CHAPTER-1

Introduction and literature review

1.1 Mathematical modeling

1.1.1 What is mathematical modeling?

A mathematical model is an abstraction or simplification that allows one to describe or summarize a system. There are many ways to define the devices or phenomena and their behaviors. In order to define, we can use words, drawings or sketches, physical models, computer programs, or mathematical formulas. In other words, the modeling can be applied in several languages, often simultaneously. Since we are particularly interested in using the language of mathematics to make models, the definition of mathematical modeling can be refined as given below:
"*Mathematical modeling is a representation in mathematical terms of the behavior of real devices and objects or phenomena*".
Mathematical modeling has very precise advantages which can be described as below:

- Mathematics is a very precise language. This helps us to formulate ideas and identify underlying assumptions.
- All the results that mathematicians have proved over hundreds of years are at our disposal.
- Mathematics is a concise language, with well-defined rules for manipulations.
- Computers can be used to perform numerical calculations.

1.1.2 Objective of mathematical modeling

Since the modeling of any device and phenomena is very essential part to both the engineering and sciences, therefore engineers and scientists have the perfect practical reasons for doing mathematical modeling. In order to formulate the mathematical model for any problem, firstly we must see its behavior in “real world” and “conceptual world”. The external world is called the real world where we observe the behaviors and various phenomena. These behaviors and phenomena are either natural or produced by artifacts. However, the conceptual world is the world of mind where we live and try to understand what is going on in the real or external world. We see the conceptual world in three different ways: observation, modeling, and prediction. The following figure is described to understand how the model works.

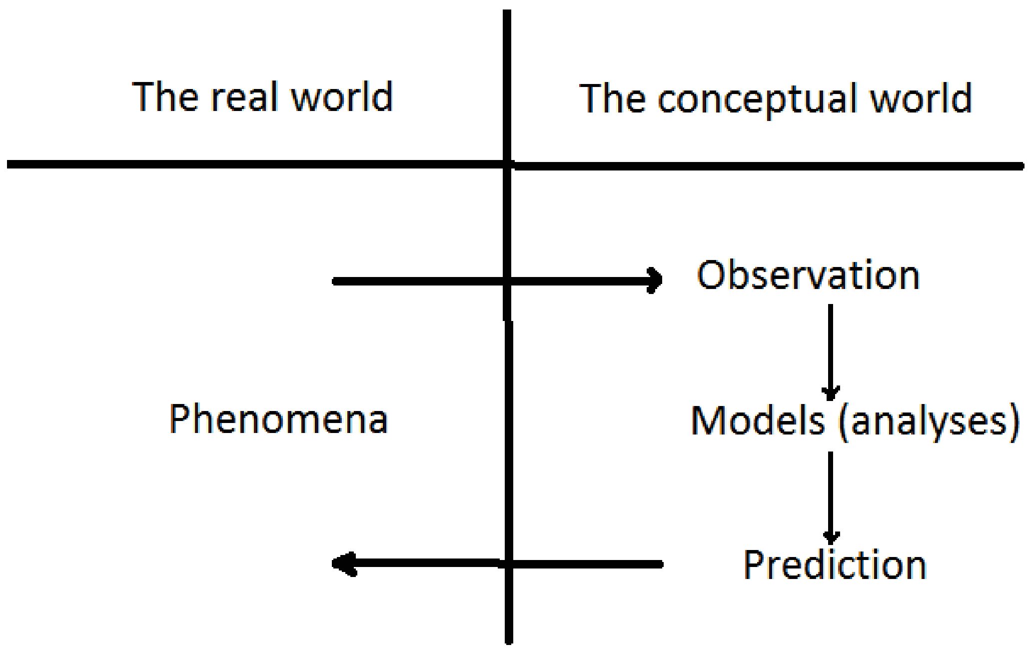

Fig. 1.1 Graphical representation of modeling

We can use mathematical modeling for a number of different reasons. The accuracy of the model depends on both the state of knowledge about a system and how well the modeling is done. Main objective of mathematical modeling underlies in developing scientific understanding through quantitative expression of current knowledge of a system. The present thesis shall use the above concept of modeling to study the physical behavior of various thermoelastic systems under different thermoelastic models and thereby to understand the basic differences between these models.

1.2 Thermoelasticity theory

Thermoelasticity takes into account of the change in the size and shape of a solid object as the temperature of that object fluctuates. Materials that are more elastic will expand and contract more than those materials that are more inelastic. Scientists use their understanding of thermoelasticity to design materials and objects that can withstand fluctuations in temperature without breaking. All materials that are elastic expand when heated and contract when cooled. The expansion that is described by thermoelasticity formulas is caused by an increase in the movement of the atoms in the material. These atoms remain linked to each other as a solid is heated but the molecular bonds grow in size, allowing the atoms to move away from each other and causing the material to grow. Conversely, when a material is cooled, the atoms move less and the bonds pull them closer to each other.

Initially, the investigations in the area of thermoelasticity were based on the "Uncoupled theory of thermoelasticity" with the simplifying assumption that the influence of the strain on the temperature field may be neglected. However, the experimental evidence shows that the deformation of a body is associated with a change of its heat content. This implies that the time varying external loading of a body causes in it not only displacements but also temperature distribution changing with time. Similarly, the heating of a body gives the deformation as well as the change in temperature. The mutual interactions between the temperature and deformation fields control the motion of the body in any situation. The classical "Uncoupled theory of thermodynamics" is, therefore, suffers form the drawback that the elastic changes have no effect on the temperature and vice versa. The domain of science that deals with the coupling between these two different fields is called coupled thermoelasticity.

In the case of coupled theory of thermoelasticity, the internal energy of a body depends on the temperature and deformation fields. Due to the coupling between these two fields, the temperature term is included in the displacement equation of motion, whereas deformation is appeared in the heat conduction equation. Firstly, Duhamel (1837, 1838) had assumed the coupling between deformation and temperature fields and proposed the first theory of thermal stresses. He had also introduced the dilatation term in the equation of thermal conductivity. Similarly, Neumann (1841) also developed stress-strain-temperature relations. Therefore, these approaches are known as Neumann-Duhamel relations. During last century, the innovative research has advanced this area to a great extent.

1.2.1 Classical coupled theory of thermoelasticity

First of all, Biot (1956) carried out the work in the thermoelasticity theory which was based on irreversible thermodynamics. He developed the constitutive relations and basic governing equations of this theory by incorporating the coupling between strain and temperature fields based on Nuemann-Duhamel approach. He has given the following system of linear equations of the theory of coupled dynamical thermoelasticity theory for anisotropic materials.

Equations of compatibility:

$$e_{ij,kl} + e_{kl,ij} - e_{ik,jl} - e_{jl,ik} = 0, \textbf{ for } i,\ j,\ k,\ l = 1,\ 2,\ 3 \tag{1.1}$$

Heat conduction equation:

$$q_{i,i} = \rho(R - \dot{S}T_0) \tag{1.2}$$

Equation of motion:

$$\sigma_{ij,j} + F_i = \rho \ddot{u}_i \tag{1.3}$$

where, F_i is the body force per unit volume.

Strain-displacement relations:

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{1.4}$$

Constitutive relations:

$$\sigma_{ij} = c_{ijkl}e_{kl} - \gamma_{ij}T \tag{1.5}$$

$$\rho S = \frac{\rho c_e}{T_0} T + \gamma_{ij} e_{ij} \tag{1.6}$$

Energy conservation law:

$$\dot{U} = \sigma_{ij}\dot{e}_{ij} - q_{i,i} + Q \tag{1.7}$$

Fourier's law:

$$q_i = -K_{ij} T_{,j} \tag{1.8}$$

where, U is the internal energy, S is the entropy, R is the strength of the internal heat source, γ_{ij} is the thermoelasticity tensor, c_{ijkl} is the elasticity tensor and K_{ij} is the thermal conductivity tensor, whereas c_e is the specific heat per mass in the isothermal state.

