n-1} \circ f_j^{-1}(W_j(K^*)). \tag{12}$$

Now from (4), we observe that if $r_1 < r_2$, then $W_i(B_{r_1}) < W_i(B_{r_2})$, for all $0 \le i \le n-1$. Thus the function $r \longmapsto f_i(r)$ is a strictly increasing function for all $0 \le i \le n-1$. Therefore inequality (12) immediately gives that

$$W_j(K) \le W_j(K^*), \text{ for all } 0 \le j < n-1. \tag{13}$$

Also the equality occurs in (13) only when K is a geodesic ball (by Proposition 2.8). Thus using (13) in (9), we have

$$P(K_\delta) \le \sum_{j=0}^{n-2} n\binom{n}{j}\Big\{W_{n-j}(K^*) + \frac{n-j-1}{j+2}W_{n-j-2}(K^*)\Big\}\left(\frac{d}{d\delta}\int_0^\delta \cosh^j(t)\sinh^{n-j-1}(t)\mathrm{d}t\right)$$

$$+ n^2 W_1(K^*) \left(\frac{d}{d\delta} \int_0^\delta \cosh^{n-1}(t)\mathrm{d}t \right)$$
$$= P(K_\delta^*).$$

Hence $P(K_\delta) \le P(K_\delta^*)$. Now the equality case follows from (13). This completes the proof. □

Remark 3.1. Note that for $n = 2$, $W_{n-1}(K) = W_{n-1}(K^*)$ implies that $P(K) = P(K^*)$ (see Section 2.1). However, if $n \ge 3$, then $P(K) < P(K^*)$.

Next, we prove an auxiliary result which is needed to prove our main result. Let us start with a notational set up. Let Ω and A_Ω be as stated in ($\mathscr{D}$), i.e., $\Omega = \Omega_N \setminus \overline{\Omega_D}$ and $A_\Omega = B_R \setminus \overline{B_r}$. Also, we have $W_{n-1}(\Omega_D) = W_{n-1}(B_r)$, i.e., $\Omega_D^* = B_r$. Define

$$\mathcal{B}(\delta) = \partial\Omega_{D_\delta} \cap \Omega,\ L(\delta) = |\mathcal{B}(\delta)|, \text{ for } \delta > 0,$$
$$\delta_0 = \sup\{\delta > 0 : \mathcal{B}(\delta) \ne \emptyset\}.$$

Note that $L(\delta) \le P(\Omega_{D_\delta})$, $\forall\ \delta > 0$. We set $\widetilde{L}(\delta) := P(B_{r_\delta})$ for simplicity. For $p \in (1, \infty)$, we construct M and $\widetilde{M}$ as follows:

$$M(\delta) = \int_0^\delta \frac{1}{L(r)^{p'-1}}\mathrm{d}r, \quad \widetilde{M}(\delta) = \int_0^\delta \frac{1}{\widetilde{L}(r)^{p'-1}}\mathrm{d}r, \tag{14}$$

where $p' = \frac{p}{p-1}$ is the holder conjugate of p.

Remark 3.2.

(i) Note that, both $\delta \longmapsto M(\delta)$ and $\delta \longmapsto \widetilde{M}(\delta)$ are strictly increasing functions on $[0, \delta_0]$ and $[0, R - r]$, respectively. Moreover, $M(\delta_0)$ can be infinite also.
(ii) Observe that, by Theorem 1.3, $L(\delta) \le \widetilde{L}(\delta)$ for all $\delta > 0$. Therefore, by the definitions of M and $\widetilde{M}$, we immediately have $\widetilde{M}(\delta) \le M(\delta)$ for all $\delta > 0$.

The euclidean version of the following lemma has been proved in [1, Lemma 2.7] when $\Omega_D \subset \mathbb{R}^n$ is a ball, and in [2, Lemma 5.2] when $\Omega_D \subset \mathbb{R}^n$ is convex. We generalize these results for $\Omega_D \subset \mathbb{H}^n$ using the hyperbolic analogue of Nagy's inequality.

