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FRÉCHET SUBDIFFERENTIAL CALCULUS FOR INTERVAL-VALUED FUNCTIONS AND ITS APPLICATIONS IN NONSMOOTH INTERVAL OPTIMIZATION

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Abstract. To deal with nondifferentiable interval-valued functions (IVFs) (not necessarily convex), we present the notion of Fréchet subdifferentiability or gH-Fréchet subdifferentiability. We explore its relationship with gH-differentiability and develop various calculus results for gH-Fréchet subgradients of extended IVFs. By using the proposed notion of subdifferentiability, we derive new necessary optimality conditions for unconstrained interval optimization problems (IOPs) with nondifferentiable IVFs. We also provide a necessary condition for unconstrained weak sharp minima of an extended IVF in terms of the proposed notion of subdifferentiability. Examples are pesented to support the main results.
Keywords. Interval-valued functions; Interval optimization; gH-Fréchet subgradient; Weak efficient solution; Weak sharp minima.

1. INTRODUCTION

It is well-known that the presence of nonsmoothness is inevitable in modern optimization and variational analysis. Nonsmoothness naturally enters not only through initial data of optimization problems but largely via variational principles and perturbation techniques applied to problems with smooth data. In convex optimization, subgradient acts as an essential tool to deal with nonsmooth convex objective functions. Applications of subgradient–based methods in convex optimization is now vastly well-known [1, 2, 3]. Bazaraa et al. [4] extended the notion of subgradients for nonconvex functions under the name of Fréchet subgradients. Fréchet subgradients, introduced in [4], proved a striking tool to deal with nonsmooth optimization problems. By using Fréchet subdifferential calculus, Kruger and Mordukhovich [5] provided several optimality conditions for unconstrained and constrained optimization problems. Apart from optimization problems, Fréchet subgradient played a prominent role in nonsmooth analysis, stochastic control, differential games, etc. There is abundant literature on applications of Fréchet subdifferential calculus (for instance, see [6, 7, 8, 9, 10], and the references therein).

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In practice, the coefficients of objective functions in practical optimization problems in engineering, economics and computer models are often uncertain or imprecise due to measurement errors or some unexpected factors [11, 12, 13, 14]. Sometimes one can only determine a range of values for the coefficients of the objective functions in these optimization problems. These problems are called as interval optimization problems (IOPs). Interval optimization techniques provide an alternative choice to deal with uncertainty in optimization problems. In this article, we aim to deal with IOPs with nonsmooth and nonconvex IVFs.

1.1. **Literature survey.** To the best of the knowledge of the authors, Kruger [15] was the first to use Fréchet subgradients in nonconvex optimization problems. In [15], Kruger proposed necessary optimality conditions for optimization problems with nonconvex objective functions. Later, several other articles also reported optimality conditions with the help of Fréchet subgradients [16, 17, 18]. To enlarge the existing Fréchet subdifferential calculus, Mordukhovich et al. [19] derived exact calculus results for Fréchet subgradients. In [19], exact calculus results were further used to obtain necessary optimality conditions for nonconvex optimization problems with geometric constraints. In 2002, Mordukhovich and Wang [20] proposed Fréchet subdifferential variational principle to effectively deal with general variational problems.

IOPs with nonsmooth and nonconvex IVFs are not extensively studied yet. Many authors proposed several optimality conditions to solve unconstrained and constrained IOPs with differentiable objective function (see [21, 22, 23, 24, 25, 26, 27, 28]). The authors of [29, 30, 31, 32] presented several optimality conditions for nonsmooth convex IOPs by assuming that the lower function $\underline{f}$ and the upper function $\overline{f}$ are explicitly known for the objective function $\mathbf{F}(x) = [\underline{f}(x), \overline{f}(x)]$. It is to observe that even for a very simple IVF $\mathbf{F}$, it is not always an easy task to find the expressions of $\underline{f}(x)$ and $\overline{f}(x)$, for instance, take

$$\mathbf{F}(x_1, x_2) = \frac{[-2,3] \odot \cos x_1 + [-1,2] \odot x_2}{[-1,2] \odot \sin x_2 + [-1,2] \odot x_1}.$$

In [33], optimality conditions and duality results for nonsmooth convex IOPs using the parametric representation of its objective and constraint functions are found. However, the parametric process is also practically difficult, because in the parametric process, the number of variables increases with the number of intervals involved in the IVFs, and to verify any property of an IVF one has to verify it for an infinite number of its corresponding real-valued functions (for instance, see Definition 9 in [33]). To overcome these drawbacks, Chauhan et al. [34] coined a new notion of gH-subgradients for convex IVFs. This new notion of gH-subgradients neither requires the parametric form nor the explicit form of the objective function $\mathbf{F}$. However, the optimality results given in [34] assume that the objective function of the IOP is convex, which is also not a mild condition. For instance, a very simple IVF, $\mathbf{F}(x) = [-1,2] \odot x^2$ is not convex. So, the optimization problems with even such a simple objective function cannot be analyzed using the available techniques of IOPs. Surprisingly, till date there are no methods available to solve nonsmooth IOPs with *nonconvex* objective function.

1.2. **Motivation and contribution.** We thus, in this study, introduce the notion of gH-Fréchet subdifferentiability for general IVFs, i.e., we impose no restriction of convexity, parametric form, etc. on the IVF. The major contributions of this article are the following:

- The notion of Fréchet subgradients is introduced for general IVFs

- various Fréchet subdifferential calculus results are developed for nonconvex IVFs
- necessary optimality conditions for unconstrained nonconvex IOPs are derived
- a necessary condition for unconstrained weak sharp minima is given.

1.2.1. *Comparison with existing subdifferential calculus for IVFs.* From the literature of IVFs, it is evident that the subdifferential calculus of IVFs developed so far assumes the following restrictions on the objective function $\mathbf{F}$.

(i) Parametric form of $\mathbf{F}$ (see [21, 33],
(ii) the explicit form of $\underline{f}$ and $\overline{f}$ of $\mathbf{F}$ (see [29, 30, 31, 33]), and
(iii) convexity of $\mathbf{F}$ (see [33, 34]).

It is to be mentioned that the results derived in this article does not assume any of these assumptions. Hence, the results of this article are applicable for general IVFs.

1.3. **Delineation.** The article is organized as follows. In Section 2, we provide the required preliminaries to follow the rest of the article. In Section 3, we introduce the notion of gH-Fréchet subdifferentiability and explore its relationship with gH-differentiability. We also derive various calculus results for gH-Fréchet subgradients in Section 3. Further, we provide optimality conditions for unconstrained IOPs in terms of gH-Fréchet subgradients in Section 4. A necessary condition for unconstrained weak sharp minima of an extended IVF is also given in Section 4. Lastly, the conclusion and future directions of research are given in Section 5.

2. Preliminaries and Terminologies

In this section, we discuss interval arithmetic and calculus results for IVFs. The following notations are used throughout the article.

- $\mathbb{R}$ denotes the set of real numbers
- $I(\mathbb{R})$ represents the set of all closed and bounded intervals
- Bold capital letters are used to represent the elements of $I(\mathbb{R})$
- $\overline{I(\mathbb{R})} = I(\mathbb{R}) \cup \{-\infty, +\infty\}$
- $\mathbf{0}$ represents the interval $[0,0]$
- $\|\cdot\|$ denotes the Euclidean norm
- $\mathbb{B}$ denotes the closed unit ball in $\mathbb{R}^n$
- X is a nonempty subset of $\mathbb{R}^n$
- $B_\delta(\bar{x})$ is an open ball of radius δ centered at $\bar{x}$.

2.1. **Interval arithmetic.** Consider two intervals $\mathbf{A} = [\underline{a}, \bar{a}]$ and $\mathbf{B} = [\underline{b}, \overline{b}]$, and a real number μ. The addition and subtraction of $\mathbf{A}$ and $\mathbf{B}$ are denoted by $\mathbf{A} \oplus \mathbf{B}$ and $\mathbf{A} \ominus \mathbf{B}$, respectively. The multiplication of μ with $\mathbf{A}$ is denoted by $\mu \odot \mathbf{A}$.

The *norm* of an interval $\mathbf{A} = [\underline{a}, \bar{a}]$ is defined by (see [35])

$$\|\mathbf{A}\|_{I(\mathbb{R})} = \max\{|\underline{a}|, |\bar{a}|\}.$$

It is noteworthy that the set $I(\mathbb{R})$ equipped with the norm $\|.\|_{I(\mathbb{R})}$ is a normed quasilinear space with respect to the operations $\oplus, \ominus_{gH}$ and $\odot$ (see [36]), where $\ominus_{gH}$ is defined as below.

Definition 2.1. (*gH-difference of intervals* [37]). Let $\mathbf{A} = [\underline{a}, \bar{a}]$ and $\mathbf{B} = [\underline{b}, \overline{b}]$ be two elements of $I(\mathbb{R})$. The gH-difference between $\mathbf{A}$ and $\mathbf{B}$ is defined by

$$\mathbf{A} \ominus_{gH} \mathbf{B} = [\min\{\underline{a} - \underline{b}, \bar{a} - \overline{b}\}, \max\{\underline{a} - \underline{b}, \bar{a} - \overline{b}\}].$$

Similarly, $\mathbf{A} \ominus_{gH} \mathbf{B} \ominus_{gH} \mathbf{C}$ is defined by

$$\mathbf{A} \ominus_{gH} \mathbf{B} \ominus_{gH} \mathbf{C} = [\min\{\underline{a} - \underline{b} - \underline{c}, \overline{a} - \overline{b} - \overline{c}\}, \max\{\underline{a} - \underline{b} - \underline{c}, \overline{a} - \overline{b} - \overline{c}\}].$$

Note 2.1. It is to note that $\mathbf{A} \ominus_{gH} \mathbf{B} \ominus_{gH} \mathbf{A} \oplus \mathbf{B} \neq ((\mathbf{A} \ominus_{gH} \mathbf{B}) \ominus_{gH} \mathbf{A}) \oplus \mathbf{B}$. For instance, take $\mathbf{A} = [1,2]$ and $\mathbf{B} = [-1,4]$. Then,

$$\begin{aligned} \mathbf{A} \ominus_{gH} \mathbf{B} \ominus_{gH} \mathbf{A} \oplus \mathbf{B} &= [\min\{1-(-1)-1+(-1), 2-4-2+4\}, \\ &\quad\ \max\{1-(-1)-1+(-1), 2-4-2+4\}] \\ &= \mathbf{0}. \end{aligned}$$

However,

$$\begin{aligned} ((\mathbf{A} \ominus_{gH} \mathbf{B}) \ominus_{gH} \mathbf{A}) \oplus \mathbf{B} &= (([\min\{1-(-1), 2-4\}, \\ &\quad\ \max\{1-(-1), 2-4\}]) \ominus_{gH} [1,2]) \oplus [-1,4] \\ &= ([-2,2] \ominus_{gH} [1,2]) \oplus [-1,4] \\ &= [-3,0] \oplus [-1,4] \\ &= [-4,4]. \end{aligned}$$

Definition 2.2. (*Algebraic operations on $I(\mathbb{R})^n$* [38]). Let $\widehat{\mathbf{A}} = (\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n)$ and $\widehat{\mathbf{B}} = (\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_n)$ be two elements in $I(\mathbb{R})^n$. An algebraic operation $\star$ between $\widehat{\mathbf{A}}$ and $\widehat{\mathbf{B}}$, denoted by $\widehat{\mathbf{A}} \star \widehat{\mathbf{B}}$, is defined by

$$\widehat{\mathbf{A}} \star \widehat{\mathbf{B}} = (\mathbf{A}_1 \star \mathbf{B}_1, \mathbf{A}_2 \star \mathbf{B}_2, \ldots, \mathbf{A}_n \star \mathbf{B}_n),$$

where $\star \in \{\oplus, \ominus, \ominus_{gH}\}$.

The product $d^\top \odot \widehat{\mathbf{G}}$, where $d = (d_1, d_2, \ldots, d_n) \in \mathbb{R}^n$ and $\widehat{\mathbf{G}} = (\mathbf{G}_1, \mathbf{G}_2, \ldots, \mathbf{G}_n) \in I(\mathbb{R})^n$, is given by

$$d^\top \odot \widehat{\mathbf{G}} = d_1 \odot \mathbf{G}_1 \oplus d_2 \odot \mathbf{G}_2 \oplus \cdots \oplus d_n \odot \mathbf{G}_n.$$

Since $I(\mathbb{R})$ is not a linearly ordered set (see [39]), we use the following dominance relation throughout the article.

Definition 2.3. (*Dominance of intervals* [28]). Let $\mathbf{A} = [\underline{a}, \overline{a}]$ and $\mathbf{B} = [\underline{b}, \overline{b}]$ be two elements of $I(\mathbb{R})$. Then,

(i) $\mathbf{B}$ is said to be dominated by $\mathbf{A}$ if $\underline{a} \leq \underline{b}$ and $\overline{a} \leq \overline{b}$, and then we write $\mathbf{A} \preceq \mathbf{B}$;
(ii) $\mathbf{B}$ is said to be strictly dominated by $\mathbf{A}$ if $\mathbf{A} \preceq \mathbf{B}$ and $\mathbf{A} \neq \mathbf{B}$, and then we write $\mathbf{A} \prec \mathbf{B}$. Equivalently, $\mathbf{A} \prec \mathbf{B}$ if and only if any of the following cases hold:
   - Case 1. $\underline{a} < \underline{b}$ and $\overline{a} \leq \overline{b}$,
   - Case 2. $\underline{a} \leq \underline{b}$ and $\overline{a} < \overline{b}$,
   - Case 3. $\underline{a} < \underline{b}$ and $\overline{a} < \overline{b}$;

(iii) if neither $\mathbf{A} \preceq \mathbf{B}$ nor $\mathbf{B} \preceq \mathbf{A}$, we say that none of $\mathbf{A}$ and $\mathbf{B}$ dominates the other, or $\mathbf{A}$ and $\mathbf{B}$ are not comparable. Equivalently, $\mathbf{A}$ and $\mathbf{B}$ are not comparable if either '$\underline{a} < \underline{b}$ and $\overline{a} > \overline{b}$' or '$\underline{a} > \underline{b}$ and $\overline{a} < \overline{b}$'.

Definition 2.4. (*Infimum of a subset of $\overline{I(\mathbb{R})}$* [40]). Let $\mathbf{S} \subseteq \overline{I(\mathbb{R})}$. An interval $\bar{\mathbf{A}} \in I(\mathbb{R})$ is said to be a lower bound of $\mathbf{S}$ if $\bar{\mathbf{A}} \preceq \mathbf{B}$ for all $\mathbf{B}$ in $\mathbf{S}$. A lower bound $\bar{\mathbf{A}}$ of $\mathbf{S}$ is called an infimum of $\mathbf{S}$ if for all lower bounds $\mathbf{C}$ of $\mathbf{S}$ in $I(\mathbb{R})$, $\mathbf{C} \preceq \bar{\mathbf{A}}$. We denote infimum of $\mathbf{S}$ by $\inf \mathbf{S}$.