From Eqs. (1.3) and (1.5), we have

$$c_{ijkl} u_{k,lj} - \gamma_{ij} T_{,j} + F_i = \rho \ddot{u}_i \tag{1.9}$$

Now, from Eqs. (1.2), (1.6) and (1.8), we have the following relations:

$$K_{ij} \dot{T}_{,ij} + \rho R = \rho c_e \dot{T} + T_0 \gamma_{ij} \dot{u}_{i,j} \tag{1.10}$$

If the material is assumed to be isotropic then the above equations can be written as

$$\sigma_{ij} = 2\mu e_{ij} + (\lambda e_{kk} - \gamma T)\delta_{ij} \tag{1.11}$$

$$\rho S = \frac{\rho c_e}{T_0} T + \gamma e_{kk} \tag{1.12}$$

$$q_{i,i} = Q - \rho c_e \dot{T} \tag{1.13}$$

$$q_i = -KT_{,i} \tag{1.14}$$

where, Eq. (1.14) is identical to Fourier's law of heat conduction. Eqs. (1.9) and (1.10) can also be written for an isotropic medium in the following forms:

$$\mu\nabla^2 u_i + (\lambda+\mu)u_{k,ki} - \gamma T_{,i} + F_i = \rho \ddot{u}_i \tag{1.15}$$

$$K\nabla^2 T + \rho R = \rho c_e \dot{T} + T_0 \gamma \dot{u}_{k,k} \tag{1.16}$$

A wide and detailed discussion with interesting applications and theorems based on the Biot's theory can be available in pioneering work reported by Chadwick (1960), Boley and Weiner (1960), Nowacki (1962, 1975), Parkus (1976), Nowinski (1978), Dhaliwal and Singh (1980), Chandrasekharaiah (1986, 1998), etc. The classical theory of thermodynamics was based on Fourier's law of heat conduction and as a result, the heat conduction equation for this model was a parabolic type partial differential equation that described the fact that this theory involved the wave type equation of motion and the diffusion type equation of heat conduction. This implied that if an elastic medium is subjected to thermal or mechanical disturbance, the effects in both the temperature and displacement fields are felt instantaneously at an infinite distance far from the source of disturbances. This appears to be physically unrealistic. Therefore, Biot's theory although removed the drawback of uncoupled theory of thermoelasticity, but it suffered from paradox of infinite propagation speed, this theory also offered either unsatisfactory or poor description of a solid's response to fast transient loading, like short laser pulses, and at low temperature. Due to shortcomings of this theory in the several cases, researchers had put their efforts in the last few decades to modify the concept of this theory which were basically arose from the inherent limitations in the Fourier's law of heat conduction. In the next section, we present a brief history in this respect.

1.2.2 Drawbacks of Fourier's law and its modifications

The Fourier's law of heat conduction in the isotropic and homogeneous medium can be defined as

$$\vec{q}(\vec{r},t) = -K\vec{\nabla}T(\vec{r},t) \tag{1.17}$$

From above it is clear that heat flux $\vec{q}(\vec{r},t)$ is the instantaneous result of temperature gradient, $\vec{\nabla}T(\vec{r},t)$ established at a point $\vec{r}$ of body.

After combining the Eq. (1.17) with the energy equation (1.13), we obtain the heat conduction equation:

$$K\nabla^2 T(\vec{r},t) = \rho c_e \dot{T} - Q \tag{1.18}$$

From above heat conduction equation, it is clear that the Fourier's law is successfully applicable to the problems which have large spatial dimension and long time response. However, due to the parabolic nature of the heat conduction equation (1.18), it predicts an infinite speed for thermal disturbance. Therefore, it is physically unrealistic for the transient behavior of heat conduction at extremely short time, say, on the order of picoseconds (10^{-12}s) to femtoseconds (10^{-15}s). It has also been observed that Fourier's law is in contradiction with Einstein's theory of relativity and for situations involving temperature near absolute zero, extreme thermal gradients, high heat flux conduction and short time behavior, such as laser-material interaction, this law is not acceptable. With the rapid advancement of nanotechnology, nano-scale devices have been developed and it has been realized that the heat conduction of these tiny devices demonstrates many distinct phenomena such as the size effect and wave phenomena, which are not captured by the conventional Fourier's law.

It is worth to recall that in 1867, Maxwell was carrying out some experiments related to the kinetic theory of gasses. During his experiment, he interestingly postulated the appearance of a wave type flow of thermal signal. He thereby proposed that its disturbance shows a wavelike behavior instead of diffusive type. These results are indicated to modify the Fourier's law (see Chandrasekharaiah (1986)). Subsequently, Nernst (1917) also speculated the possible appearance of temperature waves in good thermal conducting materials at low temperature (see also Ward and Wilks (1951)). This wave type behavior of thermal signal is now called “second sound “ effect. For super fluid helium, Landau (1941) indicated the ’second sound’ as the propagation of phonon density disturbance, and concluded that its speed should be equal to $v_p/\sqrt{3}$ at 0 K, where v_p is the speed of the ordinary sound (first sound). Also Tisza (1947) predicted the possibility of extremely small heat propagation rates in liquid helium. Efforts were, therefore, being carried out by several experiments as well as theoretical researchers during last few decades to remove the physical paradox of Fourier’s law. It must be mentioned that second sound was firstly detected experimentally in liquid helium by Peshkov (1944), who found its speed to be equal to 19 m/s at 1.4 K. Tisza’s and Landau’s conclusions were re-examined experimentally by Atkins and Osborne (1950), Pellam and Scott (1949), and Maurer and Herlin (1949). Lifshitz (1958) concluded that second sound occurs in fluid helium at low temperatures. Subsequently, it was also re-examined by Ackerman *et al.* (1966), Ackerman and Overton (1969) and Bertman and Standiford (1970) in solid helium and by McNelly *et al.* (1970), Jackson *et al.* (1970), Jackson and Walker (1971), Rogers (1971) etc. in other materials. We refer the review article by Chandrasekharaiah (1986) for a detailed review in this respect.