Lemma 3.3. *Suppose Ω, A_Ω, M and $\widetilde{M}$ are as mentioned above. Assume that Ω_D is convex for $n = 2$ and Ω_D is h-convex for $n \ge 3$. Then the followings hold:*

(i) $R - r \le \delta_0$, with equality occurs only when Ω is a concentric annular region.
(ii) Define

$$G(\beta) = L(M^{-1}(\beta)),\ \textit{for}\ \beta \in [0, M(\delta_0)]$$
$$\textit{and} \qquad \widetilde{G}(\beta) = \widetilde{L}(\widetilde{M}^{-1}(\beta)),\ \textit{for}\ \beta \in [0, \widetilde{M}(R - r)].$$

Then $G(\beta) \le \widetilde{G}(\beta)$, for all $\beta \in [0, \widetilde{M}(R - r)]$ and equality holds if and only if Ω is a concentric annulus. Furthermore, if Ω is not a concentric annulus, then $G(\beta) < \widetilde{G}(\beta)$ on $[\beta', \widetilde{M}(R - r)]$, for some $\beta' \in [0, \widetilde{M}(R - r)]$.

Proof. (i) If possible, let $R-r>\delta_0$. Now by Theorem 1.3, we have $L(\delta)\le\widetilde{L}(\delta)$. Therefore,

$$|A_\Omega| = \int_0^{R-r}\widetilde{L}(\delta)\mathrm{d}\delta \ge \int_0^{\delta_0} L(\delta)\mathrm{d}\delta + \int_{\delta_0}^{R-r}\widetilde{L}(\delta)\mathrm{d}\delta = |\Omega| + \int_{\delta_0}^{R-r}\widetilde{L}(\delta)\mathrm{d}\delta > |\Omega|,$$

which is a contradiction as $|\Omega|=|A_\Omega|$ (by assumption). Hence $R-r\le\delta_0$. Now if $\delta_0=R-r$, then

$$\int_0^{R-r}\widetilde{L}(\delta)\mathrm{d}\delta = \int_0^{\delta_0} L(\delta)\mathrm{d}\delta \implies \int_0^{R-r}(\widetilde{L}(\delta)-L(\delta))\mathrm{d}\delta = 0.$$

Observe that both L and $\widetilde{L}$ are continuous function and hence $L(\delta)=\widetilde{L}(\delta)$, for all $\delta\in[0,R-r]$. Further, by applying Theorem 1.3 for Ω_D, we get

$$L(\delta)\le P(\Omega_{D_\delta})\le P(B_{r_\delta})=\widetilde{L}(\delta),\quad \text{for all } \delta\in[0,R-r].$$

Thus we have $P(\Omega_{D_\delta})=P(B_{r_\delta})$. Therefore, by Theorem 1.3, it follows that Ω_D must be a geodesic ball. Also since $\delta_0=R-r$, Γ_N has to be a geodesic sphere. Hence Ω must be a concentric annulus.

(ii) Let $M_*=M(\delta_0)$ and $\widetilde{M}_*=\widetilde{M}(R-r)$. Then using (i) and Remark 3.2, we have $\widetilde{M}_*\le M_*$. Therefore,

$$M^{-1}(\beta)\le\widetilde{M}^{-1}(\beta),\quad \text{for all}\quad \beta\in[0,\widetilde{M}_*].$$

Since $\delta\longmapsto\widetilde{L}(\delta)$ is an strictly increasing function on $[0,R-r]$, we get

$$G(\beta)=L(M^{-1}(\beta))\le\widetilde{L}(M^{-1}(\beta))\le\widetilde{L}(\widetilde{M}^{-1}(\beta))=\widetilde{G}(\beta).$$

Moreover, if $G(\beta)=\widetilde{G}(\beta)$, then $L(\delta)=\widetilde{L}(\delta)$, for all $\delta\in[0,R-r]$. Thus the equality case follows immediately from (i). Now if Ω is not a concentric annulus, then by (i), there exists $\delta'\in[0,R-r]$ such that $L(\delta')<\widetilde{L}(\delta')$. Thus $\widetilde{M}(\delta')<M(\delta')$ and hence $\widetilde{M}(\delta)<M(\delta)$ for all $\delta'\le\delta\le R-r$. Now substituting $\beta'=M(\delta')$ gives the desired conclusion. □

Now we state few properties of a first eigenfunction of $(\mathscr{P})$ associated to τ_1.