Definition 2.5. (*Supremum of a subset of* $\overline{I(\mathbb{R})}$ [40]). Let $\mathbf{S} \subseteq \overline{I(\mathbb{R})}$. An interval $\bar{\mathbf{A}} \in I(\mathbb{R})$ is said to be an upper bound of $\mathbf{S}$ if $\mathbf{B} \preceq \bar{\mathbf{A}}$ for all $\mathbf{B}$ in $\mathbf{S}$. An upper bound $\bar{\mathbf{A}}$ of $\mathbf{S}$ is called a supremum of $\mathbf{S}$ if for all upper bounds $\mathbf{C}$ of $\mathbf{S}$ in $I(\mathbb{R})$, $\bar{\mathbf{A}} \preceq \mathbf{C}$. We denote supremum of $\mathbf{S}$ by $\sup \mathbf{S}$.

Remark 2.1. Let $\mathbf{S} = \left\{[a_\alpha, b_\alpha] \in \overline{I(\mathbb{R})} : \alpha \in \Lambda \text{ and } \Lambda \text{ being an index set }\right\}$. Then, by Definitions 2.4 and 2.5, it follows that $\inf \mathbf{S} = \left[\inf_{\alpha\in\Lambda} a_\alpha,\ \inf_{\alpha\in\Lambda} b_\alpha\right]$ and $\sup \mathbf{S} = \left[\sup_{\alpha\in\Lambda} a_\alpha,\ \sup_{\alpha\in\Lambda} b_\alpha\right]$. It is evident that if $\inf \mathbf{S}$ and $\sup \mathbf{S}$ exist for an $\mathbf{S}$, then they are unique.

Lemma 2.1. (See [41]). *For* $\boldsymbol{A} \in I(\mathbb{R})$ *and* $x, y \in \mathbb{R}$,

$$\boldsymbol{A} \odot (x+y) \subseteq \boldsymbol{A} \odot x \oplus \boldsymbol{A} \odot y.$$

Lemma 2.2. *For* $\alpha \in [0,1]$ *and* $\boldsymbol{A} \in I(\mathbb{R}), (1-\alpha) \odot \boldsymbol{A} = \boldsymbol{A} \ominus_{gH} \alpha \odot \boldsymbol{A}$.

Proof. Let $\mathbf{A} = [\underline{a}, \overline{a}]$. Then, $(1-\alpha) \odot \mathbf{A} = [(1-\alpha)\underline{a}, (1-\alpha)\overline{a}]$ because $1-\alpha \geq 0$. Also,

$$\begin{aligned}
\mathbf{A} \ominus_{gH} \alpha \odot \mathbf{A} &= [\min\{\underline{a} - \alpha\underline{a}, \overline{a} - \alpha\overline{a}\}, \max\{\underline{a} - \alpha\underline{a}, \overline{a} - \alpha\overline{a}\}] \\
&= [\min\{(1-\alpha)\underline{a}, (1-\alpha)\overline{a}\}, \max\{(1-\alpha)\underline{a}, (1-\alpha)\overline{a}\}] \\
&= [(1-\alpha)\underline{a}, (1-\alpha)\overline{a}] \text{ because } 1-\alpha \geq 0.
\end{aligned}$$

□

Lemma 2.3. *For two elements* $\boldsymbol{A}, \boldsymbol{B} \in I(\mathbb{R})$, *we have*

(i) $\boldsymbol{0} \preceq \boldsymbol{A} \ominus_{gH} \boldsymbol{B} \iff \boldsymbol{B} \preceq \boldsymbol{A}$, *and*

(ii) $-1 \odot (\boldsymbol{A} \ominus_{gH} \boldsymbol{B}) = \boldsymbol{B} \ominus_{gH} \boldsymbol{A}$.

Proof. See Appendix A. □

2.2. **Calculus of IVFs.** In this section and later throughout this paper, let X be a nonempty subset of $\mathbb{R}^n$. A function $\mathbf{F} : X \to I(\mathbb{R})$ is known as an IVF; $\mathbf{F}$ can be presented by

$$\mathbf{F}(x) = [\underline{f}(x), \overline{f}(x)],$$

where $\underline{f}$ and $\overline{f}$ are real-valued functions on X such that $\underline{f}(x) \leq \overline{f}(x)$ for all $x \in X$.

Similarly as in the conventional optimization theory, the need to study extended IVFs arises when we seek to convert a constrained IOP into an unconstrained IOP. For instance, consider the following IOP.

$$\min_{x\in X} \mathbf{F}(x), \tag{2.1}$$

where $\mathbf{F} : X \to I(\mathbb{R})$ is an IVF. Then (2.1) can be restated as

$$\min_{x\in\mathbb{R}^n} \mathbf{F}_0(x),$$

where

$$\mathbf{F}_0(x) = \begin{cases} \mathbf{F}(x), & x \in X \\ [+\infty, +\infty], & \text{otherwise.} \end{cases}$$

Most rules with infinity are intuitively clear except possibly $0 \times (+\infty)$ and $\infty - \infty$. Throughout the article, we are dealing with minimization problems, we follow the following convention adopted in [18].

$$0 \times (+\infty) = (+\infty) \times 0 = 0 \text{ and } \infty - \infty = \infty.$$

However, we would like to ascertain that we really need not get worried about $\infty - \infty$ as the IVFs considered in this article are proper IVFs. The definition of proper IVFs is given below.

Definition 2.6. (*Proper IVF* [40]). An extended IVF $\mathbf{F} : X \to \overline{I(\mathbb{R})}$ is called a proper function if there exists an $\bar{x} \in X$ such that $\mathbf{F}(\bar{x}) \prec [+\infty, +\infty]$ and $[-\infty, -\infty] \prec \mathbf{F}(x)$ for all $x \in X$.

Definition 2.7. (*Convex IVF* [42]). A proper extended IVF $\mathbf{F} : X \to \overline{I(\mathbb{R})}$ is said to be convex on X if for any x_1 and x_2 in X,

$$\mathbf{F}(\lambda_1 x_1 + \lambda_2 x_2) \preceq \lambda_1 \odot \mathbf{F}(x_1) \oplus \lambda_2 \odot \mathbf{F}(x_2) \text{ for all } \lambda_1, \lambda_2 \in [0,1] \text{ with } \lambda_1 + \lambda_2 = 1.$$

Remark 2.2. (See [42]). A proper extended IVF $\mathbf{F}(x) = [\underline{f}(x), \overline{f}(x)]$ is convex on X if and only if $\underline{f}$ and $\overline{f}$ are convex on X.

Definition 2.8. (*Linear IVF* [24]). A proper extended IVF $\mathbf{F} : X \to \overline{I(\mathbb{R})}$ is said to be linear if the following two conditions hold:

(i) $\mathbf{F}(c\, x) = c \odot \mathbf{F}(x)$ for all $x \in X$ and $c \in \mathbb{R}$, and
(ii) for all $x,\ y \in X$,

either $\mathbf{F}(x) \oplus \mathbf{F}(y) = \mathbf{F}(x+y)$ or 'none of $\mathbf{F}(x) \oplus \mathbf{F}(y)$ and $\mathbf{F}(x+y)$ dominates the other'.

Definition 2.9. (*Epigraph of a proper extended IVF*). The epigraph of a proper extended IVF $\mathbf{F} : \mathbb{R}^n \to \overline{I(\mathbb{R})}$, denoted $\operatorname{epi} \mathbf{F}$, is defined by

$$\operatorname{epi} \mathbf{F} = \{(x, \mathbf{A}) \in \mathbb{R}^n \times I(\mathbb{R}) : \mathbf{F}(x) \preceq \mathbf{A}\}.$$

Definition 2.10. (*Lower limit and gH-lower semicontinuity of a proper extended IVF* [40]). The lower limit of a proper extended IVF $\mathbf{F} : \mathbb{R}^n \to \overline{I(\mathbb{R})}$ at $\bar{x} \in \mathbb{R}^n$, denoted $\liminf\limits_{x \to \bar{x}} \mathbf{F}(x)$, is defined by

$$\begin{aligned} \liminf_{x \to \bar{x}} \mathbf{F}(x) &= \lim_{\delta \downarrow 0} \left(\inf\{\mathbf{F}(x) : x \in B_\delta(\bar{x})\} \right) \\ &= \sup_{\delta > 0} \left(\inf\{\mathbf{F}(x) : x \in B_\delta(\bar{x})\} \right). \end{aligned}$$

$\mathbf{F}$ is called gH-lower semicontinuous (gH-lsc) at a point $\bar{x}$ if

$$\mathbf{F}(\bar{x}) \preceq \liminf_{x \to \bar{x}} \mathbf{F}(x). \tag{2.2}$$

Further, $\mathbf{F}$ is called gH-lsc on $\mathbb{R}^n$ if (2.2) holds for every $\bar{x} \in \mathbb{R}^n$.

Note 2.2. (See [40]). Let $\mathbf{F}$ be a proper extended IVF with $\mathbf{F}(x) = \left[\underline{f}(x), \overline{f}(x)\right]$, where $\underline{f},\ \overline{f} : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty, +\infty\}$ be two extended real-valued functions. Then, $\mathbf{F}$ is gH-lsc at $\bar{x} \in \mathbb{R}^n$ if and only if $\underline{f}$ and $\overline{f}$ both are lsc at $\bar{x}$.

Definition 2.11. (*Upper limit and gH-upper semicontinuity of a proper extended IVF* [40]). The upper limit of a proper extended IVF $\mathbf{F} : \mathbb{R}^n \to \overline{I(\mathbb{R})}$ at $\bar{x} \in \mathbb{R}^n$, denoted $\limsup\limits_{x \to \bar{x}} \mathbf{F}(x)$, is defined as

$$\begin{aligned} \limsup_{x \to \bar{x}} \mathbf{F}(x) &= \lim_{\delta \downarrow 0} \left(\sup\{\mathbf{F}(x) : x \in B_\delta(\bar{x})\} \right) \\ &= \inf_{\delta > 0} \left(\sup\{\mathbf{F}(x) : x \in B_\delta(\bar{x})\} \right). \end{aligned}$$

$\mathbf{F}$ is called gH-upper semicontinuous (gH-usc) at $\bar{x}$ if

$$\limsup_{x\to\bar{x}} \mathbf{F}(x) \preceq \mathbf{F}(\bar{x}). \tag{2.3}$$

Further, $\mathbf{F}$ is called gH-usc on $\mathbb{R}^n$ if (2.3) holds for every $\bar{x} \in \mathbb{R}^n$.

Note 2.3. (See [40]). Let $\mathbf{F}$ be a proper extended IVF with $\mathbf{F}(x) = \left[\underline{f}(x), \overline{f}(x)\right]$, where $\underline{f}$, $\overline{f}$: $\mathbb{R}^n \to \mathbb{R} \cup \{-\infty, +\infty\}$ be two extended real-valued functions. Then, $\mathbf{F}$ is gH-usc at $\bar{x} \in \mathbb{R}^n$ if and only if $\underline{f}$ and $\overline{f}$ are usc at $\bar{x}$.

Remark 2.3. With the help of Notes 2.2 and 2.3, it is easy to observe that

$$\limsup_{x\to\bar{x}} \mathbf{F}(x) = -1 \odot \liminf_{x\to\bar{x}}(-1 \odot \mathbf{F}(x)).$$

Theorem 2.1. (See [40]). *Let* $\mathbf{F}_1,\ \mathbf{F}_2 : \mathbb{R}^n \to \overline{I(\mathbb{R})}$ *be two proper extended IVFs and* $\bar{x} \in \mathbb{R}^n$. *Then,*

(i) $\liminf_{x\to\bar{x}} \mathbf{F}_1(x) \oplus \liminf_{x\to\bar{x}} \mathbf{F}_2(x) \preceq \liminf_{x\to\bar{x}}(\mathbf{F}_1 \oplus \mathbf{F}_2)(x)$,

(ii) $\limsup_{x\to\bar{x}}(\mathbf{F}_1 \oplus \mathbf{F}_2)(x) \preceq \limsup_{x\to\bar{x}} \mathbf{F}_1(x) \oplus \limsup_{x\to\bar{x}} \mathbf{F}_2(x)$, *and*

(iii) $\liminf_{x\to\bar{x}} \mathbf{F}_1(x) \oplus \limsup_{x\to\bar{x}} \mathbf{F}_2(x) \preceq \limsup_{x\to\bar{x}}(\mathbf{F}_1 \oplus \mathbf{F}_2)(x)$.

Definition 2.12. (*gH-limit of an IVF* [42]). Let $\mathbf{F} : X \to \overline{I(\mathbb{R})}$ be a proper IVF on a nonempty subset X of $\mathbb{R}^n$. The function $\mathbf{F}$ is called tending to a limit $\mathbf{L} \in I(\mathbb{R})$ as x tends to $\bar{x}$, denoted by $\lim_{x\to\bar{x}} \mathbf{F}(x)$, if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\|\mathbf{F}(x) \ominus_{gH} \mathbf{L}\|_{I(\mathbb{R})} < \varepsilon \text{ whenever } 0 < \|x - \bar{x}\| < \delta.$$

Definition 2.13. (*gH-continuity* [42]). Let $\mathbf{F} : X \to \overline{I(\mathbb{R})}$ be a proper IVF on a nonempty subset X of $\mathbb{R}^n$. The function $\mathbf{F}$ is said to be gH-continuous at $\bar{x} \in X$ if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\|\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x})\|_{I(\mathbb{R})} < \varepsilon \text{ whenever } \|x - \bar{x}\| < \delta.$$

Definition 2.14. (*gH-derivative* [22]). Let $X \subseteq \mathbb{R}$. The gH-derivative of a proper extended IVF $\mathbf{F} : X \to \overline{I(\mathbb{R})}$ at $\bar{x} \in X$ is defined by

$$\mathbf{F}'(\bar{x}) = \lim_{d\to 0} \frac{1}{d} \odot (\mathbf{F}(\bar{x} + d) \ominus_{gH} \mathbf{F}(\bar{x})), \text{ provided the limit exists.}$$

Definition 2.15. (*gH-partial derivative* [43]). Let $x_0 = (x_1^0, x_2^0, \ldots, x_n^0)$ be an interior point of $X \subseteq \mathbb{R}^n$ and $h = (h_1, h_2, \ldots, h_n) \in \mathbb{R}^n$ be such that $x_0 + h \in X$. Define a function

$$\phi_i(x_i) = \mathbf{F}(x_1^0, x_2^0, \ldots, x_{i-1}^0, x_i, x_{i+1}^0, \ldots, x_n^0).$$

If the generalized Hukuhara derivative (gH-derivative) of ϕ_i exists at x_i^0, i.e.,

$$\lim_{h_i\to 0} \frac{1}{h_i} \odot (\phi_i(x_i^0 + h_i) \ominus_{gH} \phi_i(x_i^0))$$

exists, then we say that $\mathbf{F}$ has the ith partial derivative at x_0 and it is denoted by $D_i\mathbf{F}(x_0), i = 1, 2, \ldots, n$.