1.2.3 Development of non-Fourier heat conduction models

In order to tackle the insufficiency of Fourier’s law, several researchers have made their contributions and modified this law. During the modification of this law, several non-Fourier’s heat conduction models are established which are described in the following forms in a detailed way:

1.2.3.1 Cattaneo-Vernotte law

Cattaneo (1958) and Vernotte (1958,1961) independently proposed a modification to Fourier’s law by incorporating a flux-rate term as follows:

$$\vec{q}(\vec{r},t)+\tau\frac{\partial\vec{q}(\vec{r},t)}{\partial t}=-K\vec{\nabla}T(\vec{r},t) \tag{1.19}$$

where, τ is a non-negative parameter referred as a thermal relaxation time parameter which is defined as the time lag needed to establish the steady state of heat conduction in a material element when a temperature gradient is suddenly imposed on that element. This phenomena concludes the fact that heat flux is not an instantaneous results of temperature gradient. The above law is called the Cattaneo and Vernotte's law (CV law). When Eq. (1.19) is combined with the energy equation, the following parabolic type heat conduction equation is derived as

$$K\nabla^2 T(\overrightarrow{r}, t) = (1 + \tau\frac{\partial}{\partial t})(\rho c_e \dot{T} - Q) \tag{1.20}$$

This modified heat conduction equation describes the combined diffusion and wave like behavior of heat propagation and indicates a wave like signal propagating with a finite speed. This law is also called the Maxwell-Cattaneo's law. It gives the successful results in the cases in which localized moving heat sources are involved with high intensity, rapidly propagating crack tip, shock wave propagation, laser material processing, laser surgery, etc. Mengi and Turhan (1978) done an experiment where they found the actual value of τ for a given material and concluded that its values for gases have the ranges from 10^{-19}s, for metals to 10^{-14}s, with the values for τ for liquids and insulators falling within this range. The values for this parameter for some materials were also determined by Francis (1972). During the study of practically relevant problems of heat transfer that involve extreme thermal gradients, high heat flux conduction and short time behavior, some researchers have found significantly different results by employing Eq. (1.20) instead of Eq. (1.18) and observed that Eq. (1.20) can be employed as heat conduction equation to study the problems, particularly, to those that contain the elapsed time during a transient less than, say about 10^{-5}s or heat flux greater than, say about 10^5W/cm^2. Laser penetration and welding, explosive bonding and melting and nucleate boiling are the areas where the transient process of heat conduction is at extremely short time or heat flux involved is very high. The details in this respect can be found in the review article developed by Chandrasekharaiah (1998*a*), Hetnarski and Ignaczak (1999) and in the recent books by Wang *et al.* (2008), Straughan (2011), Ignaczak and Ostoja-Starzewski (2010).

1.2.3.2 Dual phase-lag heat conduction model

Now a days, the advancement of short-pulse laser technologies and their applications to modern micro-fabrication technologies have attracted the researchers to draw a great attention in high rate heating on thin film structures (Tzou (1995*a*)). Furthermore, it is recommended that the laser pulses can be reduced shorter to the range of femtoseconds (10^{-15}s). It has been realized that when the response time becomes shorter, the non-equilibrium thermodynamic transition and the microscopic effects in the energy exchange during heat transport procedure become significant. The model formulation, therefore, becomes microscopic in nature (Tzou (1997)). In view of recent experiments, the heat conduction theory of Cattaneo and Vernotte also fails in some cases, specially during the heating of thin films. Therefore, in order to remove the drawbacks of classical heat conduction theory and Cattaneo and Vernotte's theory, Tzou (1995*a,b*) has developed the dual phase-lag (DPL) theory of heat conduction. This theory is established on the basis of that either the temperature gradient may dominate the heat flux or the heat flux may dominate the temperature gradient. Here, it is worth mentioning that the dual phase-lag model is motivated from some prior established models namely phonon-scattering model, phonon-electron interaction model, micro-scopic two-step model, etc. Joseph and Preziosi (1989,1990) and Guyer and Krumhansl (1966) put forward the phonon-scattering model in order to capture the microscopic effects in heat transport mechanism. The phonon-electron interaction model is developed by Brorson *et al.* (1987), Anisimov *et al.* (1974) and Fujimoto *et al.* (1984), whereas the micro-scopic two step model is reported by Qiu and Tien (1992, 1993).

In view of all these micro-scopic models, Tzou (1995*a*) included the effects of micro-structural interactions in the fast transient process of heat transport phenomena and proposed a more generalized and modified law of heat conduction, known as dual phase-lag model which can be written as

$$\overrightarrow{q}(\overrightarrow{r}, t+\tau_q) = -K\overrightarrow{\nabla}T(\overrightarrow{r}, t+\tau_T) \tag{1.21}$$

where, τ_q and τ_T are the two delay times in which τ_q is the phase-lag in the heat flux vector while τ_T is the phase-lag in temperature gradient vector. The phase-lag τ_q captures the thermal wave behavior, a small-scale response in time for heat flux whereas phase-lag τ_T captures the effect of phonon-electron interactions, a micro-scale response in space. Therefore, the dual phase-lag concept is capable of measuring the small-scale response in both space and time. Both of the phase-lags are positive and they are the intrinsic properties of the medium (see Tzou (1997)).

The dual phase-lag model is a universal model which is able to explain all the fundamental properties in diffusion, thermal wave, phonon-electron scattering associated with small response time. Many researchers like Quintanilla (2002), Kothari and Mukhopadhyay (2013), Mukhopadhyay *et al.* (2014), Abdallah (2009), Tzou (1997), Al-Nimr and Al-Huniti (2000), Chen *et al.* (2002), Lee and Tsai (2008) studied various aspects on this theory. Some important features of this theory can be listed in the following forms:

1. The Jeffery-type heat flux equation (Joseph and Preziosi (1989,1990)) can be extracted by using the Taylor's series approximation of Eq. (1.21) upto first order of τ_q and τ_T:

$$\overrightarrow{q} + \tau_q \frac{\partial \vec{q}}{\partial t} = -K(\overrightarrow{\nabla} T + \tau_T \frac{\partial \overrightarrow{\nabla} T}{\partial t}) \tag{1.22}$$

   With the help of energy equation, the corresponding heat conduction equation is obtained as

$$(1 + \tau_T \frac{\partial}{\partial t})\nabla^2 T = \frac{\rho c_e}{K}(1 + \tau_q \frac{\partial}{\partial t})\frac{\partial T}{\partial t} - \frac{1}{K}(1 + \tau_q \frac{\partial}{\partial t})\frac{\partial Q}{\partial t} \tag{1.23}$$

2. The dual phase-lag model (1.21) can be reduced to the following:

   (a) Classical Fourier's law of heat conduction if we take $\tau_q = \tau_T = 0$.

   (b) Hyperbolic heat conduction if we assume $\tau_T = 0$ and $\tau_q > 0$.

   (c) The energy equation in phonon-scattering model (see Joseph and Preziosi (1989), Guyer and Krumhansal (1966) when we put $\frac{K}{\rho c_e} = \frac{\tau_R c^2}{3}$, $\tau_T = \frac{9\tau_N}{5}$ and $\tau_q = \tau_R$, where, τ_R is the relaxation time for Umklapp process in which momentum is lost from phonon system and τ_N is the relaxation time for normal processes in which momentum is conserved for phonon system (see Tzou (1995a)).

3. If we take the Taylor's series expansion of Eq. (1.21) by using the second order in τ_q whereas first order in τ_T (see Tzou (1995b)), we obtain the following:

$$\vec{q} + \tau_q \frac{\partial \vec{q}}{\partial t} + \frac{\tau_q^2}{2}\frac{\partial^2 \vec{q}}{\partial t^2} = -K(\vec{\nabla} T + \tau_T \frac{\vec{\nabla} T}{\partial t}) \tag{1.24}$$

   and using energy equation, the corresponding heat conduction equation can be written as

$$K(1 + \tau_T \frac{\partial}{\partial t})\nabla^2 T = (1 + \tau_q \frac{\partial}{\partial t} + + \frac{\tau_q^2}{2}\frac{\partial^2}{\partial t^2})(\rho c_e \frac{\partial T}{\partial t} + \frac{\partial Q}{\partial t}) \tag{1.25}$$

Above equation is a hyperbolic type heat conduction equation which is showing that the thermal wave travels with finite speed, $V_T = \frac{1}{\tau_q}\sqrt{\frac{2K\tau_T}{\rho c_e}}$.

Both the single phase-lag and dual phase-lag heat conduction models have been discussed to be acceptable by the second law of extended irreversible thermodynamics in Tzou (1997) and Xu (2011) and by the Boltzmann heat transport equation in Xu and Wang (2005).