Proposition 3.4. *Let Ω be as mentioned in $(\mathscr{D})$ and $\tau_1(\Omega)$ be the first eigenvalue of $(\mathscr{P})$ on Ω. Suppose that u is an eigenfunction associated to $\tau_1(\Omega)$. Then*

(i) u has constant sign;
(ii) $u\in C^1(\Omega)$;
(iii) if v is an eigenfunction associated to $\tau_1(A_\Omega)$, v is radially constant and radially increasing.

Proof. (i) Notice that, if u is a minimizer for (2), then $|u|$ is so. Since $\tau_1(\Omega)$ is simple, there exists $c\in\mathbb{R}$ such that $|u|=cu$. Hence the conclusion follows.

(ii) Proof follows along the same line as [3, Theorem 1.3] or [27].

(iii) Let v be a positive eigenfunction associated to $\tau_1(A_\Omega)$ such that $||v||_p^p=\int_{A_\Omega}|v|^p\mathrm{d}V_g=1$. Also let H be a geodesic hyperplane passing through the center of A_Ω and σ_H denotes the reflection through H. Define, $w=v\circ\sigma_H$. Then it is easy to observe that $w\in W^{1,p}_{\Gamma_D}(A_\Omega)$, $\int_{A_\Omega}|w|^p\mathrm{d}V_g=1$, and $\int_{A_\Omega}|w|^p\mathrm{d}V_g=\int_{A_\Omega}|v|^p\mathrm{d}V_g$. Therefore, w is also an eigenfunction associated to $\tau_1(A_\Omega)$. Hence by the simplicity of $\tau_1(A_\Omega)$, there exists

$c > 0$ such that $w = cv$. Using $||w||_p = ||v||_p$, we get $c = 1$. Since H is arbitrary, we conclude that v is symmetric with respect to any geodesic hyperplane passing through the center of A_Ω. Hence v is radial.

Recall that $A_\Omega = B_R \backslash \overline{B_r}$, where B_r, B_R are two concentric open geodesic balls of radius r, R $(0 < r < R)$, respectively, in $\mathbb{H}^n$. Since v is radial, we can write $v(x) = f(s)$, where $s = |x|$ (Hyperbolic norm), for some function $f : [r, R] \to \mathbb{R}$. Now f satisfies the following ordinary differential equation associated to $(\mathscr{P})$:

$$-\left(|f'(s)|^{p-2} f'(s) s^{n-1}\right)' = \tau_1(A_\Omega) s^{n-1} f(s)^{q-1} \text{ for } s \in (r, R),$$
$$f(r) = 0\ ;\ f'(R) = 0.$$

Since f is positive (as v is so) in (r, R), we get $\left(|f'(s)|^{p-2} f'(s) s^{n-1}\right)' < 0$ for $s \in (r, R)$. Therefore, $|f'(s)|^{p-2} f'(s) s^{n-1}$ is strictly decreasing and hence using the boundary condition at R, we get

$$|f'(s)|^{p-2} f'(s) s^{n-1} > 0 \text{ for } s \in (r, R).$$

Hence $f' > 0$ in (r, R). $\square$

Now we give a proof of Theorem 1.4. To prove our result, we adapt the ideas used in [1,2,16] to the hyperbolic space.

Proof of Theorem 1.4. Let v be an eigenfunction of $(\mathscr{P})$ associated to $\tau_1(A_\Omega)$. Then by Proposition 3.4, v is radial and it can be chosen positive in A_Ω, i.e., $v > 0$ and $v(x) = v(d_{\mathbb{H}}(x, \partial B_r))$, for all $x \in A_\Omega$, where $d_{\mathbb{H}}$ is the hyperbolic distance function. Let $\widetilde{M}$ be as defined in (14). Now we represent v in terms of $\widetilde{M}$ in the following way:

$$v(x) = v(d_{\mathbb{H}}(x, \partial B_r)) = (v \circ \widetilde{M}^{-1})(\widetilde{M}(d_{\mathbb{H}}(x, \partial B_r))),\ \forall\, x \in A_\Omega.$$