Note 2.4. Observe that at x_0, the partial derivatives of $\mathbf{F}$ are given by

$$D_i\mathbf{F}(x_0) = \left[\min\left\{\frac{\partial \underline{f}}{\partial x_i}(x_0), \frac{\partial \overline{f}}{\partial x_i}(x_0)\right\}, \max\left\{\frac{\partial \underline{f}}{\partial x_i}(x_0), \frac{\partial \overline{f}}{\partial x_i}(x_0)\right\}\right],\ i = 1, 2, \ldots, n.$$

Definition 2.16. (*gH-gradient* [43]). The gH-gradient of a proper extended IVF $\mathbf{F}$ at a point $x_0 \in X$ is defined by the vector

$$(D_1\mathbf{F}(x_0), D_2\mathbf{F}(x_0), \ldots, D_n\mathbf{F}(x_0)).$$

This gH-gradient is denoted by $\nabla\mathbf{F}(x_0)$.

Definition 2.17. (*gH-differentiability* [43]). A proper extended IVF $\mathbf{F} : X \to \overline{I(\mathbb{R})}$ is said to be gH-differentiable at $x_0 \in X$ if $\nabla\mathbf{F}(x_0)$ exists and

$$\lim_{\|h\|\to 0} \frac{\mathbf{F}(\bar{x}+h) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} h^\top \odot \nabla\mathbf{F}(x_0)}{\|h\|} = \mathbf{0}.$$

Definition 2.18. (*Effective domain of an IVF*). The effective domain of an extended IVF $\mathbf{F} : X \to \overline{I(\mathbb{R})}$ is the collection of all such points at which $\mathbf{F}$ is finite. It is denoted by $\text{dom}(\mathbf{F})$, i.e.,

$$\text{dom}(\mathbf{F}) = \Big\{x \in X : \mathbf{F}(x) \prec [+\infty, +\infty]\Big\}.$$

Definition 2.19. (*gH-subgradients of convex IVFs* [34]). Let $\mathbf{F} : X \subseteq \mathbb{R}^n \to \overline{I(\mathbb{R})}$ be a proper convex IVF and $\bar{x} \in \text{dom}(\mathbf{F})$. Then, gH-subdifferential of $\mathbf{F}$ at $\bar{x}$, denoted by $\partial\mathbf{F}(\bar{x})$ is defined by

$$\partial\mathbf{F}(\bar{x}) = \Big\{\widehat{\mathbf{G}} \in I(\mathbb{R})^n : (x - \bar{x})^\top \odot \widehat{\mathbf{G}} \preceq \mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \text{ for all } x \in X\Big\}. \tag{2.4}$$

The elements of (2.4) are known as gH-subgradients of $\mathbf{F}$ at $\bar{x}$. Further, if $\partial\mathbf{F}(\bar{x}) \neq \emptyset$, we say that $\mathbf{F}$ is gH-subdifferentiable at $\bar{x}$.

Definition 2.20. (*Weak sharp minima for an IVF* [44]). Let $\mathbf{F} : \mathbb{R}^n \to \overline{I(\mathbb{R})}$ be a gH-lsc and convex IVF. Let $\bar{S}$ and S be two nonempty closed convex sets such that $\bar{S} \subseteq S \subseteq \mathbb{R}^n$. Further, let $\text{dom}(\mathbf{F}) \cap S \neq \emptyset$. Then, the set $\bar{S}$ is said to be a set of WSM of $\mathbf{F}$ over the set S with modulus $\alpha > 0$ if

$$\mathbf{F}(\bar{x}) \oplus \alpha\, \text{dist}(x, \bar{S}) \preceq \mathbf{F}(x) \text{ for all } \bar{x} \in \bar{S} \text{ and } x \in S,$$

where $\text{dist}(x, \bar{S})$ is the distance function given by

$$\text{dist}(x, \bar{S}) = \inf_{\bar{x}\in\bar{S}} \|x - \bar{x}\|.$$

Next, we state the following result from conventional convex analysis, which is used in the article.

Note 2.5. (See [45]). For any $\bar{x} \in X$ and $S \subseteq X$, we have

$$\partial_f\, \text{dist}(\bar{x}, S) = \widehat{N}(\bar{x}, S) \cap \mathbb{B},$$

where $\partial_f\, \text{dist}(\bar{x}, S)$ denotes the Fréchet subgradient of the distance function $\text{dist}(\bar{x}, S)$ and $\widehat{N}(\bar{x}, S)$ denotes the Fréchet normal cone to S at $\bar{x}$, defined by

$$\widehat{N}(\bar{x}, S) = \left\{ y \in \mathbb{R}^n : \limsup_{x \xrightarrow{S} \bar{x}} \frac{y^\top (x - \bar{x})}{\|x - \bar{x}\|} \leq 0 \right\},$$

where $x \xrightarrow{S} \bar{x}$ means that $x \to \bar{x}$ with $x \in S$.

3. Calculus of gH-Fréchet Subgradients

In this section, we introduce the notion of gH-Fréchet subgradients of IVFs and derive exact calculus results for these subgradients.

Definition 3.1. (*gH-Fréchet subdifferentiability*). Let $\mathbf{F} : \mathbb{R}^n \to \overline{I(\mathbb{R})}$ be a proper extended IVF that is finite at $\bar{x} \in \mathbb{R}^n$. Then, the gH-Fréchet subdifferential set of $\mathbf{F}$ at $\bar{x}$, denoted $\partial_f \mathbf{F}(\bar{x})$, is defined by

$$\partial_f \mathbf{F}(\bar{x}) = \left\{ \widehat{\mathbf{G}} \in I(\mathbb{R})^n : \mathbf{0} \preceq \liminf_{x \to \bar{x}} \frac{1}{\|x - \bar{x}\|} \odot \left(\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot \widehat{\mathbf{G}} \right) \right\}. \tag{3.1}$$

We call elements of $\partial_f \mathbf{F}(\bar{x})$ as gH-Fréchet subgradients of $\mathbf{F}$ at $\bar{x}$. Further, if $\partial_f \mathbf{F}(\bar{x}) \neq \emptyset$, we say that $\mathbf{F}$ is gH-Fréchet subdifferentiable at $\bar{x}$. If $\mathbf{F}(\bar{x})$ is not finite, we define $\partial_f \mathbf{F}(\bar{x}) = \emptyset$.

Remark 3.1. (Geometrical interpretation of gH-Fréchet subdifferentiability). From Definition 3.1, $\widehat{\mathbf{G}} \in \partial_f \mathbf{F}(\bar{x})$ if and only if

$$\mathbf{0} \preceq \liminf_{x \to \bar{x}} \frac{1}{\|x - \bar{x}\|} \odot \left(\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot \widehat{\mathbf{G}} \right).$$

Therefore, for any $\varepsilon > 0$, we obtain a $\delta > 0$ such that whenever $0 < \|x - \bar{x}\| < \delta$, we have

$$\begin{aligned} & \mathbf{0} \preceq \mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot \widehat{\mathbf{G}} \oplus \varepsilon \|x - \bar{x}\| \\ \implies \quad & \mathbf{F}(\bar{x}) \oplus (x - \bar{x})^\top \odot \widehat{\mathbf{G}} \preceq \mathbf{F}(x) \oplus \varepsilon \|x - \bar{x}\|. \end{aligned} \tag{3.2}$$

Since (3.2) is true for any ε arbitrarily close to 0, $\mathbf{F}(\bar{x}) \oplus (x - \bar{x})^\top \odot \widehat{\mathbf{G}}$ is a supporting function from below to the epigraph of $\mathbf{F}$ at $(\bar{x}, \mathbf{F}(\bar{x}))$. Infact, at the point of nondifferentiability, there can be an infinite number of such supporting IVFs $\mathbf{F}(\bar{x}) \oplus (x - \bar{x})^\top \odot \widehat{\mathbf{G}}$ and the collection of all such $\widehat{\mathbf{G}}$'s form the set $\partial_f \mathbf{F}(\bar{x})$. In other words, there always exists a neighbourhood of the point $\bar{x}$ such that the graph of $\mathbf{F}(\bar{x}) \oplus (x - \bar{x})^\top \odot \widehat{\mathbf{G}}$ does not completely lie above the graph of $\mathbf{F}$. To have a better understanding of this idea, we consider the following example.

Example 3.1. Consider an IVF $\mathbf{F} : \mathbb{R} \to I(\mathbb{R})$ given by $\mathbf{F}(x) = [|x|, k|x|]$, where $k > 1$ is a real number.

Let us apply Definition 3.1 to check gH-Fréchet subdifferentiability of $\mathbf{F}$ at 0.

$$\begin{aligned} \partial_f \mathbf{F}(0) &= \left\{ \mathbf{G} \in I(\mathbb{R}) : \mathbf{0} \preceq \liminf_{x \to 0} \frac{1}{|x - 0|} \odot (\mathbf{F}(x) \ominus_{gH} \mathbf{F}(0) \ominus_{gH} (x - 0) \odot \mathbf{G}) \right\} \\ &= \left\{ \mathbf{G} : \mathbf{0} \preceq \liminf_{x \to 0} \frac{1}{|x|} \odot ([|x|, k|x|] \ominus_{gH} x \odot \mathbf{G}) \right\}. \end{aligned}$$

Since $\mathbf{G} \in I(\mathbb{R})$, let $\mathbf{G} = [a, b]$ for some $a,\ b \in \mathbb{R}$ with $a \leq b$. Therefore,

$$\partial_f \mathbf{F}(0) = \left\{ [a, b] : \mathbf{0} \preceq \liminf_{x \to 0} \frac{1}{|x|} \odot ([1, k] \odot |x| \ominus_{gH} x \odot [a, b]) \right\}.$$

Let us now consider the following two possible cases.

Case 1. $x \geq 0$.

Note that

$$\begin{aligned} & \mathbf{0} \preceq \liminf_{x\to 0} \frac{1}{|x|} \odot ([1,k] \odot |x| \ominus_{gH} x \odot [a,b]) \\ \Longrightarrow\quad & \mathbf{0} \preceq \liminf_{x\to 0} \frac{1}{x} \odot ([1,k] \odot x \ominus_{gH} x \odot [a,b]) \\ \Longrightarrow\quad & \mathbf{0} \preceq [1,k] \ominus_{gH} [a,b] \\ \Longrightarrow\quad & [a,b] \preceq [1,k], \text{ by (i) of Lemma 2.3} \\ \Longrightarrow\quad & a \leq 1 \text{ and } b \leq k. \end{aligned}$$

Case 2. $x < 0$.

Note that

$$\begin{aligned} & \mathbf{0} \preceq \liminf_{x\to 0} \frac{1}{|x|} \odot ([1,k] \odot |x| \ominus_{gH} x \odot [a,b]) \\ \Longrightarrow\quad & \mathbf{0} \preceq \liminf_{x\to 0} \frac{1}{-x} \odot ([1,k] \odot (-x) \ominus_{gH} x \odot [a,b]) \\ \Longrightarrow\quad & \mathbf{0} \preceq [1,k] \ominus_{gH} (-1) \odot [a,b] \\ \Longrightarrow\quad & \mathbf{0} \preceq [1,k] \ominus_{gH} [-b,-a] \\ \Longrightarrow\quad & [-b,-a] \preceq [1,k], \text{ by (i) of Lemma 2.3} \\ \Longrightarrow\quad & b \geq -1 \text{ and } a \geq -k. \end{aligned}$$

Therefore, from Case 1 and Case 2, we have

$$-k \leq a \leq 1 \text{ and } -1 \leq b \leq k.$$

Hence,

$$\partial_f \mathbf{F}(0) = \{[a,b] : -k \leq a \leq 1 \text{ and } -1 \leq b \leq k\}. \tag{3.3}$$

The function $\mathbf{F}$ with $k = 2$ is depicted by the grey shaded region in Figure 1. We also figure out gH-Fréchet subgradient of $\mathbf{F}$ at 0 namely $\mathbf{G}'$, where $\mathbf{G}'(x) = [-0.5, 1.5] \odot x$. Since $\mathbf{G}'(x)$ belongs to the set (3.3), $\mathbf{G}'(x)$ is a gH-Fréchet subgradient of $\mathbf{F}$ at 0. $\mathbf{G}'(x)$ is depicted by the dotted region in Figure 1. Observe from Figure 1 that the graph of $\mathbf{G}'$ does not completely lie above the graph of $\mathbf{F}$ as reflected in Remark 3.1.

Note 3.1. If we take $k = 1$ in Example 3.1, then the IVF $\mathbf{F}$ reduces to a real-valued function given as $f(x) = |x|$. We now apply Definition 3.1 to find gH-Fréchet subgradients of f at 0.

$$\partial_f f(0) = \left\{ a : 0 \leq \liminf_{x\to 0} \frac{|x| - ax}{|x|}, \text{ where } a \in \mathbb{R} \right\}.$$

Similar to Example 3.1, let us now consider the following two cases.

- Case 1. $x \geq 0$.

  In this case, we obtain

$$0 \leq \liminf_{x\to 0} \frac{x - ax}{x}, \text{ i.e., } a \leq 1.$$

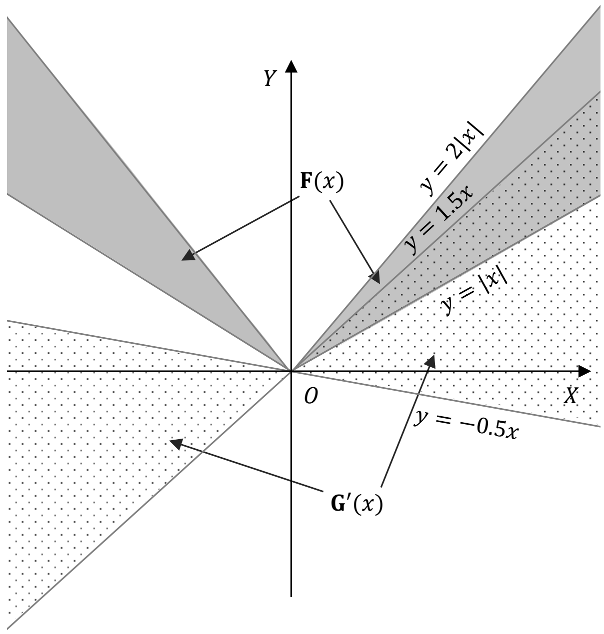

FIGURE 1. Geometrical view of gH-Fréchet subdifferentiability of $\mathbf{F}$ of Example 3.1

• Case 2. $x < 0$.

In this case, we obtain $a \geq -1$.

Therefore, from both the cases, we obtain

$$\partial_f f(0) = [-1, 1].$$

Remark 3.2. It is to observe that set (3.1) can be empty. For instance, consider an IVF $\mathbf{F} : \mathbb{R} \to I(\mathbb{R})$ given by $\mathbf{F}(x) = [-k|x|, -|x|]$, where $k > 1$ is a real number. Then, by following similar steps as in Example 3.1, it can be seen that $\partial_f \mathbf{F}(0) = \emptyset$.