1.3 Generalized thermoelasticity theory

Several researchers have contributed in the area of thermoelasticity theory in order to provide major growth in the subject by incorporating the non-Fourier's heat conduction in the elastic materials. Hence, some generalized theories are developed on the basis of non-Fourier's heat conduction models as mentioned above. The main objective of these theories is to remove the drawbacks inherent in the classical theory of thermoelasticity given by Biot (1956). These theories are called as the generalized thermoelasticity theory or hyperbolic thermoelasticity theory. A brief description is listed as given below:

1.3.1 Lord-Shulman's theory or extended thermoelasticity (ETE) theory

In 1967, Lord and Shulman have developed a generalized thermoelasticity theory in which one thermal relaxation parameter is included for isotropic thermoelastic material. In this theory, the flux rate is included in the Fourier's law of heat conduction. This theory is basically based on the Cattaneo-Vernotte law (see Eq. (1.19)) and its results show the wave type behavior, i.e., the propagation speed of both the thermal and elastic waves are finite. This theory is also called extended thermoelasticity theory which is further extended by Dhaliwal and Sherief (1980) to general anisotropic media.

1.3.2 Temperature-rate dependent thermoelasticity (TRDTE) theory

The second generalization of the coupled thermoelastic theory is said to be the thermoelasticity with two relaxation parameters or the theory of temperature-rate dependent thermoelasticity.

This theory includes the "second sound" effects and this theory is developed by Green and Lindsay (1972). It is necessary to mention here that prior to this theory, Mullar (1971) developed the entropy production inequality with some restrictions on a class of constitutive equations in a review of thermodynamics of thermoelasticity. Later on, a generalization of this inequality is re-formulated by Green and Laws (1972). Subsequently, an explicit version of the constitutive equations are obtained by Green and Lindsay (1972) and these equations are also developed independently by Suhubi (1975). Under the assumption that the medium has a center of symmetry, the classical Fourier's law of heat conduction is not violated in this model. All the equations of this coupled theory are modified by involving two constants into the constitutive equations that act as thermal relaxation time parameters. Chandrsekhasraiah (1986, 1998a) and Hetnarski and Ignaczak (1999) have given a detailed discussion about extended and temperature-rate dependent theory of thermoelasticity.

1.3.3 Green and Naghdi's theory of thermoelasticity type-I, II and III

The third generalization of thermoelasticity theory was made by Green and Naghdi (1991, 1992, 1993, 1995) and they have introduced this generalization as an alternative way. In this theory, the propagation of heat is modeled in such a way that it can produce the consistent theory of thermoelasticity. This theory is based on the thermodynamics principles. However, in order to get finite wave speed of the thermal signals, Green and Naghdi (1993) have introduced a new concept in generalized thermoelasticity theory which is known as the thermoelasticity theory with no energy dissipation. The prime characteristic of this theory is totally contrast with the classical Fourier's law of heat conduction. Basically, Green and Naghdi's theory is based on the entropy balance law rather than the usual entropy inequality. In this theory, thermal displacement (ν) , gradient of thermal displacement (∇v) and temperature (T) are considered as the constitutive variables, where $\dot{v} = T$. In the formulation of this theory, the usual entropy production inequality is replaced with an entropy balance law (see Chandrasekharaiah (1998*a*)) and the heat conduction law for GN-III model is of the form:

$$\overrightarrow{q}(\overrightarrow{r},t) = -[K\overrightarrow{\nabla}T(\overrightarrow{r},t) + K^{*}\vec{\nabla}\nu(\vec{r},t)] \tag{1.26}$$

where, K^* is the rate of thermal conductivity of material.

When we assume K is much greater than K^* , GN-III model gives its first special case: GN-I model. The linearized form of GN-I model is similar to the Biot's theory and therefore, GN-I model suffers from the drawback of infinite speed of heat propagation. When we assume $K^*$is much greater than K , GN-III model reduces into the second special case: GN-II model. In GN-II, the internal rate of production of entropy is assumed to be identically zero which implies that there is no dissipation of internal energy. In the type-II model, the thermal signals move with finite speed. In the heat conduction equation of type-III, the heat flux is the combination of type-I and type-II theories. The linearized heat conduction equations under different models of Green and Naghdi for an isotropic medium can be written as given below:

Green and Naghdi type-III (GN-III) model:

$$(K^* + K\frac{\partial}{\partial t})\nabla^2 T = \frac{\partial^2}{\partial t^2}(\rho c_e T + \gamma T_0 e) \tag{1.27}$$

Green and Naghdi type-II (GN-II) model:

$$K^*\nabla^2 T = \frac{\partial^2}{\partial t^2}(\rho c_e T + \gamma T_0 e) \tag{1.28}$$

Green and Naghdi type-I (GN-I) model:

$$K\nabla^2 T = \frac{\partial}{\partial t}(\rho c_e T + \gamma T_0 e) \tag{1.29}$$

where, e is the dilatation.

1.3.4 Thermoelasticity theory with dual phase-lags (DPLTE)

This theory is developed in the frame of extended thermoelasticity theory by incorporating the dual phase-lag heat conduction model in the place of Fourier's law. Tzou (1997) has introduced this theory, whereas Chandrasekharaiah (1998*a*) has reformulated and discussed this theory in a proper way.

1.3.5 Thermoelasticity theory with three phase-lags (TPLTE)

The thermoelasticity with three phase-lags is developed by Roychoudhury (2007*a*) with the inclusion of three different phase-lags in the heat conduction law given by Green and Naghdi (1992) (GN-III model). These three phase-lags are introduced in the heat flux vector, temperature gradient vector and thermal displacement gradient vector. Therefore, this theory is known as three phase-lag thermoelasticity theory (TPLTE). The heat conduction law under this theory can be written as given below:

$$\overrightarrow{q}(\overrightarrow{r},t+\tau_q) = -[K\overrightarrow{\nabla}T(\overrightarrow{r},t+\tau_T) + K^*\vec{\nabla}\nu(\vec{r},t+\tau_\nu)] \tag{1.30}$$

Here , τ_ν the delay time in thermal displacement gradient vector is the newly included in addition to τ_q and τ_T of dual phase-lag model. Therefore, three phase-lag theory may be considered as the generalization of GN-III thermoelasticity theory.

1.3.6 Quintanilla's thermoelasticity theory

The above mentioned thermoelasticity theories have attracted the serious attention of researchers in recent years in order to find out several features of these models. Some qualitative analysis on these models are also reported. Quintanilla and Racke (2008) have discussed the stability of three phase-lag model of heat conduction equation and the effects of considering all these three material parameters. Since, the phase-lag theories could let to parabolic-type or hyperbolic-type differential equations based on the order of Taylor's series expansion of the phase lag parameters, a big interest has also been developed to study the different Taylor's approximations to these heat conduction equations where continuous dependence and also stability can be achieved (see Horgan & Quintanilla (2005), Mukhopadhyay & Kumar (2010), Quintanilla (2002), Quintanilla (2003), Quintanilla & Racke (2006,2008)). Dreher *et al.* (2009) have reported an analysis on dual phase-lag and three phase-lag heat conduction models and showed that when we combine the constitutive equations introduced in dual phase-lags and three phase-lags heat conduction theory with the energy equation, then there exists a sequence of eigenvalues in a point spectrum in such away that its real parts tend to infinity (see Dreher *et al.* (2009), Jordan *et al.* (2008)). This implies the ill-posed behavior of the problem in Hadamard sense and we can not find the continuous dependence results of the solution with respect to initial parameters. By mentioning about these unacceptable results, Quintanilla (2011) has recently proposed to reformulate the three phase-lags heat conduction model and suggested an alternative heat conduction theory with a single delay term. Leseduarte and Quintanilla (2013) re-examined this new model given by Quintanilla (2011) and found out the stability and spatial behavior of the solutions under this model. They considered $\tau_v < \tau_q = \tau_T$ and $\tau = \tau_q - \tau_v$, so that the constitutive law of heat conduction can be taken as