Let $f = v \circ \widetilde{M}^{-1}$. Then $v(x) = (f \circ \widetilde{M})(d_{\mathbb{H}}(x, \partial B_r))$, $\forall\, x \in A_\Omega$. If $M_* = M(\delta_0)$ and $\widetilde{M}_* = \widetilde{M}(R - r)$, then $\widetilde{M}_* \le M_*$. Recall that $\Gamma_D = \partial\Omega_D$. Now define $u : \Omega \longrightarrow \mathbb{R}$ as

$$u(x) = \begin{cases} (f \circ M)(d_{\mathbb{H}}(x, \Gamma_D)), & \text{if} \quad M(d_{\mathbb{H}}(x, \Gamma_D)) \in [0, \widetilde{M}_*], \\ f(\widetilde{M}_*), & \text{if} \quad M(d_{\mathbb{H}}(x, \Gamma_D)) \in (\widetilde{M}_*, M_*]. \end{cases}$$

Note that $d_{\mathbb{H}}(\cdot, \Gamma_D)$ is a Lipschitz function. Also f is C^1 as v is so (by Proposition 3.4-(ii)). Thus $u \in W^{1,p}(\Omega)$. Further, $u(x) = 0$ for all $x \in \Gamma_D$. Hence $u \in W^{1,p}_{\Gamma_D}(\Omega)$. Now using the fact that $|\nabla d_{\mathbb{H}}(x, \Gamma_D)| = 1$, $\forall\, x \in \Omega$ and by the Coarea formula [10, Theorem 3.1], we get

$$\begin{aligned}
\int_\Omega |\nabla u(x)|^p \mathrm{d}V_g &= \int_\Omega |\nabla u(x)|^p |\nabla d_{\mathbb{H}}(x, \Gamma_D)| \mathrm{d}V_g \\
&= \int_0^{M^{-1}(\widetilde{M}_*)} \left(\int_{\{x \in \Omega:\ d_{\mathbb{H}}(x, \Gamma_D) = \delta\}} |\nabla u(x)|^p \mathrm{d}S \right) \mathrm{d}\delta \\
&= \int_0^{M^{-1}(\widetilde{M}_*)} \left(|f'(M(\delta))|^p |M'(\delta)|^p \right) \left(\int_{\mathcal{B}(\delta)} \mathrm{d}S \right) \mathrm{d}\delta \\
&= \int_0^{M^{-1}(\widetilde{M}_*)} \left(|f'(M(\delta))|^p |M'(\delta)|^p \right) L(\delta) \mathrm{d}\delta
\end{aligned}$$

$$= \int_0^{M^{-1}(\widetilde{M}_*)} \frac{|f'(M(\delta))|^p}{L(\delta)^{p'-1}} \mathrm{d}\delta = \int_0^{\widetilde{M}_*} |f'(\beta)|^p \mathrm{d}\beta,$$

where we make a change of variable $M(\delta) = \beta$ in the last step. Thus

$$\int_\Omega |\nabla u(x)|^p \mathrm{d}V_g = \int_0^{\widetilde{M}_*} |f'(\beta)|^p \mathrm{d}\beta. \tag{15}$$

Also

$$\begin{aligned}
&\int_\Omega |u(x)|^p \mathrm{d}V_g \\
&= \int_0^{M^{-1}(M_*)} \left(\int_{\{x\in\Omega:\ d_{\mathbb{H}}(x,\Gamma_D)=\delta\}} |u(x)|^p \mathrm{d}S \right) \mathrm{d}\delta \\
&= \int_0^{M^{-1}(\widetilde{M}_*)} \left(\int_{\{x\in\Omega:\ d_{\mathbb{H}}(x,\Gamma_D)=\delta\}} |u(x)|^p \mathrm{d}S \right) \mathrm{d}\delta + \int_{M^{-1}(\widetilde{M}_*)}^{\delta_0} \left(\int_{\{x\in\Omega:\ d_{\mathbb{H}}(x,\Gamma_D)=\delta\}} |u(x)|^p \mathrm{d}S \right) \mathrm{d}\delta \\
&= \int_0^{\widetilde{M}_*} f(\beta)^p L(M^{-1}(\beta))^{p'} \mathrm{d}\beta + f(\widetilde{M}_*)^p \int_{\widetilde{M}_*}^{M_*} L(M^{-1}(\beta))^{p'} \mathrm{d}\beta. \qquad [\text{putting } M(\delta) = \beta]
\end{aligned}$$