Next, in Note 3.2, we show that the notion of gH-subgradients (Definition 2.19) introduced in [34] is a special case of Definition 3.1.

Note 3.2. If $\mathbf{F} : \mathbb{R}^n \to \overline{I(\mathbb{R})}$ is convex, then $\widehat{\mathbf{G}} \in I(\mathbb{R})^n$ is a gH-subgradient of $\mathbf{F}$ at $\bar{x} \in \mathbb{R}^n$ according to Definition 3.1 if and only if $\widehat{\mathbf{G}}$ is a subgradient of $\mathbf{F}$ at $\bar{x}$ according to Definition 2.19. The reason is as follows.

Let $\widehat{\mathbf{G}} \in \partial_f \mathbf{F}(\bar{x})$. Then,

$$\mathbf{0} \preceq \liminf_{x \to \bar{x}} \frac{1}{\|x - \bar{x}\|} \odot \left(\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot \widehat{\mathbf{G}} \right).$$

Therefore, for any $\varepsilon > 0$, we obtain a $\delta > 0$ such that whenever $0 < \|x - \bar{x}\| < \delta$

$$\mathbf{0} \preceq \mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot \widehat{\mathbf{G}} \oplus \varepsilon \|x - \bar{x}\|.$$

By taking $x = \bar{x} + \lambda d,\ \lambda \downarrow 0$, we obtain

$$\mathbf{0} \preceq \mathbf{F}(\bar{x} + \lambda d) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} \lambda d^\top \odot \widehat{\mathbf{G}} \oplus \varepsilon \|\lambda d\|.$$

In particular, by taking $d = x - \bar{x}$, we obtain

$$\begin{aligned}
& \mathbf{0} \preceq \mathbf{F}(\bar{x} + \lambda(x - \bar{x})) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} \lambda(x - \bar{x})^{\top} \odot \widehat{\mathbf{G}} \oplus \varepsilon\lambda \|x - \bar{x}\| \\
\implies & \mathbf{0} \preceq \mathbf{F}(\lambda x + (1 - \lambda)\bar{x}) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} \lambda(x - \bar{x})^{\top} \odot \widehat{\mathbf{G}} \oplus \varepsilon\lambda \|x - \bar{x}\| \\
\implies & \mathbf{0} \preceq \lambda \odot \mathbf{F}(x) \oplus (1 - \lambda) \odot \mathbf{F}(\bar{x}) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} \lambda(x - \bar{x})^{\top} \odot \widehat{\mathbf{G}} \oplus \varepsilon\lambda \|x - \bar{x}\|, \\
& \text{because } \mathbf{F} \text{ is convex} \\
\implies & \mathbf{0} \preceq \lambda \odot \mathbf{F}(x) \oplus \mathbf{F}(\bar{x}) \ominus_{gH} \lambda \odot \mathbf{F}(\bar{x}) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} \lambda(x - \bar{x})^{\top} \odot \widehat{\mathbf{G}} \oplus \varepsilon\lambda \|x - \bar{x}\|, \\
& \text{by using Lemma 2.2} \\
\implies & \mathbf{0} \preceq \mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x - \bar{x})^{\top} \odot \widehat{\mathbf{G}} \oplus \varepsilon \|x - \bar{x}\|.
\end{aligned}$$

Therefore, by letting $\varepsilon \to 0$, we obtain

$$\begin{aligned}
& \mathbf{0} \preceq \mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x - \bar{x})^{\top} \odot \widehat{\mathbf{G}} \text{ for all } x \in \mathbb{R}^n \\
\implies & (x - \bar{x})^{\top} \odot \widehat{\mathbf{G}} \preceq \mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \text{ for all } x \in \mathbb{R}^n, \text{ by (i) of Lemma 2.3}.
\end{aligned}$$

Thus, $\widehat{\mathbf{G}}$ is a subgradient of $\mathbf{F}$ at $\bar{x}$ according to Definition 2.19.

Conversely, let $\widehat{\mathbf{G}}$ be a subgradient of $\mathbf{F}$ at $\bar{x}$ according to Definition 2.19. Then,

$$\begin{aligned}
& (x - \bar{x})^{\top} \odot \widehat{\mathbf{G}} \preceq \mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \text{ for all } x \in \mathbb{R}^n \\
\implies & \mathbf{0} \preceq \mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x - \bar{x})^{\top} \odot \widehat{\mathbf{G}} \text{ for all } x \in \mathbb{R}^n, \\
& \text{by (i) of Lemma 2.3} \\
\implies & \mathbf{0} \preceq \liminf_{x \to \bar{x}} \frac{1}{\|x - \bar{x}\|} \odot \left(\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x - \bar{x})^{\top} \odot \widehat{\mathbf{G}} \right).
\end{aligned}$$

That is, $\widehat{\mathbf{G}}$ is a subgradient of $\mathbf{F}$ at $\bar{x}$ according to Definition 3.1.

Remark 3.3. It is to mention that Definition 2.19 is applicable only for convex IVFs. However, Definition 3.1 is applicable to more general IVFs, which may not be convex.

Theorem 3.1. *The set (3.1) of gH-Fréchet subgradients is convex.*

Proof. If $\partial_f \mathbf{F}(\bar{x}) = \emptyset$, then set $\partial_f \mathbf{F}(\bar{x})$ is vacuously convex. So, let $\partial_f \mathbf{F}(\bar{x}) \neq \emptyset$. Consider $\widehat{\mathbf{G}}$, $\widehat{\mathbf{H}} \in \partial_f \mathbf{F}(\bar{x})$. Then,

$$\mathbf{0} \preceq \liminf_{x \to \bar{x}} \frac{1}{\|x - \bar{x}\|} \odot \left(\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x - \bar{x})^{\top} \odot \widehat{\mathbf{G}} \right) \tag{3.4}$$

and

$$\mathbf{0} \preceq \liminf_{x \to \bar{x}} \frac{1}{\|x - \bar{x}\|} \odot \left(\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x - \bar{x})^{\top} \odot \widehat{\mathbf{H}} \right). \tag{3.5}$$

On multiplying (3.4) by λ and (3.5) by μ, where $\lambda,\ \mu \in [0,1]$ with $\lambda + \mu = 1$, and adding the resultant, we obtain

$$\begin{aligned}
\mathbf{0} \preceq\ & \lambda \odot \left(\liminf_{x \to \bar{x}} \tfrac{1}{\|x - \bar{x}\|} \odot (\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x - \bar{x})^{\top} \odot \widehat{\mathbf{G}}) \right) \oplus \\
& \mu \odot \left(\liminf_{x \to \bar{x}} \tfrac{1}{\|x - \bar{x}\|} \odot (\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x - \bar{x})^{\top} \odot \widehat{\mathbf{H}}) \right).
\end{aligned}$$

Therefore, by (i) of Theorem 2.1, we obtain

$$\begin{aligned}\mathbf{0} \preceq \liminf_{x\to\bar{x}} \Big(&\tfrac{1}{\|x-\bar{x}\|} \odot (\lambda \odot (\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x-\bar{x})^\top \odot \widehat{\mathbf{G}})) \oplus \\ &\tfrac{1}{\|x-\bar{x}\|} \odot (\mu \odot (\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x-\bar{x})^\top \odot \widehat{\mathbf{H}}))\Big).\end{aligned} \tag{3.6}$$

Notice that the numerator of the right hand side of (3.6) is equal to

$$\begin{aligned} &\Big(\lambda \odot \mathbf{F}(x) \ominus_{gH} \lambda \odot \mathbf{F}(\bar{x}) \ominus_{gH} \lambda (x-\bar{x})^\top \odot \widehat{\mathbf{G}} \oplus \\ &(1-\lambda) \odot \mathbf{F}(x) \ominus_{gH} (1-\lambda) \odot \mathbf{F}(\bar{x}) \ominus_{gH} \mu (x-\bar{x})^\top \odot \widehat{\mathbf{H}}\Big) \\ = \; &\Big(\lambda \odot \mathbf{F}(x) \ominus_{gH} \lambda \odot \mathbf{F}(\bar{x}) \ominus_{gH} \lambda (x-\bar{x})^\top \odot \widehat{\mathbf{G}} \oplus \\ &\mathbf{F}(x) \ominus_{gH} \lambda \odot \mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \oplus \lambda \odot \mathbf{F}(\bar{x}) \ominus_{gH} \mu (x-\bar{x})^\top \odot \widehat{\mathbf{H}}\Big), \text{ by using Lemma 2.2} \\ = \; &\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x-\bar{x})^\top \odot (\lambda \odot \widehat{\mathbf{G}} \oplus \mu \odot \widehat{\mathbf{H}}).\end{aligned}$$

Therefore, by (3.6), we obtain

$$\mathbf{0} \preceq \liminf_{x\to\bar{x}} \frac{1}{\|x-\bar{x}\|} \odot \Big(\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x-\bar{x})^\top \odot (\lambda \odot \widehat{\mathbf{G}} \oplus \mu \odot \widehat{\mathbf{H}})\Big).$$

This implies

$$\Big(\lambda \odot \widehat{\mathbf{G}} \oplus \mu \odot \widehat{\mathbf{H}}\Big) \in \partial_f \mathbf{F}(\bar{x}),$$

and hence the set (3.1) is convex. □

Next, in Theorem 3.2, we show that a gH-differentiable IVF has only one gH-Fréchet subgradient, which is the gH-gradient of the IVF.

Theorem 3.2. *Let $\boldsymbol{F} : \mathbb{R}^n \to \overline{I(\mathbb{R})}$ be a proper extended IVF. If $\boldsymbol{F}$ is gH-differentiable at $\bar{x} \in \mathbb{R}^n$, then $\boldsymbol{F}$ is also gH-Fréchet subdifferentiable at $\bar{x}$. Moreover, $\partial_f \boldsymbol{F}(\bar{x}) = \{\nabla \boldsymbol{F}(\bar{x})\}$.*

Proof. Since $\mathbf{F}$ is gH-differentiable at $\bar{x}$, we have

$$\begin{aligned} &\lim_{x\to\bar{x}} \frac{1}{\|x-\bar{x}\|} \odot \Big(\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x-\bar{x})^\top \odot \nabla\mathbf{F}(\bar{x})\Big) = \mathbf{0} \\ \Longrightarrow \quad &\liminf_{x\to\bar{x}} \frac{1}{\|x-\bar{x}\|} \odot \Big(\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x-\bar{x})^\top \odot \nabla\mathbf{F}(\bar{x})\Big) = \mathbf{0}.\end{aligned} \tag{3.7}$$

Therefore, $\nabla\mathbf{F}(\bar{x}) \in \partial_f \mathbf{F}(\bar{x})$, and hence $\mathbf{F}$ is gH-Fréchet subdifferentiable at $\bar{x}$.

Consider $\widehat{\mathbf{G}} \in \partial_f \mathbf{F}(\bar{x})$. Then,

$$\begin{aligned}
& \mathbf{0} \preceq \liminf_{x\to\bar{x}} \frac{1}{\|x-\bar{x}\|} \odot \left(\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x-\bar{x})^\top \odot \widehat{\mathbf{G}}\right) \\
\implies & \liminf_{x\to\bar{x}} \frac{1}{\|x-\bar{x}\|} \odot \left(\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x-\bar{x})^\top \odot \nabla\mathbf{F}(\bar{x})\right) \preceq \\
& \liminf_{x\to\bar{x}} \frac{1}{\|x-\bar{x}\|} \odot \left(\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x-\bar{x})^\top \odot \widehat{\mathbf{G}}\right), \text{ by (3.7)} \\
\implies & \mathbf{0} \preceq \Bigg(\liminf_{x\to\bar{x}} \frac{1}{\|x-\bar{x}\|} \odot \left(\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x-\bar{x})^\top \odot \widehat{\mathbf{G}}\right) \ominus_{gH} \\
& \liminf_{x\to\bar{x}} \frac{1}{\|x-\bar{x}\|} \odot \left(\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x-\bar{x})^\top \odot \nabla\mathbf{F}(\bar{x})\right)\Bigg), \text{ by (i) of Lemma 2.3.}
\end{aligned}$$

Therefore, by using Remark 2.3 and (ii) of Lemma 2.3, we obtain

$$\begin{aligned}
\mathbf{0} \preceq \Bigg(& \liminf_{x\to\bar{x}} \frac{1}{\|x-\bar{x}\|} \odot \left(\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x-\bar{x})^\top \odot \widehat{\mathbf{G}}\right) \oplus \\
& \limsup_{x\to\bar{x}} \frac{1}{\|x-\bar{x}\|} \odot (\mathbf{F}(\bar{x}) \ominus_{gH} \mathbf{F}(x) \oplus (x-\bar{x})^\top \odot \nabla\mathbf{F}(\bar{x}))\Bigg),
\end{aligned}$$

which implies

$$\begin{aligned}
& \mathbf{0} \preceq \limsup_{x\to\bar{x}} \frac{1}{\|x-\bar{x}\|} \odot (\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x-\bar{x})^\top \odot \widehat{\mathbf{G}} \ominus_{gH} \mathbf{F}(x) \oplus \\
& \quad \mathbf{F}(\bar{x}) \oplus (x-\bar{x})^\top \odot \nabla\mathbf{F}(\bar{x})), \text{ by (iii) of Theorem 2.1} \\
\implies & \mathbf{0} \preceq \limsup_{x\to\bar{x}} \frac{1}{\|x-\bar{x}\|} (x-\bar{x})^\top \odot (\nabla\mathbf{F}(\bar{x}) \ominus_{gH} \widehat{\mathbf{G}}) \\
\implies & \mathbf{0} \preceq \limsup_{\lambda\to 0} \frac{1}{\|\lambda d\|} (\lambda d)^\top \odot (\nabla\mathbf{F}(\bar{x}) \ominus_{gH} \widehat{\mathbf{G}}), \text{ where } x = \bar{x} + \lambda d \text{ for any } d \in \mathbb{R}^n \text{ and} \\
& \lambda > 0 \\
\implies & \mathbf{0} \preceq \limsup_{\lambda\to 0} d^\top \odot (\nabla\mathbf{F}(\bar{x}) \ominus_{gH} \widehat{\mathbf{G}}) \text{ for any } d \in \mathbb{R}^n \\
\implies & \nabla\mathbf{F}(\bar{x}) \ominus_{gH} \widehat{\mathbf{G}} = \mathbf{0} \\
\implies & \nabla\mathbf{F}(\bar{x}) = \widehat{\mathbf{G}}.
\end{aligned}$$

Since $\widehat{\mathbf{G}}$ is an arbitrarily chosen element of $\partial_f \mathbf{F}(\bar{x})$, the result follows. □

Note 3.3. Converse of Theorem 3.2 is not true. For instance, consider the $\mathbf{F}$ as in Example 3.1. We have seen that $\mathbf{F}$ is gH-Fréchet subdifferentiable at 0. Let us now find the following limit:

$$\begin{aligned}
& \lim_{h\to 0} \tfrac{1}{h} \odot (\mathbf{F}(0+h) \ominus_{gH} \mathbf{F}(0)) \\
= & \lim_{h\to 0} \tfrac{1}{h} \odot (\mathbf{F}(h) \ominus_{gH} \mathbf{0}) = \lim_{h\to 0} \tfrac{1}{h} \odot [|h|, k|h|],
\end{aligned}$$

which does not exist. Therefore, $\mathbf{F}$ is not gH-differentiable at 0.