$$\vec{q}(t) = -[K\nabla T(t) + K^*\nabla v(t-\tau)] \tag{1.31}$$

By using the above equation, they have studied the spatial behavior of the solutions for this theory. A Phragmen-Lindelof type alternative (see Quintanilla (2011)) is found out and it is shown that the solutions either decay in an exponential way or blow up at infinity in an exponential way. The results are extended to a thermoelasticity theory by considering the Taylor's series approximation of the equation of heat conduction to the delay term and Phragmen-Lindelof type alternative is obtained for both the forward and backward in time equations. Continuous dependence results for initial data and supply terms have been proved for this case. The continuous dependence results are further extended to the thermoelastic case.

Quintanilla (2011) has further considered the Taylor's series approximation until order l in the thermal gradient part of the constitutive law and reduced it in the form

$$\vec{q}(t) = -[K\nabla T(t) + K^*\{\nabla v(t) + \tau\nabla\dot{v}(t) + ... + \frac{\tau^l}{l(l-1)...1}\nabla v^{(l)}\}]$$

If this equation is adjoined with the energy equation, the new heat conduction equation is obtained as

$$c_e\dot{T}(t) = -[K\Delta T(t) + K^*\{\Delta \nu(t) + \tau\Delta T(t) + ... + \frac{\tau^l}{l(l-1)...1}\Delta T^{(l-1)}\}] \tag{1.32}$$

where, c_e is the specific heat, $\Delta = \nabla^2$ is the Laplacian operator. Quintanilla has also shown that the solutions of this heat conduction equation are always stable (at least) whenever $l \leq 3$.

When we take $l = 0$, the Eq. (1.32) reduces to the form

$$c_e\dot{T}(t) = -[K\Delta T(t) + K^*\Delta \nu(t)] \tag{1.33}$$

This is the heat conduction equation under GN-III model.

When we take $l = 2$ in Eq. (1.32), we get the following equation of heat conduction which we refer to new model-I (i.e., Quintanilla model-I):

$$c_e\dot{T}(t) = -[K\,\Delta T(t) + K^*(1 + \tau\frac{\partial}{\partial t} + \frac{\tau^2}{2}\frac{\partial^2}{\partial t^2})\Delta \nu(t)] \tag{1.34}$$

If we neglect the term containing τ^2 for smallness in Eq. (1.34), then we get the following equation which we refer to new model-II (i.e., Quintanilla model-II):

$$c_e\dot{T}(t) = -[K\Delta T(t) + K^*(1 + \tau\frac{\partial}{\partial t})\Delta \nu(t)] \tag{1.35}$$

1.4 Fractional order thermoelasticity theory

The fractional calculus is just the generalization of the ordinary differential and integral calculus to non-integer order. The fractional calculus became a very attractive subject to mathematicians. In the recent years, many existing models have been developed interestingly by using fractional calculus to study physical processes. This is due to the intensive development of the theory of fractional calculus itself and its applications to many phenomena in various fields of science and engineering. The fractional calculus is being applied to a great extent particularly, in the area of heat conduction, diffusion, viscoelasticity, mechanics of solids, electrical theories, rheology, fluid flows, chemical physics, bio-sciences, signal processing, electrochemistry, etc. (see Rabotnov (1966), Oldham and Spanier (1974), Bagley (1983), Nishimoto (1990), Engheta

(1996), Mainardi (1998), Podlubny (1999), Hilfer (2000), Magin (2004), Oldham (2009)). It must be recalled here that firstly, Abel (1823) has applied fractional calculus in the solution of an integral equation that arises in the formulation of the tautochrone problem. Euler (1730), Lagrange (1772) and Fourier (1822) have also mentioned the concept of derivatives of arbitrary order in their studies before a systematic study of fractional calculus. Consequently, many different forms of fractional (i.e. non-integer) differential operators are introduced. They are Riemann-Liouville derivative, Caputo derivative and Riesz derivative (for details, see Oldham and Spanier (1974), Miller and Ross (1993), Podlubny (1999), Hilfer (2000), Herrmann (2011)). But two of them are popularly used which are Riemann-Liouville fractional order derivative and Caputo fractional order derivative. The corresponding formula can be listed as given below.

Riemann-Liouvlle fractional order derivatives:

Riemann-Liouville fractional order derivative $D_t^\alpha f(t)$ of order α with respect to time t of a function $f(t)$ can be written as:

$$D_t^\alpha f(t) = \frac{\partial^\alpha f(t)}{\partial t^\alpha} = \frac{\partial^n}{\partial t^n}\left(\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1} f(\tau)\,d\tau\right), \text{ for } n-1 < \alpha \leq n$$

Caputo fractional order derivatives:

Caputo fractional order derivative $D_t^\alpha f(t)$ of order α with respect to time t is defined by

$$D_t^\alpha f(t) = \frac{\partial^\alpha f(t)}{\partial t^\alpha} = \left(\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1} \frac{\partial^n f(\tau)}{\partial t^n}\,d\tau\right), \text{ for } n-1 < \alpha \leq n$$

Notable contributions have been made to both the theory and applications of fractional calculus during the 20$^{\text{th}}$ century, when some properties of fractional order derivatives were examined with respect to arbitrary functions. The generalization of the concept of fractional calculus has been subjected to several approaches. A considerable attention of several researchers has also been paid to anomalous diffusion characterized by the time-fractional diffusion wave equation introduced by Kimmich (2002).

Recently, fractional calculus approach has also been introduced in the theory of thermoelasticity. A quasi-static uncoupled theory of thermoelasticity by including time-fractional derivative in the heat conduction equation is developed by Povstenko (2005). He applied Caputo fractional derivatives and obtained the thermal stresses by finding the fundamental solution of a Cauchy problem for the fractional heat conduction equation in one-dimensional and two-dimensional cases. Subsequently, Sherief *et al.* (2010), Youssef (2010) and Ezzat *et al.* (2012*a*) have proposed some new theories on generalized thermoelasticity using the heat conduction law under fractional order time derivatives.

1.5 Thermoelasticity theory with memory dependent derivatives

Recently, many researchers have tried to modify the classical Fourier's law of heat conduction equation using the fractional calculus (see Di Paola *et al.* (2009), Ezzat (2010, 2011), Ezzat, and El-Karamany (2011*a,b*), Ezzat *et al.* (2012, 2014), Ezzat and El-Bary (2014)). Recently, Diethelm (2010) has explained that the Caputo (1967) fractional derivative can be defined as:

$$D_a^{\alpha} g(t) = \int_a^t K_{\alpha}(t-s)\, g^{(m)}(s)\, ds,$$

where,

$$K_{\alpha}(t-s) = \frac{(t-s)^{m-\alpha-1}}{\Gamma(m-\alpha)}$$

Here, $\alpha > 0$ and m is the integer which satisfies $m-1 < \alpha < m$, while $K_{\alpha}(t-s)$ is the kernel function and $g^{(m)}$ is the m-order derivative of $g(t)$ which has some interesting physical meanings. In above equation, the kernel $K_{\alpha}(t-\xi)$ is fixed and from the above definition it is clear that $\alpha-$order fractional derivative is not defined locally at time t, but it depends on the total effects of $m-$order integer derivative on the interval $[a,\, t]$. Hence, this concept of fractional order derivative can be used to describe the variation of a system in which the instantaneous change rate depends on the past state which is known as "memory effect" (Diethelm (2010)). However, we know that the memory effect of real process basically arises in a segment of time $[t-\tau,$ $t]$, where τ denotes the time delay and it is always positive. In spite of several applications of fractional calculus, it has some demerits. Due to this, the concept of fractional order derivative has been modified and a new concept of derivative has been established by Wang and Li (2011) which has been named as "memory dependent derivative".