Therefore,

$$\int_\Omega |u(x)|^p \mathrm{d}V_g = \int_0^{\widetilde{M}_*} f(\beta)^p G(\beta)^{p'} \mathrm{d}\beta + f(\widetilde{M}_*)^p \int_{\widetilde{M}_*}^{M_*} G(\beta)^{p'} \mathrm{d}\beta, \tag{16}$$

where G is as defined in Lemma 3.3-(ii). By similar methods, we can show that

$$\int_{A_\Omega} |\nabla v(x)|^p \mathrm{d}V_g = \int_0^{\widetilde{M}_*} |f'(\beta)|^p \mathrm{d}\beta, \tag{17}$$

$$\int_{A_\Omega} |v(x)|^p \mathrm{d}V_g = \int_0^{\widetilde{M}_*} f(\beta)^p \widetilde{G}(\beta)^{p'} \mathrm{d}\beta. \tag{18}$$

Observe that, by Proposition 3.4-(iii), v attains its maxima on ∂B_R and hence we have $f(\beta) \le f(\widetilde{M}_*)$ for all $\beta \in [0, \widetilde{M}_*]$. Thus from (16), (18) and using Lemma 3.3-(ii), we get

$$\begin{aligned}
&\int_{A_\Omega} |v(x)|^p \mathrm{d}V_g - \int_\Omega |u(x)|^p \mathrm{d}V_g \\
&\le f(\widetilde{M}_*)^p \int_0^{\widetilde{M}_*} \left(\widetilde{G}(\beta)^{p'} - G(\beta)^{p'} \right) \mathrm{d}\beta - f(\widetilde{M}_*)^p \int_{\widetilde{M}_*}^{M_*} G(\beta)^{p'} \mathrm{d}\beta
\end{aligned}$$

$$= f(\widetilde{M}_*)^p \int_0^{\widetilde{M}_*} \widetilde{G}(\beta)^{p'} \mathrm{d}\beta - f(\widetilde{M}_*)^p \int_0^{M_*} G(\beta)^{p'} \mathrm{d}\beta$$

$$= f(\widetilde{M}_*)^p \int_0^{\widetilde{M}_*} \Big(\widetilde{L}(\widetilde{M}^{-1}(\beta))\Big)^{p'} \mathrm{d}\beta - f(\widetilde{M}_*)^p \int_0^{M_*} \Big(L(M^{-1}(\beta))\Big)^{p'} \mathrm{d}\beta$$

$$= f(\widetilde{M}_*)^p \int_0^{R-r} \widetilde{L}(\delta)\mathrm{d}\delta - f(\widetilde{M}_*)^p \int_0^{\delta_0} L(\delta)\mathrm{d}\delta = f(\widetilde{M}_*)^p \Big(|A_\Omega| - |\Omega|\Big).$$

Since by assumption $|\Omega| = |A_\Omega|$, we have

$$\int_{A_\Omega} |v(x)|^p \mathrm{d}V_g \leq \int_\Omega |u(x)|^p \mathrm{d}V_g, \tag{19}$$

where equality occurs only when Ω is a concentric annular region (by Lemma 3.3-(ii)). Now the assertion follows substituting (15), (17) and (19) in the variational characterization (2) of τ_1. Further, equality case immediately comes from the equality case in (19). This completes the proof. □

Thermal insulation problem: Let $\Omega = \Omega_N \setminus \overline{\Omega_D}$ be a smooth, doubly connected domain in $\mathbb{H}^n$ as defined in ($\mathscr{D}$). For $p \in (1, \infty)$, let us consider the following boundary value problem on Ω:

$$\left.\begin{aligned} -\Delta_p u &= 0 && \text{in} \quad \Omega, \\ u &= 1 && \text{on} \quad \Omega_D, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \eta} + \beta |u|^{p-2} u &= 0 && \text{on} \quad \partial\Omega_N, \end{aligned}\right\} \tag{$\mathscr{T}$}$$

where $\beta > 0$ is a real parameter and η is the outward unit normal to $\partial\Omega_N$. Then the energy functional $\mathcal{E}(\Omega_D, \Omega)$ associated to ($\mathscr{T}$) is given by

$$\mathcal{E}(\Omega_D, \Omega) = \inf_{v \in W^{1,p}(\Omega_N), v\equiv 1 \text{ in } \Omega_D} \left\{ \int_\Omega |\nabla v|^p + \beta \int_{\partial\Omega_N} |v|^p \right\}. \tag{20}$$