In Theorems 3.3 and 3.4 below, we show that the scalar multiplication of a gH-Fréchet subdifferentiable IVF with any $\lambda > 0$ and the sum of two Fréchet subdifferentiable IVFs are again gH-Fréchet subdifferentiable.

Theorem 3.3. *Let $\boldsymbol{F} : \mathbb{R}^n \to \overline{I(\mathbb{R})}$ be gH-Fréchet subdifferentiable at $\bar{x} \in \mathbb{R}^n$. Then,*

$$\partial_f(\lambda \odot \boldsymbol{F})(\bar{x}) = \lambda \odot \partial_f \boldsymbol{F}(\bar{x}) \text{ for any } \lambda > 0.$$

Proof. Proof follows directly from Definitions 2.10 and 3.1. □

Theorem 3.4. *Let $\boldsymbol{F}_1,\ \boldsymbol{F}_2 : \mathbb{R}^n \to \overline{I(\mathbb{R})}$ be gH-Fréchet subdifferentiable IVFs at $\bar{x} \in \mathbb{R}^n$. Then, $\boldsymbol{F}_1 \oplus \boldsymbol{F}_2$ is gH-Fréchet subdifferentiable at $\bar{x} \in \mathbb{R}^n$, and*

$$\partial_f \boldsymbol{F}_1(\bar{x}) \oplus \partial_f \boldsymbol{F}_2(\bar{x}) \subseteq \partial_f(\boldsymbol{F}_1 \oplus \boldsymbol{F}_2)(\bar{x}).$$

Proof. Let $\widehat{\mathbf{G}} \in (\partial_f \mathbf{F}_1(\bar{x}) \oplus \partial_f \mathbf{F}_2(\bar{x}))$. Then, $\widehat{\mathbf{G}} = \widehat{\mathbf{H}} \oplus \widehat{\mathbf{K}}$ for some $\widehat{\mathbf{H}} \in \partial_f \mathbf{F}_1(\bar{x})$ and $\widehat{\mathbf{K}} \in \partial_f \mathbf{F}_2(\bar{x})$. Therefore,

$$\mathbf{0} \preceq \liminf_{x \to \bar{x}} \frac{1}{\|x - \bar{x}\|} \odot (\mathbf{F}_1(x) \ominus_{gH} \mathbf{F}_1(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot \widehat{\mathbf{H}}) \tag{3.8}$$

and

$$\mathbf{0} \preceq \liminf_{x \to \bar{x}} \frac{1}{\|x - \bar{x}\|} \odot (\mathbf{F}_2(x) \ominus_{gH} \mathbf{F}_2(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot \widehat{\mathbf{K}}). \tag{3.9}$$

By adding (3.8) and (3.9), we obtain

$$\begin{aligned} \mathbf{0} \ &\preceq \ \liminf_{x \to \bar{x}} \frac{1}{\|x - \bar{x}\|} \odot (\mathbf{F}_1(x) \ominus_{gH} \mathbf{F}_1(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot \widehat{\mathbf{H}}) \oplus \\ &\quad\ \liminf_{x \to \bar{x}} \frac{1}{\|x - \bar{x}\|} \odot (\mathbf{F}_2(x) \ominus_{gH} \mathbf{F}_2(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot \widehat{\mathbf{K}}) \\ &\preceq \ \liminf_{x \to \bar{x}} \frac{1}{\|x - \bar{x}\|} \odot ((\mathbf{F}_1 \oplus \mathbf{F}_2)(x) \ominus_{gH} (\mathbf{F}_1 \oplus \mathbf{F}_2)(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot (\widehat{\mathbf{H}} \oplus \widehat{\mathbf{K}})), \\ &\quad\ \text{by (i) of Theorem 2.1.} \end{aligned}$$

This implies

$$\widehat{\mathbf{H}} \oplus \widehat{\mathbf{K}} = \widehat{\mathbf{G}} \in \partial_f(\mathbf{F}_1 \oplus \mathbf{F}_2)(\bar{x}),$$

and hence $\partial_f \mathbf{F}_1(\bar{x}) \oplus \partial_f \mathbf{F}_2(\bar{x}) \subseteq \partial_f(\mathbf{F}_1 \oplus \mathbf{F}_2)(\bar{x})$. □

In the next theorem (Theorem 3.5), we show that every gH-Fréchet subgradient $\widehat{\mathbf{G}}$ of an arbitrary IVF $\mathbf{F} : \mathbb{R}^n \to \overline{I(\mathbb{R})}$ at $\bar{x} \in \mathbb{R}^n$ can be equivalently described via the gH-derivative of another IVF $\mathbf{H}$ such that the difference $\mathbf{F} \ominus_{gH} \mathbf{H}$ attains its local minimum at $\bar{x}$. This property of gH-Fréchet subgradients of $\mathbf{F}$ is used to prove the difference rule for gH-Fréchet subgradients (Theorem 3.6).

Theorem 3.5. *Let $\boldsymbol{F} : \mathbb{R}^n \to \overline{I(\mathbb{R})}$ be a proper extended IVF and $\bar{x} \in \mathbb{R}^n$. Then, $\widehat{\boldsymbol{G}} \in \partial_f \boldsymbol{F}(\bar{x})$ if and only if there exists a function $\boldsymbol{H} : X \to I(\mathbb{R})$ such that*

(i) *$\boldsymbol{H}(x) \preceq \boldsymbol{F}(x)$ for any $x \in \mathbb{R}^n$, $\boldsymbol{H}(\bar{x}) = \boldsymbol{F}(\bar{x})$, and*
(ii) *$\boldsymbol{H}$ is gH-differentiable at $\bar{x}$ with $\boldsymbol{H}'(\bar{x}) = \widehat{\boldsymbol{G}}$.*

Proof. Let us first prove the sufficient part. Since $\mathbf{H}$ is gH-differentiable at $\bar{x}$ and $\mathbf{H}'(\bar{x}) = \widehat{\mathbf{G}}$, we have

$$\begin{aligned}\mathbf{0} &= \lim_{x\to\bar{x}} \frac{1}{\|x-\bar{x}\|} \odot (\mathbf{H}(x) \ominus_{gH} \mathbf{H}(\bar{x}) \ominus_{gH} (x-\bar{x})^\top \odot \widehat{\mathbf{G}}) \\ &= \liminf_{x\to\bar{x}} \frac{1}{\|x-\bar{x}\|} \odot (\mathbf{H}(x) \ominus_{gH} \mathbf{H}(\bar{x}) \ominus_{gH} (x-\bar{x})^\top \odot \widehat{\mathbf{G}}).\end{aligned} \tag{3.10}$$

Notice the following two points.

(a) By (i), we have

$$\mathbf{H}(x) \preceq \mathbf{F}(x) \text{ for any } x \in \mathbb{R}^n,\ \mathbf{H}(\bar{x}) = \mathbf{F}(\bar{x}), \text{ and}$$

(b) for all $\mathbf{A},\ \mathbf{B},\ \mathbf{C},\ \mathbf{D} \in I(\mathbb{R})$, we have

$$\mathbf{A} \ominus_{gH} \mathbf{B} \ominus_{gH} \mathbf{C} \preceq \mathbf{D} \ominus_{gH} \mathbf{B} \ominus_{gH} \mathbf{C} \text{ whenever } \mathbf{A} \preceq \mathbf{D}.$$

Therefore, (3.10) gives

$$\mathbf{0} \preceq \liminf_{x\to\bar{x}} \frac{1}{\|x-\bar{x}\|} \odot (\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x-\bar{x})^\top \odot \widehat{\mathbf{G}}),$$

and hence $\widehat{\mathbf{G}} \in \partial_f \mathbf{F}(\bar{x})$.

To prove the necessary part, consider $\mathbf{H}(x) = \inf\{\mathbf{F}(x), \mathbf{F}(\bar{x}) \oplus (x-\bar{x})^\top \odot \widehat{\mathbf{G}}\}$ for all $x \in \mathbb{R}^n$. Clearly, $\mathbf{H}(x) \preceq \mathbf{F}(x)$ for any $x \in \mathbb{R}^n$ and $\mathbf{H}(\bar{x}) = \mathbf{F}(\bar{x})$. Next, to see that $\mathbf{H}$ is gH-differentiable at $\bar{x}$ and $\mathbf{H}'(\bar{x}) = \widehat{\mathbf{G}}$, we evaluate the following limit:

$$\lim_{x\to\bar{x}} \frac{1}{\|x-\bar{x}\|} \odot (\mathbf{H}(x) \ominus_{gH} \mathbf{H}(\bar{x}) \ominus_{gH} (x-\bar{x})^\top \odot \widehat{\mathbf{G}}).$$

- Case 1. If $\mathbf{H}(x) = \mathbf{F}(\bar{x}) \oplus (x-\bar{x})^\top \odot \widehat{\mathbf{G}}$. Then,

$$\begin{aligned}&\lim_{x\to\bar{x}} \frac{1}{\|x-\bar{x}\|} \odot (\mathbf{H}(x) \ominus_{gH} \mathbf{H}(\bar{x}) \ominus_{gH} (x-\bar{x})^\top \odot \widehat{\mathbf{G}}) \\ = \ &\lim_{x\to\bar{x}} \frac{1}{\|x-\bar{x}\|} \odot (\mathbf{F}(\bar{x}) \oplus \mathbf{G}(x-\bar{x}) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x-\bar{x})^\top \odot \widehat{\mathbf{G}}) \\ = \ &\mathbf{0}.\end{aligned}$$

- Case 2. If $\mathbf{H}(x) = \mathbf{F}(x)$.

  Since $\widehat{\mathbf{G}} \in \partial_f \mathbf{F}(\bar{x})$, we have

$$\begin{aligned}\mathbf{0} \ &\preceq \ \liminf_{x\to\bar{x}} \frac{1}{\|x-\bar{x}\|} \odot (\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x-\bar{x})^\top \odot \widehat{\mathbf{G}}) \\ &= \ \liminf_{x\to\bar{x}} \frac{1}{\|x-\bar{x}\|} \odot (\mathbf{H}(x) \ominus_{gH} \mathbf{H}(\bar{x}) \ominus_{gH} (x-\bar{x})^\top \odot \widehat{\mathbf{G}}).\end{aligned} \tag{3.11}$$

  Observe that

$$\begin{aligned}&\liminf_{x\to\bar{x}} \frac{1}{\|x-\bar{x}\|} \odot (\mathbf{H}(x) \ominus_{gH} \mathbf{H}(\bar{x}) \ominus_{gH} (x-\bar{x})^\top \odot \widehat{\mathbf{G}}) \\ = \ &\liminf_{x\to\bar{x}} \frac{1}{\|x-\bar{x}\|} \odot (\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x-\bar{x})^\top \odot \widehat{\mathbf{G}}) \\ \preceq \ &\lim_{x\to\bar{x}} \frac{1}{\|x-\bar{x}\|} \odot (\mathbf{F}(\bar{x}) \oplus \mathbf{G}(x-\bar{x}) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x-\bar{x})^\top \odot \widehat{\mathbf{G}})\end{aligned}$$

$$\text{because } \mathbf{F}(x) \preceq \mathbf{F}(\bar{x}) \oplus (x-\bar{x})^\top \odot \widehat{\mathbf{G}}$$
$$= \mathbf{0}.$$

Thus,

$$\liminf_{x\to\bar{x}} \frac{1}{\|x-\bar{x}\|} \odot (\mathbf{H}(x) \ominus_{gH} \mathbf{H}(\bar{x}) \ominus_{gH} (x-\bar{x})^\top \odot \widehat{\mathbf{G}}) \preceq \mathbf{0}. \tag{3.12}$$

Therefore, by (3.11) and (3.12), we obtain

$$\liminf_{x\to\bar{x}} \frac{1}{\|x-\bar{x}\|} \odot (\mathbf{H}(x) \ominus_{gH} \mathbf{H}(\bar{x}) \ominus_{gH} (x-\bar{x})^\top \odot \widehat{\mathbf{G}}) = \mathbf{0}. \tag{3.13}$$

It is clear from (3.13) that

$$\mathbf{0} \preceq \limsup_{x\to\bar{x}} \frac{1}{\|x-\bar{x}\|} \odot (\mathbf{H}(x) \ominus_{gH} \mathbf{H}(\bar{x}) \ominus_{gH} (x-\bar{x})^\top \odot \widehat{\mathbf{G}}). \tag{3.14}$$

Again, since

$$\begin{aligned}
&\limsup_{x\to\bar{x}} \frac{1}{\|x-\bar{x}\|} \odot (\mathbf{H}(x) \ominus_{gH} \mathbf{H}(\bar{x}) \ominus_{gH} (x-\bar{x})^\top \odot \widehat{\mathbf{G}}) \\
=\ &\limsup_{x\to\bar{x}} \frac{1}{\|x-\bar{x}\|} \odot (\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x-\bar{x})^\top \odot \widehat{\mathbf{G}}) \\
\preceq\ &\lim_{x\to\bar{x}} \frac{1}{\|x-\bar{x}\|} \odot (\mathbf{F}(\bar{x}) \oplus \mathbf{G}(x-\bar{x}) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x-\bar{x})^\top \odot \widehat{\mathbf{G}}) \\
=\ &\mathbf{0},
\end{aligned}$$

we obtain

$$\limsup_{x\to\bar{x}} \frac{1}{\|x-\bar{x}\|} \odot (\mathbf{H}(x) \ominus_{gH} \mathbf{H}(\bar{x}) \ominus_{gH} (x-\bar{x})^\top \odot \widehat{\mathbf{G}}) \preceq \mathbf{0}. \tag{3.15}$$

Therefore, by (3.14) and (3.15), we obtain

$$\limsup_{x\to\bar{x}} \frac{1}{\|x-\bar{x}\|} \odot (\mathbf{H}(x) \ominus_{gH} \mathbf{H}(\bar{x}) \ominus_{gH} (x-\bar{x})^\top \odot \widehat{\mathbf{G}}) = \mathbf{0}. \tag{3.16}$$

Hence, from (3.13) and (3.16), we have

$$\lim_{x\to\bar{x}} \frac{1}{\|x-\bar{x}\|} \odot (\mathbf{H}(x) \ominus_{gH} \mathbf{H}(\bar{x}) \ominus_{gH} (x-\bar{x})^\top \odot \widehat{\mathbf{G}} = \mathbf{0}.$$

That is, $\mathbf{H}'(\bar{x}) = \widehat{\mathbf{G}}$. Hence, from Case 1 and Case 2, $\mathbf{H}'(\bar{x}) = \widehat{\mathbf{G}}$, which completes the proof. □

Apart from Theorem 3.5, we also need the following lemma to prove Theorem 3.6.