They defined first order memory dependent derivative of a function $g(t)$ in an integral form of a common derivative with a kernel function $K(t-s)$ on a slipping interval $[t-\tau, t]$ in the following manner:

$$D_\tau g(t) = \frac{1}{\tau}\int_{t-\tau}^{t} K(t-s)\, g^{(1)}(s)\, ds$$

where, τ is the time delay and the kernel function, $K(t-s)$ is differentiable w.r.t. t and s.

The definition given above can reflect the memory effect on the delayed interval $[t-\tau,\ t]$ which varies along with time. The kernel function $K_\alpha(t-s)$ is fixed for a given α in the case of fractional derivative. However, not only the delay parameter τ, but the kernel function $K(t-s)$ can also be chosen freely in the case of "memory dependent derivative" such as 1, $1+s-t$, $1+\frac{(s-t)}{\tau}$, and $[1+\frac{(s-t)}{\tau}]^2$, etc. The kernel function is a monotone function with $K=0$ for the past time $t-\tau$ and $K=1$ for the present time t. Generally from the view point of applications, the memory effect needs the weights $K(t-s)$ which have the values between 0 and 1 for $s \in [t-\tau, t)$. Similarly, m-order "memory dependent derivative" of $g(.)$ at t relative to time delay τ can be defined as

$$D_\tau^m g(t) = \frac{1}{\tau}\int_{t-\tau}^{t} K(t-s)\, g^{(m)}(s)\, ds$$

where, $K(t-s)$ is $m-$ times differentiable w.r.t. t and s.

1.6 Stochastic processes and its applications in thermoelasticity

A stochastic or random process can be defined as a collection of random variables that is indexed by some mathematical set. Sometimes, it is called random function which is just the counterpart of a deterministic process. It is very useful to tackle separately both the cases of discrete and continuous. A discrete time stochastic process can be denoted as $r=\{r_n, n=0, 1, 2...\}$which is a countable collection of random variables indexed by the non-negative integers, while a continuous time stochastic process $r=\{r_t, 0 \le t < \infty\}$ is an uncountable collection of random variables indexed by the non negative real numbers (see Hoel *et al.* (1972)). We can also index the random variables by negative time.

A stochastic process, $r(t)$ is said to be stationary if for all n and for every set of time instants $(t_i \in T,\ i = 1, 2, 3, ..., n)$, its joint probability density function, $G_r(.)$ does not change with a shift of the time parameter scale which can be expressed as

$$G_r(r_1, r_2, ..., r_n; t_1, t_2, ..., t_n) = G_r(r_1, r_2, ..., r_n; t_1 + \varsigma, t_2 + \varsigma, ..., t_n + \varsigma)$$

where, ς is a time interval.

We also would like to recall here that the Brownian paths, also called as Wiener process, are not differentiable point-wise. We may interpret their time derivative in a distributional sense to get a generalized stochastic process called white noise. We denote it by

$$\phi_0(t, u) = \dot{W}(t, u)$$

We can also use the notation $\phi_0 dt = dW$, where $W(.)$ is a Wiener process and $\phi_0(.)$ is a stochastic process. The term white noise arises from the spectral theory of stationary stochastic processes, according to which white noise has a flat power spectrum that is uniformly distributed over all frequencies (like white light). Since, Brownian motion has Gaussian independent increments with mean zero, its time derivative is a Gaussian stochastic process with mean zero whose values at different times are independent.

In order to take into account of the noise or error in a system, it is suggested that the deterministic model can be replaced with the stochastic one. This is due to the fact that instead of dealing with only one possible reality of how the process might evolve under time, in a stochastic process there is some indeterminacy in its future evolution described by probability distributions. This means that even if the initial conditions are known, there are many possibilities the process might go to, but some paths may be more probable than the others. Hence, deterministic models that represent idealized situations are often improved by including stochastic effects (see Kloeden and Platen (1992), Lawler (2006), Sherief *et al.* (2013, 2016),). As discussed by Sherief *et al.* (2013, 2016), Bellomo and Flandoli (1989) and Omar (2009), there are many reasons which allow to replace the deterministic cases with the stochastic simulations in which stationary stochastic processes are used. We must recall here some of them as

1. The system is not fully isolated, thus background fields give rise to additional noise.
2. Not all the variables that characterize the system are included in the model and these variables give rise to additional noise.
3. The accuracy of the measuring devices for the mechanical load, temperature, etc. are not 100% accurate.

Recently, stochastic simulations are carried out by Sherief *et al.* (2013) to analyze the thermoelastic problem in which the effects of the stochastic thermal shock at the boundary of the medium. Sherief *et al.* (2016) have also discussed the wave propagation in the theory of generalized thermoelastic diffusion using stochastic simulation technique.

1.7 Literature review

In last few decades, several researchers have been attracted towards the thermoelasticity theory due to its wide applications in science, engineering and technology. Several problems based on the thermoelastic interactions in different types of media have been investigated. We can find wide research work carried out on the various theories of thermoelasticity in the review articles by Chandrasekharaiah (1986, 1998*a*), Joseph and Preziosi (1989, 1990), Hetnarski and Ignaczak (1999) etc. The recent books by Wang *et al.* (2008), Hetnarski and Eslami (2010), Ignaczak and Ostoja-Starzewski (2010) as well as the Ph.D. theses of Kumar (2010), Prasad (2012), Kothari (2014), and Tiwari (2017) may also be referred in this regard. We give below some important work which are relevant to the present study.

Danilovskaya (1950) firstly studied a problem on elastic half space under the theory of classical thermoelasticity by neglecting the coupling term. This problem is thereafter called as Danilovskaya's problem. Subsequently, Hetnarski (1964*b*) discussed the solution of this problem under the classical coupled thermoelasticity theory for small times. From above, it is noted that the classical theory is not acceptable physically. Therefore, several efforts have been made to remove this paradox. The half space problems are investigated under different boundary conditions in the context of Lord-Shulman (LS) theory by several researchers, like Chattopadhyay *et al*. (1982) and Ramamurthy and Sharma (1991), etc. However, several other problems based on this theory are studied by researchers, like Wadhawan (1972, 1973), Sharma (1987*b*), Furukawa *et al*. (1990), Sharma and Chand (1991, 1996), and Roychoudhuri and Bhatta (1983). Misra *et al.* (1987), Mukhopadhyay and Bera (1989), Mukhopadhyay *et al.* (1991), Banerjee and Roychoudhuri (1995) have investigated the problems of viscoelastic medium based on the LS-theory. Chattopadhyay *et al.* (1982) investigated the stresses produced in an initially stressed elastic half-space due to a moving load. A coupled thermoelastic problem for an infinite aelotropic medium having a cylindrical hole is discussed by Chattopadhyay *et al.* (1985). The boundary initiated axi-symmetric waves in an annular cylinder under different boundary conditions are discussed by Sherief and Anwar (1988, 1989). Sherief (1986) has studied the fundamental solutions for spherically symmetric space. El-Maghraby (2010) has discussed a generalized thermoelasticity problem for a half-space with heat sources and body forces. Elhagary (2014*a*) has discussed an asymptotic expansions of solutions for a problem on coupled and generalized thermoelasticity theories. A two-dimensional problem for a thick plate with heat sources in generalized thermoelasticity with one thermal relaxation time has been studied by El-Maghraby (2005). Kulkarni and Deshmukh (2008) determined the thermal stresses in a thick circular plate under steady temperature field. A problem of non-homogeneous steady-state heat conduction in a thin circular plate and its thermal stresses has been reported by Deshmukh *et al.* (2009). Recently, Sherief and Hussein (2016) have also studied a two-dimensional problem for a thick plate with axi-symmetric temperature distribution in the theory of generalized thermoelastic diffusion. Further, Kiani and Eslami (2017) have studied nonlinear generalized thermoelasticity of an isotropic layer based on Lord-Shulman theory.