These types of problems arise in the study of thermal insulation, where a body Ω_D of constant temperature remains surrounded by an insulating material $\Omega_N \setminus \overline{\Omega_D}$ and $\mathcal{E}(\Omega_D, \Omega)$ represents the energy of the system; we refer to the book [5] for an overview of such problems. Now it is natural to look for the critical configurations of Ω_D and Ω_N so that the energy $\mathcal{E}(\Omega_D, \Omega)$ is optimized. For planar Euclidean domains, in [7, Theorem 3.1], authors proved that

$$\mathcal{E}(\Omega_D, \Omega_D + \delta B_1) \leq \mathcal{E}(\Omega_D^{\#}, \Omega_D^{\#} + \delta B_1),$$

where $\delta > 0$, $\Omega_D^{\#}$ is an open ball with the same perimeter as Ω_D, and B_1 is the open Euclidean ball of radius one centered at the origin. Here $\Omega_D + \delta B_1 := \{x + \delta y : x \in \Omega_D, y \in B_1\}$. The similar result holds in higher dimensions also if Ω_D is convex and $\Omega_D^{\#}$ is replaced by Ω_D^*, where Ω_D^* is the open Euclidean ball centered at the origin such that $W_{n-1}(\Omega_D^*) = W_{n-1}(\Omega_D)$; cf. [7, Theorem 4.1]. We would like to stress that the hyperbolic analogue of these results can be proved using a similar method developed in this article. To be precise, we can prove the following result.

Theorem 3.5. *Let $\Omega_D \subset \mathbb{H}^n$ be a smooth, simply connected domain and $\Omega_N = \Omega_D + \delta B_1$, for some $\delta > 0$, i.e., $\Omega = (\Omega_D + \delta B_1) \setminus \overline{\Omega_D}$. Let $\mathcal{E}(\Omega_D, \Omega)$ be the energy associated to ($\mathscr{T}$) as defined in (20). Then the following holds:*

(i) if $n = 2$ and Ω_D is convex, then $\mathcal{E}(\Omega_D, \Omega) \leq \mathcal{E}(\Omega_D^{\#}, \Omega_D^{\#} + \delta B_1)$, where $\Omega_D^{\#}$ is an open geodesic ball with same perimeter as Ω_D.

(ii) if $n \geq 2$ and Ω_D is h-convex, then $\mathcal{E}(\Omega_D, \Omega) \leq \mathcal{E}(\Omega_D^{*}, \Omega_D^{*} + \delta B_1)$, where Ω_D^{*} is an open geodesic ball with same $(n-1)$-th quermassintegral as Ω_D.

Moreover, equality occurs in both the above cases when Ω_D is an open geodesic ball in $\mathbb{H}^n$.

4. Final comments and open problems

Remark 4.1. We conclude this article by introducing some immediate open questions.

(i) As pointed out in Remark 3.1, for $n \geq 3$, K^* has a larger perimeter than K. It is not known whether Theorem 1.3-(ii) will hold or not if K^* is replaced by $K^{\#}$, where $K^{\#}$ is an open geodesic ball such that $P(K) = P(K^{\#})$.

(ii) For $n = 2$, we have established the reverse Faber-Krahn inequality (Theorem 1.4) under the assumption that Ω_D is a convex (geodesically) domain in $\mathbb{H}^n$. This assumption is necessary to apply the hyperbolic Steiner formula (7). Therefore, our approach of proofs is not applicable if Ω_D is not convex. Of course, it could be an interesting problem to study when Ω_D is a non-convex domain. However, this seems to be a challenging problem at this moment.

(iii) For $n \geq 3$, we proved Theorem 1.4 when Ω_D is a h-convex domain in $\mathbb{H}^n$. Such assumption is essential in order to use the hyperbolic Alexandrov-Fenchel inequality (Theorem 2.8) which is a crucial tool in proving Nagy's type inequality (Theorem 1.3-(ii)). To the best of our knowledge, a similar version of Alexandrov-Fenchel inequality is not available in $\mathbb{H}^n$ if the domain is not h-convex.

Acknowledgments

The authors are grateful to Prof. T. V. Anoop for his valuable comments and many fruitful discussions about the results of this article. The authors would also like to thank the anonymous reviewer for their valuable comments and suggestions.

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