Lemma 3.1. *Let $\boldsymbol{F}_1$, $\boldsymbol{F}_2 : \mathbb{R}^n \to \overline{I(\mathbb{R})}$ be two proper extended IVFs, which are finite at $\bar{x} \in \mathbb{R}^n$. Further, let $\boldsymbol{F}_1 \ominus_{gH} \boldsymbol{F}_2$ and $\boldsymbol{F}_2$ be gH-Fréchet subdifferentiable at $\bar{x}$. Then, $\boldsymbol{F}_1$ is gH-Fréchet subdifferentiable at $\bar{x}$, and*

$$\partial_f(\boldsymbol{F}_1 \ominus_{gH} \boldsymbol{F}_2)(\bar{x}) \oplus \partial_f \boldsymbol{F}_2(\bar{x}) \subseteq \partial_f \boldsymbol{F}_1(\bar{x}).$$

Proof. We are given that $\mathbf{F}_1 \ominus_{gH} \mathbf{F}_2$ and $\mathbf{F}_2$ are gH-Fréchet subdifferentiable at $\bar{x}$. So, by Theorem 3.4, their sum is gH-Fréchet subdifferentiable at $\bar{x}$. That is, $\mathbf{F}_1$ is gH-Fréchet subdifferentiable at $\bar{x}$. Also, by applying Theorem 3.4 to $\mathbf{F}_1 \ominus_{gH} \mathbf{F}_2$ and $\mathbf{F}_2$, we obtain

$$\partial_f(\mathbf{F}_1 \ominus_{gH} \mathbf{F}_2)(\bar{x}) \oplus \partial \mathbf{F}_2(\bar{x}) \subseteq \partial_f(\mathbf{F}_1 \ominus_{gH} \mathbf{F}_2 \oplus \mathbf{F}_2)(\bar{x}) = \partial_f \mathbf{F}_1(\bar{x}).$$

□

Theorem 3.6. (Difference rule for gH-Fréchet subgradients). *Let $\boldsymbol{F}_1,\ \boldsymbol{F}_2 : \mathbb{R}^n \to \overline{I(\mathbb{R})}$ be two proper extended IVFs, finite at $\bar{x} \in \mathbb{R}^n$. Assume that $\partial_f \boldsymbol{F}_2(\bar{x}) \neq \emptyset$. Then,*

$$\partial_f(\boldsymbol{F}_1 \ominus_{gH} \boldsymbol{F}_2)(\bar{x}) \subseteq \bigcap_{\widehat{\boldsymbol{G}} \in \partial_f \boldsymbol{F}_2(\bar{x})} (\partial_f \boldsymbol{F}_1(\bar{x}) \ominus_{gH} \widehat{\mathbf{G}}) \subseteq \partial_f \boldsymbol{F}_1(\bar{x}) \ominus_{gH} \partial_f \boldsymbol{F}_2(\bar{x}). \tag{3.17}$$

Proof. To prove (3.17), fix any $\widehat{\mathbf{H}} \in \partial_f(\mathbf{F}_1 \ominus_{gH} \mathbf{F}_2)(\bar{x})$ and $\widehat{\mathbf{K}} \in \partial_f \mathbf{F}_2(\bar{x})$. By applying Theorem 3.5, for gH-Fréchet subgradient $\widehat{\mathbf{K}} \in \partial_f \mathbf{F}_2(\bar{x})$, we obtain an IVF $\mathbf{H}$ such that

$$\mathbf{H}(x) \preceq \mathbf{F}_2(x) \text{ for any } x \in \mathbb{R}^n,\ \mathbf{H}(\bar{x}) = \mathbf{F}_2(\bar{x}) \text{ and } \mathbf{H}'(\bar{x}) = \widehat{\mathbf{K}}. \tag{3.18}$$

Since $\widehat{\mathbf{H}} \in \partial_f(\mathbf{F}_1 \ominus_{gH} \mathbf{F}_2)(\bar{x})$, we obtain

$$\mathbf{0} \preceq \liminf_{x \to \bar{x}} \frac{1}{\|x - \bar{x}\|} \odot ((\mathbf{F}_1 \ominus_{gH} \mathbf{F}_2)(x) \ominus_{gH} (\mathbf{F}_1 \ominus_{gH} \mathbf{F}_2)(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot \widehat{\mathbf{H}}).$$

Therefore, for any $\varepsilon > 0$, we obtain a $\delta > 0$ such that whenever $0 < \|x - \bar{x}\| < \delta$, we have

$$\begin{aligned} & \mathbf{0} \preceq \mathbf{F}_1(x) \ominus_{gH} \mathbf{F}_2(x) \ominus_{gH} (\mathbf{F}_1(\bar{x}) \ominus_{gH} \mathbf{F}_2(\bar{x})) \ominus_{gH} (x - \bar{x})^\top \odot \widehat{\mathbf{H}} \oplus \varepsilon \|x - \bar{x}\| \\ \implies & (x - \bar{x})^\top \odot \widehat{\mathbf{H}} \preceq \mathbf{F}_1(x) \ominus_{gH} \mathbf{F}_2(x) \ominus_{gH} (\mathbf{F}_1(\bar{x}) \ominus_{gH} \mathbf{F}_2(\bar{x})) \oplus \varepsilon \|x - \bar{x}\|, \\ & \text{by (i) of Lemma 2.3} \\ \implies & (x - \bar{x})^\top \odot \widehat{\mathbf{H}} \preceq \mathbf{F}_1(x) \ominus_{gH} \mathbf{H}(x) \ominus_{gH} (\mathbf{F}_1(\bar{x}) \ominus_{gH} \mathbf{H}(\bar{x})) \oplus \varepsilon \|x - \bar{x}\|, \\ & \text{because } \mathbf{H}(x) \preceq \mathbf{F}_2(x) \text{ for any } x \in X \text{ and } \mathbf{H}(\bar{x}) = \mathbf{F}_2(\bar{x}). \end{aligned}$$

Thus,

$$\widehat{\mathbf{H}} \in \partial_f(\mathbf{F}_1 \ominus_{gH} \mathbf{H})(\bar{x}).$$

Also, by Lemma 3.1, we obtain $\partial_f(\mathbf{F}_1 \ominus_{gH} \mathbf{H})(\bar{x}) \subseteq \partial_f \mathbf{F}_1(\bar{x}) \ominus_{gH} \partial_f \mathbf{H}(\bar{x})$.
Hence, by using (3.18), we obtain

$$\widehat{\mathbf{H}} \in \partial_f \mathbf{F}_1(\bar{x}) \ominus_{gH} \widehat{\mathbf{K}},$$

which proves the difference rule (3.17). □

To conclude this section, we derive a rule for calculating gH-Fréchet subgradients of the minimum function,

$(\wedge \mathbf{F}_i)(x) := \inf \{\mathbf{F}_i | i = 1, 2, \ldots, n\}$, where $\mathbf{F}_i : \mathbb{R}^n \to \overline{I(\mathbb{R})}$ is a proper IVF for each i and $n \geq 2$.

Denote

$$I(x) := \{j \in \{1, 2, \ldots, n\} | \mathbf{F}_j(x) = (\wedge \mathbf{F}_i)(x)\}.$$

Theorem 3.7. *The following inclusion holds:*

$$\partial_f(\wedge \boldsymbol{F}_i)(\bar{x}) \subseteq \bigcap_{j \in I(\bar{x})} \partial_f \boldsymbol{F}_j(\bar{x}).$$

Proof. Take $\widehat{\mathbf{G}} \in \partial_f(\wedge \mathbf{F}_i)(\bar{x})$. Then, by Definition 3.1, we have

$$\mathbf{0} \preceq \liminf_{x \to \bar{x}} \frac{1}{\|x - \bar{x}\|} \odot ((\wedge \mathbf{F}_i)(x) \ominus_{gH} (\wedge \mathbf{F}_i)(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot \widehat{\mathbf{G}}).$$

That is, for any $\varepsilon > 0$, we obtain a $\delta > 0$ such that whenever $0 < \|x - \bar{x}\| < \delta$, we have

$$\mathbf{0} \preceq (\wedge \mathbf{F}_i)(x) \ominus_{gH} (\wedge \mathbf{F}_i)(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot \widehat{\mathbf{G}} \oplus \varepsilon \|x - \bar{x}\|.$$

Therefore, for x such that $0 < \|x - \bar{x}\| < \delta$ and for any $j \in I(\bar{x})$, we have

$$\begin{aligned} (x - \bar{x})^\top \odot \widehat{\mathbf{G}} &\preceq (\wedge \mathbf{F}_i)(x) \ominus_{gH} (\wedge \mathbf{F}_i)(\bar{x}) \oplus \varepsilon \|x - \bar{x}\|, \text{ by (i) of Lemma 2.3} \\ &= (\wedge \mathbf{F}_i)(x) \ominus_{gH} (\mathbf{F}_j)(\bar{x}) \oplus \varepsilon \|x - \bar{x}\| \\ &\preceq \mathbf{F}_j(x) \ominus_{gH} \mathbf{F}_j(\bar{x}) \oplus \varepsilon \|x - \bar{x}\|, \end{aligned}$$

which proves that $\widehat{\mathbf{G}} \in \partial_f \mathbf{F}_j(\bar{x})$, and hence the proof is complete. □

4. Necessary Optimality Conditions for IOPs with Nondifferentiable IVFs

With the help of the studied concepts in Section 3, we now proceed to identify optimality conditions for the following unconstrained IOP:

$$\min_{x \in \mathbb{R}^n} \mathbf{F}(x), \tag{4.1}$$

where $\mathbf{F} : \mathbb{R}^n \to \overline{I(\mathbb{R})}$ is a proper extended IVF. By an optimum solution of (4.1), we refer to the following concept. A point $\bar{x} \in \mathbb{R}^n$ is called a *weak efficient solution* of (4.1) if $\mathbf{F}(\bar{x}) \preceq \mathbf{F}(x)$ for all $x \in \mathbb{R}^n$ (see [44]).

Theorem 4.1. *If $\bar{x}$ is a weak efficient solution of (4.1), then $\widehat{\mathbf{0}} \in \partial_f \mathbf{F}(\bar{x})$.*

Proof. Since $\bar{x}$ is a weak efficient solution of (4.1),

$$\begin{aligned} & \mathbf{F}(\bar{x}) \preceq \mathbf{F}(x) \text{ for all } x \in \mathbb{R}^n \\ \Longrightarrow\ & \mathbf{0} \preceq \mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \text{ for all } x \in \mathbb{R}^n, \text{ by (i) of Lemma 2.3} \\ \Longrightarrow\ & \mathbf{0} \preceq \frac{1}{\|x - \bar{x}\|} \odot (\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot \widehat{\mathbf{0}}) \text{ for all } x \in \mathbb{R}^n \\ \Longrightarrow\ & \mathbf{0} \preceq \liminf_{x \to \bar{x}} \frac{1}{\|x - \bar{x}\|} \odot (\mathbf{F}(x) \ominus_{gH} \mathbf{F}(\bar{x}) \ominus_{gH} (x - \bar{x})^\top \odot \widehat{\mathbf{0}}) \\ \Longrightarrow\ & \widehat{\mathbf{0}} \in \partial_f \mathbf{F}(\bar{x}). \end{aligned}$$

□

We next consider the following example to verify Theorem 4.1.

Example 4.1. Consider the following IOP:

$$\min_{x \in \mathbb{R}} \mathbf{F}(x) = \begin{cases} [-(x+1), x^2 - 1], & x \leq -1 \\ [0, 1 - x^2|], & -1 \leq x \leq 1 \\ [x - 1, x^2 - 1], & x \geq 1. \end{cases} \tag{4.2}$$

The graph of the IVF $\mathbf{F}$ is illustrated in Figure 2 by the grey shaded region. From Figure 2, it is clear that $\mathbf{F}$ is not convex. Also, observe that $\mathbf{F}(-1) = \mathbf{F}(1) \preceq \mathbf{F}(x)$ for all $x \in \mathbb{R}$. Hence, -1

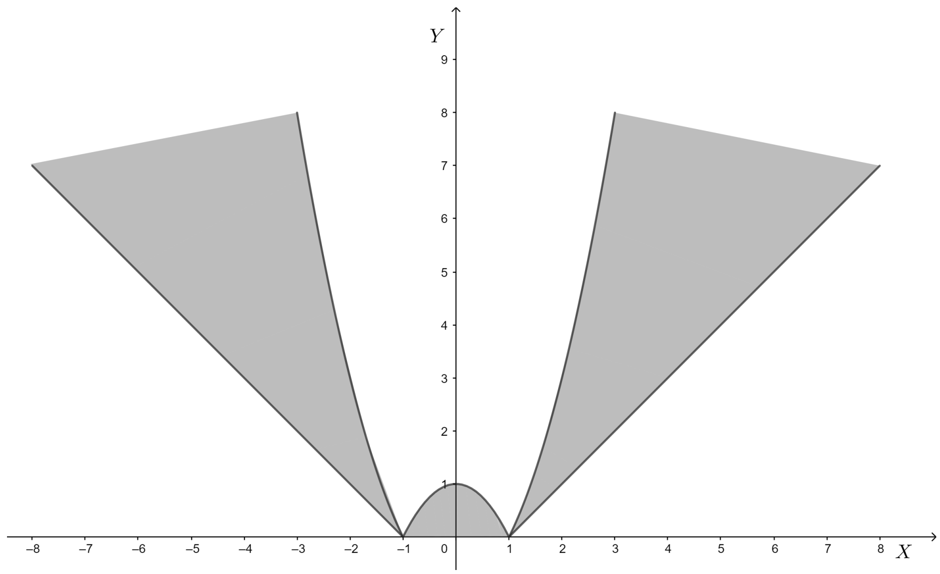

FIGURE 2. Objective function of the IOP (4.2) of Example 4.1

and 1 are two weak efficient solutions of (4.2).
At $\bar{x} = -1$,

$$\begin{aligned}(\mathbf{F}'(x))_+ : &= \lim_{d \to 0^+} \frac{1}{d} \odot (\mathbf{F}(\bar{x}+d) \ominus_{gH} \mathbf{F}(\bar{x}))\\ &= \lim_{d \to 0^+} \frac{1}{d} \odot (\mathbf{F}(d-1) \ominus_{gH} \mathbf{F}(-1))\\ &= \lim_{d \to 0^+} \frac{1}{d} \odot [0, 1-(d-1)^2]\\ &= \lim_{d \to 0^+} \frac{1}{d} \odot [0, -d^2+2d]\\ &= [0,2].\end{aligned}$$

and

$$\begin{aligned}(\mathbf{F}'(\bar{x}))_- : &= \lim_{d \to 0^-} \frac{1}{d} \odot (\mathbf{F}(\bar{x}+d) \ominus_{gH} \mathbf{F}(\bar{x}))\\ &= \lim_{d \to 0^-} \frac{1}{d} \odot (\mathbf{F}(d-1) \ominus_{gH} \mathbf{F}(-1))\\ &= \lim_{d \to 0^-} \frac{1}{d} \odot [-d, d^2-2d]\\ &= [-2,-1] \neq (\mathbf{F}'(x))_+.\end{aligned}$$

Therefore, at the point $x = -1$, $\mathbf{F}$ is not gH-differentiable. Similarly, it can be proved that $\mathbf{F}$ is not gH-differentiable at the point $x = 1$. Note that

$$\begin{aligned}
&\liminf_{x\to -1} \frac{1}{|x+1|} \odot (\mathbf{F}(x) \ominus_{gH} \mathbf{F}(-1) \ominus_{gH} (x+1)^\top \odot \mathbf{0}) \\
= &\liminf_{x\to -1} \frac{1}{|x+1|} \odot \mathbf{F}(x) \\
= &\liminf_{x\to -1} \frac{1}{|x+1|} \odot [\underline{f}(x), \overline{f}(x)], \text{ where} \\
&\underline{f}(x) = \begin{cases} -(x+1), & x \leq -1 \\ 0, & -1 \leq x \leq 1 \\ x-1, & x \geq 1 \end{cases} \text{ and } \overline{f}(x) = \begin{cases} x^2-1, & x \leq -1 \\ 1-x^2, & -1 \leq x \leq 1 \\ x^2-1, & x \geq 1. \end{cases}
\end{aligned}$$

It can be easily seen that

$$\liminf_{x\to -1} \frac{1}{|x+1|} \underline{f}(x) = 0 \text{ and } \liminf_{x\to -1} \frac{1}{|x+1|} \overline{f}(x) = 2.$$

Therefore,

$$\liminf_{x\to -1} \frac{1}{|x+1|} \odot \mathbf{F}(x) = [0,2].$$

Clearly,

$$\mathbf{0} \preceq [0,2] = \liminf_{x\to -1} \frac{1}{|x+1|} \odot \mathbf{F}(x) = \liminf_{x\to -1} \frac{1}{|x+1|} \odot (\mathbf{F}(x) \ominus_{gH} \mathbf{F}(-1) \ominus_{gH} (x+1)^\top \odot \mathbf{0}).$$

Thus, by Definition 3.1, $\mathbf{0} \in \partial_f \mathbf{F}(-1)$, which verifies Theorem 4.1 for the weak efficient solution $x = -1$. Similarly, Theorem 4.1 can be verified for the weak efficient solution $x = 1$.