In the context of Green-Lindsay model, a problem of an infinite allotropic medium having a cylindrical hole is studied by Chattopadhyay *et al.* (1985). However, the problems of thermoelastic interactions due to heat sources in an unbounded elastic medium have been investigated by many researchers including Roychoudhuri and Bhatta (1981), Roychoudhuri and Sain (1982), Sherief and Anwar (1986), Sharma (1986), Misra *et al.* (1987), and Chandrashekharaiah (1988).

Elhagary (2014*b*) has reported the results of a two-dimensional generalized thermoelastic diffusion problem for a thick plate subjected to thermal loading due to laser pulse. Freed (2017) has given a note on stress/strain conjugate pairs: explicit and implicit theories of thermoelasticity for anisotropic materials. Further, Ignaczak and Domański (2017) have studied an one dimensional model of nonlinear thermoelasticity at low temperatures and small strains. Lotfy (2014) has studied two temperature generalized magneto-thermoelastic interactions in an elastic medium under three theories. Mukhopadhyay and Kumar (2016) have investigated a problem of annular cylinder under two-temperature thermoelasticity theory of Lord-Shulman model. Further, Kumar *et al.* (2016) have investigated a problem on thermoelastic interactions under two-temperature theory of thermoelasticity with two relaxation parameters. Recently, Mukhopadhyay *et al.* (2017) have given a note on a two-temperature model in linear thermoelasticity with one relaxation parameter. Sherief and Hussein (2017) have found the fundamental solution of thermoelasticity with two relaxation times for an infinite spherically symmetric space. Sherief and Allam (2017) have also studied two-dimensional axi-symmetric problem for a sphere with heat sources in the theory of generalized thermo-viscoelasticity. Various generalized thermoelastic problems on thick circular plates have been investigated by Tripathi *et al.* (2016 *a,b*,c) and also by Tripathi *et al.* (2017).

The three theories of Green and Naghdi (1991, 1992, 1993, 1995) have drawn the attention of several researchers. Chandrashekharaiah (1996*a*) has studied one dimensional wave propagation in an elastic medium under Green and Naghdi-II (GN-II) theory of thermoelasticity. Chandrashekharaiah (1996*b*) has also studied free plane harmonic waves in an unbounded medium in the context of GN-II thermoelasticity theory. Later on, this problem was extended for rotating body by Chandrashekharaiah and Srinath (1997). A problem of cylindrical and spherical cavity is studied by Chandrashekharaiah and Srinath (1997) in an unbounded medium subjected to some loads on the boundary and due to heat source in an unbounded medium under GN-II theory of thermoelasticity. A half space problem is also investigated by Misra *et al.* (2000) in the context of GN theory of thermeolasticity. Subsequently, Quintanilla (2001*a,b*), Quintanilla (2003), and Quintanilla and Straughan (2004) have reported the qualitative research works based on the Green and Naghdi theory of thermoelasticity. Various problems under GN-III model have also been reported by several researchers including Taheri *et al.* (2005), Mallik and Kanoria (2006), Kar and Kanoria (2006, 2007*a*), Roychoudhuri and Bandyopadhyay (2007), Banik *et al.* (2007), Kar and Kanoria (2007*b*), and Mukhopadhyay and Kumar (2008*a*, 2008*b*). Mallik and Kanoria (2008) have investigated a two dimensional problem of transversely isotropic problem based on GN-II and GN-III theories. Bijelonja *et al.* (2017) have discussed mixed finite volume method for linear thermoelasticity at all Poisson's ratios. Harmonic plane wave propagation in thermoelastic medium under GN-III model is reported in a detailed study by Puri and Jordan (2004) and later on, by Kovalev and Radayev (2010) and also by Kothari and Mukhopadhyay (2012). Recently, the convolutional type variational and reciprocity theorems in the context of linear theory of GN-II and GN-III are reported by Chirita and Ciarletta (2010), and Mukhopadhyay and Prasad (2011). Apalara *et al.* (2017) have given a stability result for the vibrations given by the standard linear model with thermoelasticity of type-III. El-Karamany and Ezzat (2016) have given a note on the phase-lag and Green-Naghdi thermoelasticity theories in which they have proposed three models of generalized thermoelasticity: a single phase-lag Green-Naghdi theory of type-III, a dual phase-lag Green-Naghdi theory of type-II and type-III. Recently, Kumari and Mukhopadhyay (2016, 2017) have discussed the domain of influence theorems for thermoelasticity of type-II. Further, Tiwari and Mukhopadhyay (2017) have studied a problem on electromagneto-thermoelastic plane waves under Green-Naghdi theory of thermoelasticity-II. Yasinskyy and Tokova (2017) have studied an inverse problem on the identification of temperature and thermal stresses in an FGM hollow cylinder by the surface displacements. Further, Wang *et al.* (2017) have studied the energy decay rate of transmission problem between thermoelasticity of type-I and type-II. Sherief and Raslan (2016*a*) have investigated a thermoelastic problem of spherical shell with and without energy dissipation. Sherief and Raslan (2016*b*) have also discussed the thermoelastic interactions without energy dissipation in an unbounded body with a cylindrical cavity. Recently, Sherief and Raslan (2017) have studied a two dimensional problem of thermoelasticity without energy dissipation for a sphere subjected to axi-symmetric temperature distribution.

Phase-lag theories have gained much interest among the scientists in order to understand them. We recall few recent contributions for the exact dual-phase-lag heat equation by Kulish and

Novozhilov (2004), Ordo˜nez-Miranda and Alvarado-Gil, 2010), Dreher et al. (2009), Jordan *et al*. (2008). Magana and Quintanilla (2017) have discussed the existence and uniqueness in the phase-lag thermoelasticity. Singh and Renu (2017) have studied the displacement field due to cylindrical inclusion in a thermoelastic half space. Kar and Kanoria (2009) have investigated a problem on generalized thermoelasticity for functionally graded orthotropic hollow sphere under thermal shock with three phase-lag effect. Tiwari *et al.* (2016) have studied a problem on magneto-thermoelastic disturbances induced by thermal shock in an elastic half space having finite conductivity under dual phase-lag heat conduction. Mukhopadhyay *et al.* (2016) have reported a detailed mathematical analysis of various models in linear thermoelasticity with rational material laws. Sherief *et al.* (2017) have given a general formula for the drag on a solid of revolution body at low Reynolds numbers in a micro-stretch fluid.