Remark 4.1. One may think that the optimality condition given in Theorem 4.1 is useful only to solve unconstrained IOPs. However, this is not the case. Since every constrained IOP can be converted into unconstrained IOP with the help of extended IVFs (for details, see subsection 2.2), Theorem 4.1 can be useful to solve constrained IOPs as well.

Note 4.1. The converse of Theorem 4.1 is not true. For example, consider $\mathbf{F} : \mathbb{R} \to I(\mathbb{R})$ as $\mathbf{F}(x) = [1,2] \odot x^3$. Then,

$$\liminf_{x\to 0} \frac{1}{|x-0|} \odot (\mathbf{F}(x) \ominus_{gH} \mathbf{F}(0) \ominus_{gH} (x-0) \odot \mathbf{0}) = \liminf_{x\to 0} \frac{1}{|x|} \odot \left([1,2] \odot x^3\right) = \mathbf{0}.$$

Therefore, $\mathbf{0} \in \partial_f \mathbf{F}(0)$. However, 0 is not a weak efficient solution of $\mathbf{F}$ as $\mathbf{F}(-1) = [-2,-1] \prec \mathbf{F}(0)$.

In the next theorem (Theorem 4.2), we provide a necessary condition for $\bar{x} \in \mathbb{R}^n$ to be a weak efficient solution of an unconstrained IOP whose objective function is given as difference of two IVFs.

Theorem 4.2. (Necessary optimality condition for minimizing difference IVFs). *Let $\bar{x} \in \mathbb{R}^n$ be a weak efficient solution of the difference IVF $\boldsymbol{F} = \boldsymbol{F}_1 \ominus_{gH} \boldsymbol{F}_2$, where both $\boldsymbol{F}_1, \boldsymbol{F}_2 : \mathbb{R}^n \to \overline{I(\mathbb{R})}$ are proper extended IVFs, finite at $\bar{x}$. Then, the following inclusion holds*

$$\partial_f \boldsymbol{F}_2(\bar{x}) \subseteq \partial_f \boldsymbol{F}_1(\bar{x}).$$

Proof. Since $\bar{x}$ is a weak efficient solution of $\mathbf{F}$, by Theorem 4.1, $\widehat{\mathbf{0}} \in \partial_f \mathbf{F}(\bar{x})$. Therefore,

$$\mathbf{0} \preceq \liminf_{x \to \bar{x}} \frac{1}{\|x-\bar{x}\|} \odot \left((\mathbf{F}_1 \ominus_{gH} \mathbf{F}_2)(x) \ominus_{gH} (\mathbf{F}_1 \ominus_{gH} \mathbf{F}_2)(\bar{x}) \ominus_{gH} (x-\bar{x})^\top \odot \widehat{\mathbf{0}} \right),$$

$$\text{i.e.,} \quad \mathbf{0} \preceq \liminf_{x \to \bar{x}} \frac{\mathbf{F}_1(x) \ominus_{gH} \mathbf{F}_2(x) \ominus_{gH} (\mathbf{F}_1(\bar{x}) \ominus_{gH} \mathbf{F}_2(\bar{x}))}{\|x-\bar{x}\|}.$$

Thus, for each $\varepsilon > 0$, we obtain a $\delta_1 > 0$ such that whenever $0 < \|x-\bar{x}\| < \delta_1$, we have

$$\mathbf{0} \preceq \mathbf{F}_1(x) \ominus_{gH} \mathbf{F}_2(x) \ominus_{gH} (\mathbf{F}_1(\bar{x}) \ominus_{gH} \mathbf{F}_2(\bar{x})) \oplus \varepsilon \|x-\bar{x}\|.$$

This implies

$$\mathbf{F}_2(x) \ominus_{gH} \mathbf{F}_2(\bar{x}) \preceq \mathbf{F}_1(x) \ominus_{gH} \mathbf{F}_1(\bar{x}) \oplus \varepsilon \|x-\bar{x}\|. \tag{4.3}$$

Next to prove that $\partial_f \mathbf{F}_2(\bar{x}) \subseteq \partial_f \mathbf{F}_1(\bar{x})$, consider $\widehat{\mathbf{G}} \in \partial_f \mathbf{F}_2(\bar{x})$. Then,

$$\mathbf{0} \preceq \liminf_{x \to \bar{x}} \frac{\mathbf{F}_2(x) \ominus_{gH} \mathbf{F}_2(\bar{x}) \ominus_{gH} (x-\bar{x})^\top \odot \widehat{\mathbf{G}}}{\|x-\bar{x}\|}.$$

Thus, for each $\varepsilon > 0$, we obtain a $\delta_2 > 0$ such that whenever $0 < \|x-\bar{x}\| < \delta_2$, we have

$$\mathbf{0} \preceq \mathbf{F}_2(x) \ominus_{gH} \mathbf{F}_2(\bar{x}) \ominus_{gH} (x-\bar{x})^\top \odot \widehat{\mathbf{G}} \oplus \varepsilon \|x-\bar{x}\|.$$

Therefore, by using (i) of Lemma 2.3, we obtain

$$(x-\bar{x})^\top \odot \widehat{\mathbf{G}} \preceq \mathbf{F}_2(x) \ominus_{gH} \mathbf{F}_2(\bar{x}) \oplus \varepsilon \|x-\bar{x}\|. \tag{4.4}$$

By taking $\delta = \min\{\delta_1, \ \delta_2\}$ and using (4.3) and (4.4), we obtain

$$(x-\bar{x})^\top \odot \widehat{\mathbf{G}} \preceq \mathbf{F}_1(x) \ominus_{gH} \mathbf{F}_1(\bar{x}) \oplus \varepsilon \|x-\bar{x}\| \text{ whenever } 0 < \|x-\bar{x}\| < \delta.$$

This implies $\widehat{\mathbf{G}} \in \partial_f \mathbf{F}_1(\bar{x})$. Hence, $\partial_f \mathbf{F}_2(\bar{x}) \subseteq \partial_f \mathbf{F}_1(\bar{x})$. □

It is well known that the notion of conventional WSM introduced in Burke and Ferris [46], plays an important role in the sensitivity analysis and convergence analysis of conventional optimization problems. Recently, Krishan et al. [44] extended the notion of WSM for IVFs and showed its applications to solve linear and quadratic IOPs. Adding to the literature of WSM for IVFs, in Corollary 4.1, we provide a necessary condition for a subset S of $\mathbb{R}^n$ to be a set of WSM of an extended IVF $\mathbf{F}$.

Corollary 4.1. (Necessary condition for unconstrained weak sharp minima). *Let S be the set of WSM for the function $\boldsymbol{F} : \mathbb{R}^n \to \overline{I(\mathbb{R})}$ relative to the whole space $\mathbb{R}^n$ with modulus α. Then, for every $\bar{x} \in S$, we have*

$$\alpha \mathbb{B} \cap \widehat{N}(\bar{x}, S) \subseteq \partial_f \boldsymbol{F}(\bar{x}),$$

where $\mathbb{B} \subseteq \mathbb{R}^n$ stands for the closed unit ball and $\widehat{N}(\bar{x}, S)$ denotes the Fréchet normal cone to S at $\bar{x}$.

Proof. By definition of WSM, we have

$$\mathbf{F}(y) \oplus \alpha \operatorname{dist}(x, S) \preceq \mathbf{F}(x) \text{ for all } x \in \mathbb{R}^n \text{ and } y \in S.$$

Thus, every $y \in S$ is a weak efficient solution to the unconstrained problem of minimizing the difference function $\mathbf{G}(x) := \mathbf{F}(x) \ominus_{gH} \alpha \operatorname{dist}(x, S)$. Therefore, by Theorem 4.2, we obtain

$$\alpha \partial_f \operatorname{dist}(\bar{x}, S) \subseteq \partial_f \mathbf{F}(\bar{x}). \tag{4.5}$$

By Note 2.5, we have

$$\partial_f \operatorname{dist}(\bar{x}, S) = \widehat{N}(\bar{x}, S) \cap \mathbb{B}.$$

Thus, by (4.5), we obtain $\alpha \mathbb{B} \cap \widehat{N}(\bar{x}, S) \subseteq \partial_f \mathbf{F}(\bar{x})$. □

Example 4.2. In this Example, we verify corollary 4.1 for the IVF

$$\mathbf{F}(x) = \begin{cases} [-(x+2), -(x-3)], & x < -2 \\ [0, 5], & x \in [-2, 2] \\ [x-2, x+3], & x > 2. \end{cases}$$

In Fig. 3, we have depicted the graph of $\mathbf{F}(x)$ by light grey-shaded region. The dark grey-

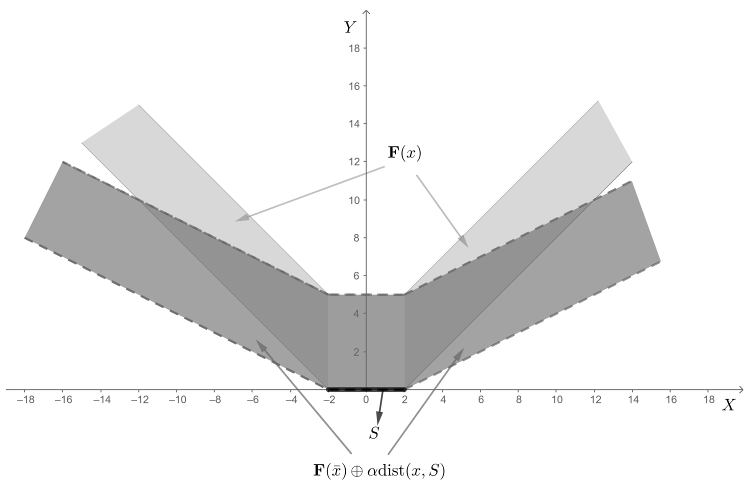

FIGURE 3. Objective function **F** of Example 4.2 and the location of the set of WSM of **F**

shaded region with dashed lines show the graph of $\mathbf{F}(\bar{x}) \oplus \alpha \operatorname{dist}(x, S)$. From the graphs, notice that for any $x \in \mathbb{R}$,

$$\mathbf{F}(\bar{x}) \oplus \alpha \operatorname{dist}(x, S) \preceq \mathbf{F}(x) \text{ for all } \bar{x} \in S \text{ and } \alpha = \frac{1}{2}.$$

Hence, by Definition 2.20 of WSM, $S = [-2, 2]$ is the set of WSM of given $\mathbf{F}$ relative to the whole space $\mathbb{R}$ with modulus $\alpha = \frac{1}{2}$. Note that

$$\widehat{N}(\bar{x}, S) = \begin{cases} 0, & \text{if } x \in (-2, 2) \\ [0, \infty), & \text{if } x = 2 \\ (-\infty, 0], & \text{if } x = -2. \end{cases}$$

Therefore, for all $\bar{x} \in (-2, 2)$, $\alpha \mathbb{B} \cap \widehat{N}(\bar{x}, S) = \{0\}$. Also, $\partial_f \mathbf{F}(\bar{x}) = 0$ for all $\bar{x} \in (-2, 2)$. Therefore,

$$\alpha \mathbb{B} \cap \widehat{N}(\bar{x}, S) \subseteq \partial_f \mathbf{F}(\bar{x}) \text{ for all } \bar{x} \in (-2, 2). \tag{4.6}$$

At $\bar{x} = 2$, $\alpha \mathbb{B} \cap \widehat{N}(\bar{x}, S) = [0, \frac{1}{2}]$. It can be easily seen that

$$(x-2) \odot [0, 1] \preceq \mathbf{F}(x) \ominus_{gH} \mathbf{F}(2) \text{ for all } x \in \mathbb{R}.$$

Therefore, $[0,1] \in \partial_f \mathbf{F}(2)$. Thus,

$$\alpha\mathbb{B} \cap \widehat{N}(2,S) \subseteq \partial_f \mathbf{F}(2). \tag{4.7}$$

At $\bar{x} = -2$, $\alpha\mathbb{B} \cap \widehat{N}(\bar{x},S) = [-\frac{1}{2},0]$. It can be easily seen that

$$(x+2) \odot [-1,0] \preceq \mathbf{F}(x) \ominus_{gH} \mathbf{F}(-2) \text{ for all } x \in \mathbb{R}.$$

Therefore, $[-1,0] \in \partial_f \mathbf{F}(-2)$. Thus,

$$\alpha\mathbb{B} \cap \widehat{N}(-2,S) \subseteq \mathbf{F}(-2). \tag{4.8}$$

Hence, by (4.6), (4.7) and (4.8), we have

$$\alpha\mathbb{B} \cap \widehat{N}(\bar{x},S) \subseteq \partial_f \mathbf{F}(\bar{x}) \text{ for all } \bar{x} \in S.$$

5. Discussion and Conclusion

In this article, the concept of gH-Fréchet subdifferentiability has been introduced (Definition 3.1). Various calculus results for gH-Fréchet subgradients has been provided. It has been shown that for a gH- Fréchet differentiable IVF, gH-Fréchet subdifferentiable set reduces to a singleton, i.e., $\partial_f \mathbf{F}(\bar{x}) = \{\nabla \mathbf{F}(\bar{x})\}$ (Theorem 3.2). A smooth variational description of gH-Fréchet subgradients has been given (Theorem 3.5). By using the proposed notion of subdifferentiability, necessary optimality condition for unconstrained IOPs with nondifferentiable IVFs has been given (Theorems 4.1 and 4.2). A necessary condition for unconstrained WSM has been given (Corollary 4.1).

Based on the proposed research, in future, we have the following directions to work on.

- (i) In the literature, several optimality conditions are provided for unconstrained and constrained smooth IOPs (for instance, see ([42, 47]), and references therein). However, the optimality conditions for nonsmooth IOPs are not much explored. Therefore, in our future research, we shall work on the application of gH-Fréchet subdifferentiability in constrained interval optimization with nondifferentiable IVFs.
- (ii) Recently, the authors of [38] presented gH-gradient based algorithms to solve convex IOPs. However, for nonsmooth IOPs there are no algorithms available in the literature of IOPs. Thus, in future, we shall try to develop a gH-Fréchet subgradient method to solve IOPs with nonconvex and nondifferentiable IVFs.
- (iii) As IVFs are the special case of set-valued functions and IOPs are the special case of set optimization problems, similar results can be extended for set-valued functions and set optimization problems.

Appendix A. Proof of the Lemma 2.3

Proof.
Proof of (i). Let $\mathbf{A} = [\underline{a},\overline{a}]$ and $\mathbf{B} = [\underline{b},\overline{b}]$.
Then, $\mathbf{A} \ominus_{gH} \mathbf{B} = [\underline{a}-\underline{b}, \overline{a}-\overline{b}]$ or $[\overline{a}-\overline{b}, \underline{a}-\underline{b}]$. Let us now consider the following two possible cases.

• Case 1. $\mathbf{A} \ominus_{gH} \mathbf{B} = [\underline{a} - \underline{b}, \overline{a} - \overline{b}]$.
We see that

$$\begin{aligned} \mathbf{0} \preceq \mathbf{A} \ominus_{gH} \mathbf{B} &\iff \mathbf{0} \preceq [\underline{a} - \underline{b}, \overline{a} - \overline{b}] \iff 0 \leq \underline{a} - \underline{b} \text{ and } 0 \leq \overline{a} - \overline{b} \\ &\iff \underline{b} \leq \underline{a} \text{ and } \overline{b} \leq \overline{a} \iff \mathbf{B} \preceq \mathbf{A}. \end{aligned}$$

• Case 2. $\mathbf{A} \ominus_{gH} \mathbf{B} = [\overline{a} - \overline{b}, \underline{a} - \underline{b}]$.
Note that

$$\begin{aligned} \mathbf{0} \preceq \mathbf{A} \ominus_{gH} \mathbf{B} &\iff \mathbf{0} \preceq [\overline{a} - \overline{b}, \underline{a} - \underline{b}] \iff 0 \leq \overline{a} - \overline{b} \text{ and } 0 \leq \underline{a} - \underline{b} \\ &\iff \overline{b} \leq \overline{a} \text{ and } \underline{b} \leq \underline{a} \iff \mathbf{B} \preceq \mathbf{A}. \end{aligned}$$

Hence, $\mathbf{0} \preceq \mathbf{A} \ominus_{gH} \mathbf{B} \iff \mathbf{B} \preceq \mathbf{A}$.

Proof of (ii). Let $\mathbf{A} = [\underline{a}, \overline{a}]$ and $\mathbf{B} = [\underline{b}, \overline{b}]$.
Then, $\mathbf{A} \ominus_{gH} \mathbf{B} = [\underline{a} - \underline{b}, \overline{a} - \overline{b}]$ or $[\overline{a} - \overline{b}, \underline{a} - \underline{b}]$. Let us now consider the following two possible cases.

• Case 1. $\mathbf{A} \ominus_{gH} \mathbf{B} = [\underline{a} - \underline{b}, \overline{a} - \overline{b}]$.
In this case, we have $\underline{a} - \underline{b} \leq \overline{a} - \overline{b}$. This implies

$$\underline{b} - \underline{a} \geq \overline{b} - \overline{a}. \tag{A.1}$$

Also, in this case, we have $(-1) \odot (\mathbf{A} \ominus_{gH} \mathbf{B}) = [\overline{b} - \overline{a}, \underline{b} - \underline{a}]$. Notice that

$$\begin{aligned} \mathbf{B} \ominus_{gH} \mathbf{A} &= [\min\{\underline{b} - \underline{a}, \overline{b} - \overline{a}\}, \max\{\underline{b} - \underline{a}, \overline{b} - \overline{a}\}] \\ &= [\overline{b} - \overline{a}, \underline{b} - \underline{a}], \text{ by (A.1)} \\ &= (-1) \odot (\mathbf{A} \ominus_{gH} \mathbf{B}). \end{aligned}$$

• Case 2. $\mathbf{A} \ominus_{gH} \mathbf{B} = [\overline{a} - \overline{b}, \underline{a} - \underline{b}]$.
In this case, we have $\overline{a} - \overline{b} \leq \underline{a} - \underline{b}$. Therefore,

$$\overline{b} - \overline{a} \geq \underline{b} - \underline{a}. \tag{A.2}$$

Also, in this case, we have $(-1) \odot (\mathbf{A} \ominus_{gH} \mathbf{B}) = [\underline{b} - \underline{a}, \overline{b} - \overline{a}]$. Note that

$$\begin{aligned} \mathbf{B} \ominus_{gH} \mathbf{A} &= [\min\{\underline{b} - \underline{a}, \overline{b} - \overline{a}\}, \max\{\underline{b} - \underline{a}, \overline{b} - \overline{a}\}] \\ &= [\underline{b} - \underline{a}, \overline{b} - \overline{a}], \text{ by (A.2)} \\ &= (-1) \odot (\mathbf{A} \ominus_{gH} \mathbf{B}). \end{aligned}$$

Hence, from Case 1 and 2, we obtain

$$(-1) \odot (\mathbf{A} \ominus_{gH} \mathbf{B}) = \mathbf{B} \ominus_{gH} \mathbf{A}.$$

□

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REFERENCES

[1] D.P. Bertsekas, Incremental gradient, subgradient, and proximal methods for convex optimization: A survey, Optimization for Machine Learning, 3 (2010), 1-38.

[2] D. Mifflin, L. Qi, R. Sun, Quasi-Newton bundle-type methods for nondifferentiable convex optimization, SIAM Journal on Optimization, 8 (1998), 583-603.

[3] A. Nedić, A. Ozdaglar, Subgradient methods for saddle-point problems, Journal of Optimization Theory and Applications, 142 (2009), 205-228.

[4] M.S. Bazaraa, J.J. Goode, M.Z. Nashed, On the cones of tangents with applications to mathematical programming, Journal of Optimization Theory and Applications, 13 (1974), 389-426.

[5] A. Y. Kruger, B.S. Mordukhovich, Extremal points and the Euler equation in nonsmooth optimization problems, In Doklady Akademii Nauk Belarusi, 24(8) (1980).

[6] E.N. Barron, R. Jensen, Optimal control and semicontinuous viscosity solutions, Proceedings of the American Mathematical Society, 113 (1991), 397-402.

[7] P. Cannarsa, G. Da Prato, Second-order Hamilton–Jacobi equations in infinite dimensions, SIAM Journal on Control and Optimization, 29 (1991), 474-492.

[8] F.H. Clarke, Y.S. Ledyaev, R.J. Stern, P.R. Wolenski, Nonsmooth analysis and control theory (Vol. 178), Springer Science and Business Media, 2008.

[9] M. Falcone, R. Ferretti, Discrete time high-order schemes for viscosity solutions of Hamilton-Jacobi-Bellman equations, Numerische Mathematik, 67(3) (1994), 315-344.

[10] A. Świch, Sub-and superoptimality principles of dynamic programming revisited, Nonlinear Analysis: Theory, Methods and Applications, 26 (1996), 1429-1436.

[11] C. Fu, Y. Liu, Z. Xiao, Interval differential evolution with dimension-reduction interval analysis method for uncertain optimization problems, Applied Mathematical Modelling, 69 (2019), 441-452.

[12] J. Armengol, P. Herrero, M. Á. Sainz, J. Vehí, Continuous minimax optimization using modal intervals, Journal of Mathematical Analysis and Applications, 339 (2008), 18-30.

[13] Y. Che, C. Li, P. Li, Y. Li, J. Wang, Y. Zhou, MPC-based interval number optimization for electric water heater scheduling in uncertain environments, Frontiers in Energy, 15 (2021), 186-200.

[14] T. Cheong, S. Choi, Y. Son, X. Zhang, Affine-arithmetic-based microgrid interval optimization considering uncertainty and battery energy storage system degradation, Energy, 242 (2022), 123015.

[15] A.Y. Kruger, Necessary conditions of an extremum in problems of nonsmooth optimization, Deposited at VINITI, (1333-81) (1981).

[16] B.S. Mordukhovich, Approximation methods in problems of optimization and control, Nauka, Moscow, 1988.

[17] B.S. Mordukhovich, Y. Shao, Extremal characterizations of Asplund spaces, Proceedings of the American Mathematical Society, 124 (1996), 197-205.

[18] R.T. Rockafellar, R. J. Wets, Variational Analysis, Springer–Verlag, New York, 1998.

[19] B.S. Mordukhovich, N.M. Nam, N.D. Yen, Fréchet subdifferential calculus and optimality conditions in nondifferentiable programming, Optimization, 55 (2006), 685-708.

[20] B.S. Mordukhovich, B. Wang, Necessary suboptimality and optimality conditions via variational principles, SIAM Journal on Control and Optimization, 41 (2002), 623-640.

[21] A.K. Bhurjee, G. Panda, Efficient solution of interval optimization problem, Mathematical Methods of Operations Research, 76 (2012), 273-288.

[22] Y. Chalco-Cano, W.A. Lodwick, A. Rufián-Lizana, Optimality conditions of type KKT for optimization problem with interval-valued objective function via generalized derivative, Fuzzy Optimization and Decision Making, 12 (2013), 305-322.

[23] S. Chanas, D. Kuchta, Multiobjective programming in optimization of interval objective functions—a generalized approach, European Journal of Operational Research, 94 (1996), 594-598.

[24] R.S. Chauhan, A.K. Debnath, D. Ghosh, R. Mesiar, Generalized Hukuhara Gâteaux and Fréchet derivatives of interval-valued functions and their application in optimization with interval-valued functions, Information Sciences, 510 (2020), 317-340.

[25] M. Hladík, T. Masařík, J. Novotná, Duality gap in interval linear programming, Journal of Optimization Theory and Applications, 184 (2020), 565-580.
[26] Y. Chalco-Cano, B. Hernández-Jiménez, R. Osuna-Gómez, G. Ruiz-Garzón, New efficiency conditions for multiobjective interval-valued programming problems, Information Sciences, 420 (2017), 235-248.
[27] B.A. Dar, D.S. Kim, D. Singh, KKT optimality conditions in interval valued multiobjective programming with generalized differentiable functions, European Journal of Operational Research, 254 (2016), 29-39.
[28] H.C. Wu, Wolfe duality for interval-valued optimization, Journal of Optimization Theory and Applications, 138 (2008), 497-509.
[29] I. Ahmad, B.A. Dar, D. Singh, Optimality and duality in non-differentiable interval valued multiobjective programming, Int. J. Math. Oper. Res., 11(3) (2017), 332-356.
[30] T. Antczak, Optimality conditions and duality results for nonsmooth vector optimization problems with the multiple interval-valued objective function, Acta Mathematica Scientia, 37 (2017), 1133-1150.
[31] I. Ahmad, J. Banerjee, A. Jayswal, Nonsmooth interval-valued optimization and saddle-point optimality criteria, Bulletin of the Malaysian Mathematical Sciences Society, 39 (2016), 1391-1411.
[32] D.H. Dinh, T. Su Van, Duality results for interval-valued pseudoconvex optimization problem with equilibrium constraints with applications, Computational and Applied Mathematics, 39 (2020), 1-24.
[33] A.K. Bhurjee, S.K. Padhan, Optimality conditions and duality results for non-differentiable interval optimization problems, Journal of Applied Mathematics and Computing, 50 (2016), 59-71.
[34] R.S. Chauhan, A.K. Debnath, D. Ghosh, R. Mesiar, Generalized-Hukuhara subgradient and its application in optimization problem with interval-valued functions, Sādhanā, 47 (2022), 1-16.
[35] R. E. Moore, Interval analysis. Prentice-Hall, 1966.
[36] V. Lupulescu, Fractional calculus for interval-valued functions, Fuzzy Sets and Systems, 265 (2015), 63-85.
[37] B. Bede, L. Stefanini, Generalized Hukuhara differentiability of interval-valued functions and interval differential equations, Nonlinear Analysis: Theory, Methods and Applications, 71 (2009), 1311-1328.
[38] O. Castillo, R.S. Chauhan, A.K. Debnath, D. Ghosh, Generalized-Hukuhara-gradient efficient-direction method to solve optimization problems with interval-valued functions and its application in least-squares problems, International Journal of Fuzzy Systems, 24 (2022), 1275-1300.
[39] A.K. Debnath, D. Ghosh, W. Pedrycz, A variable and a fixed ordering of intervals and their application in optimization with interval-valued functions, International Journal of Approximate Reasoning, 121 (2020), 187-205.
[40] D. Ghosh, G. Kumar, Ekeland's Variational Principle for Interval-valued Functions, (2021), arXiv preprint arXiv:2104.11167.
[41] M.J. Cloud, R.B. Kearfott, R.E. Moore, Introduction to interval analysis, Society for Industrial and Applied Mathematics, Philadelphia, 2009.
[42] H. C. Wu, The Karush–Kuhn–Tucker optimality conditions in an optimization problem with interval-valued objective function, European Journal of operational research, 176 (2007), 46-59.
[43] G. Panda, P. Roy, Existence of solution of constrained interval optimization problems with regularity concept, RAIRO-Operations Research, 55 (2021), S1997-S2011.
[44] D. Ghosh, G. Kumar, K. Kumar, Weak sharp minima for interval-valued functions and its primal-dual characterizations using generalized Hukuhara subdifferentiability, Soft Computing, (2021), https://doi.org/10.1007/s00500-022-07332-0.
[45] B.S. Mordukhovich, Variational analysis and generalized differentiation II, Applications (Vol. 331), Springer, Berlin, 2006.
[46] J.V. Burke, M.C. Ferris, Weak sharp minima in mathematical programming, SIAM Journal on Control and Optimization, 31 (1993), 1340-1359.
[47] D. Ghosh, K. Manchanda, K.K. Shukla, A. Singh, Extended Karush-Kuhn-Tucker condition for constrained interval optimization problems and its application in support vector machines, Information Sciences, 504 (2019), 276-292.