Fractional order thermoelasticity theory has been employed by some researchers in recent years to study various problems. Some problems are investigated by Povstenko (*2008a, 2008b, 2009a, 2009b, 2010, 2011a, 2011b*) under fractional order thermoelasticity in the framework of quasi-static uncoupled theory of thermoelasticity and he has discussed the effects of fractional order parameter. Youssef (2010) has proved a uniqueness theorem and studied a problem under fractional order thermoelasticity. Youssef and Al-Lehaibi (2010) and Youssef (2012) have investigated some problems on thermoelastic interactions in the context of the model given by Youssef (2010). Sherief *et al.* (2010) have established a uniqueness theorem and a reciprocity theorem as well as a variational principle on fractional order thermoelasticity theory. Sarkar and Lahiri (2012) have investigated a two-dimensional problem of a homogeneous isotropic and thermally conducting thermoelastic rotating medium based on a fractional order thermoelasticity theory. Ezzat *et al.* (2012) have derived an ultrafast fractional thermoelasticity model utilizing the modified hyperbolic heat conduction model and derived a generalized fractional order thermoelasticity theory to describe the thermoelastic behavior of a thin metal film irradiated by a femtosecond laser pulse. Tiwari and Mukhopadhyay (2016) have studied a problem on harmonic plane wave propagation under fractional order thermoelasticity and reported a detailed analysis of fractional order heat conduction equation. Sherief and Raslan (2016*c*) have also studied two dimensional problem for a long cylinder in the fractional theory of thermoelasticity.

Some interesting discussions on fractional order thermoelasticity have also been reported by Warbhe *et al.* (2017*a,b*), Tripathi *et al.* (2017) and Tripathi *et al.* (2017). Recently, Ezzat *et al.* (2015) have proposed a new model of magneto thermoelasticity theory in the context of a new consideration of heat conduction with memory dependent derivative and compared it with the dynamical classical coupled thermoelasticity (see Biot (1956)). Subsequently, some researchers (see Ezzat *et al.* (2016*a,b*) Ezzat and El-bary (2015)) have considered some problems on the thermoelasticity theory using memory dependent derivatives and discussed the effects of memory dependent derivatives as compared to ordinary time derivatives. Shaw (2017) has given a note on the generalized thermoelasticity theory with memory dependent derivatives in which the discontinuity solutions of generalized thermoelasticity are discussed and he has also proposed a suitable Lyapunov function which will be an important tool to show several qualitative properties. Tiwari and Mukhopadhyay (2017) have also used the new concept of a memory dependent derivative in a heat transfer process in a solid to investigate the problem of wave propagation in a homogeneous, isotropic and unbounded solid due to a continuous line heat source and they have attempted to exhibit the significance of kernel function and time-delay parameter, that are characteristics of memory dependent derivative heat transfer, in the behavior of field variables.

We find in literature that some researchers have used stochastic simulation techniques for analysis of heat conduction and thermoelastic problems. By considering uncertainty in the thermal conductivity, some problems have been studied by Ahmadi (1978), Chen and Tien (1967), Keller *et al.* (1978), and Tzou (1988). Chen and Tien (1967), Samuels (1966), and Val'kovskaya and Lenyuk (1996) have discussed the results of some problems involving stochastic internal heat generation. Chiba and Sugano (2007) have investigated a stochastic thermoelastic problem of a functionally graded plate subjected to random external temperature load. Recently, Sherief *et al.* (2013) have discussed the effects of stochastic thermal shock at the boundary of an elastic medium. Sherief *et al.* (2016) have further discussed the wave propagation in the theory of generalized thermoelastic diffusion using stochastic simulation technique. Hosseini and Shahabian (2013) have discussed the stochastic hybrid numerical method for transient analysis of stress field in functionally graded thick hollow cylinders subjected to shock loading. Allam *et al.* (2016) have studied a problem of stochastic thermoelastic diffusion interaction in an infinitely long annular cylinder.

Problems on cracks and failures in solid have been explored by several researchers due to the wide applications of these problems in the industry, particularly in the fabrication of electronic components, geophysics and earthquake engineering etc. It is noted that Griffith (1921) has firstly studied the theory of the cracks in two dimensional thermoelastic medium. We would also like to mention here that the thermal stresses play an important role in the building of structural elements. The flow induced thermal stresses in the infinite isotropic solids has been studied by Florence and Goodier (1963). The crack problems in thermoelastic media have also been discussed by Sih (1962), Kassir and Bergman (1971), Prasad and Aliabadi (1996), Raveendra and Banerjee (1992), Elfalaky and Abdel-Halim (2006), Hosseini-Teherani and Eslami (2000), Chaoudhuri and Ray (2006). Mallik and Kanoria (2009) discussed an understanding of thermally induced stresses in solids which is necessary for a detailed study of the manufacturing stages. Work reported by Sherief and El-Maghraby (2005) and Prasad and Mukhopadhyay (2013) are also worth to be mentioned in this respect.

Very recently, thermoelasticty theory with single delay term proposed by Quintanilla (2011) has been studied by some researchers. Kumari and Mukhopadhyay (2017*a*) have studied the fundamental solutions of thermoelasticity with this recent heat conduction model with a single delay term. Further, Kumari and Mukhopadhyay (2017*b*) have established some important theorems on linear theory of thermoelasticity for an anisotropic medium under this new heat conduction model. Subsequently, a uniqueness theorem and instability of solutions under the relaxed assumption that the elasticity tensor can be negative is established by Quintanilla (2016). Kumar and Mukhopadhyay (2016) have investigated a problem of thermoelastic interactions on this theory in which state-space approach is used to formulate the problem and the formulation is then applied to a problem of an isotropic elastic half space with its plane boundary subjected to sudden increase in temperature and zero stress. Later on, Kumar and Mukhopadhyay (2017) have further carried out an investigation on the effects of temperature dependency of material parameters on a thermoelastic loading problem.

1.8 Objective of the present thesis

Thermoelasticity involves a large category of phenomena and it comprises of the general theory of heat conduction, thermal stresses, and strains set up by thermal flow in elastic bodies and the reverse result of temperature distribution caused by the elastic deformation itself. The absence of any elasticity term in the heat conduction equation in uncoupled thermoelasticity appears to be unrealistic, since the produced strain causes variation in the temperature field due to the mechanical loading of an elastic body. The classical theories of thermoelasticity also have infinite speed of propagation of thermal signals, that contradict physical facts. During the last few decades, various generalizations of the classical theory have been addressed to overcome this paradox. An appreciable progress in the field of aircraft and machine structure has given rise to numerous problems where thermal stress plays a very important role. In-depth research has been carried out on generalized thermoelasticity theories in solving thermoelastic problems in place of the classical uncoupled/coupled theory of thermoelasticity. It has been seen that the wave type thermal transport is physically realistic than the diffusion type equation in analyzing a problem on highly intensive heat transfer. Furthermore, due to the advancement of pulsed lasers, fast burst nuclear reactors, and particle accelerators, which can supply heat pulses with a very fast time-rise, generalized thermoelasticity theory is receiving serious attention as they predict more realistic results.

The present thesis is concerned with the mathematical modeling on various unsolved problems involving thermoelastic interactions. It is aimed at investigation of the behavior of physical field variables of various thermoelastic systems under recently proposed thermoelastic models and thereby to understand the basic differences of these models with respect to the responses of the field variables due to thermoelastic interactions. It is concerned with the analysis of various aspects of these thermoelasticity theories by investigating some problems involving thermoelastic interactions inside different media and due to various types of thermo-mechanical loads.