CHAPTER-6

An investigation on a two dimensional problem of Mode-I crack in a thermoelastic medium

6.1 Introduction

Cracks and failures in solid has been a topic of active research due to its wide applications in the industry and particularly in the fabrication of electronic components, geophysics and earthquake engineering, etc. The present Chapter aims at investigating a two dimensional Griffith crack problem represented by a line segment. It is in reality a long flat ribbon shaped cavity in a solid and stressed in such a way that the stress pattern remains unaltered while passing in a direction parallel to the plane of the crack. It is worth to recall that the theory of cracks in two dimensional medium was first studied by Griffith (1921). The fracture mode of any material shows the separation geometrically. In two dimensions, there are three basic problems of crack corresponding to three different modes (Mode-I, II and III) of displacement which are useful to study. A Griffith crack having the length $2r$ in a solid medium in the case of Mode-I is shown in Fig. 6.1 under the action of the tension which is in the direction perpendicular to the line of the crack. The Mode-I crack denotes a symmetric opening with the relative displacements of the medium being normal to the fracture surface (see Irwin (1958)). It can be noted that crack growth usually takes place in Mode-I or close to it.

It must be mentioned that thermal stresses play a very important role in building structural elements, like machines, gas or stream turbines, aircrafts, etc. In almost all structures crack may occur as manufacturing defects or because of service loading which can either be mechanical or thermal. If the load is frequently applied, the crack may grow in fatigue to a final fracture. The

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result of any unaccounted induced thermal stress may also be catastrophic in many cases. Hence, an understanding of thermally induced stresses in solids is necessary for a comprehensive study at the manufacturing stages. The flow induced thermal stresses in infinite isotropic solids has been studied by Florence and Goodier (1963). The crack problems in thermoelastic media are discussed by Sih (1962), Kassir and Bergman (1971), Prasad and Aliabadi (1996), Raveendra and Banerjee (1992), Elfalaky and Abdel-Halim (2006), Hosseini-Teherani and Eslami (2000), Chaoudhuri and Ray (2006), Sherief and El-Maghraby (2003). Mallik and Kanoria (2009), Abdel-Halim and Elfalaky (2005) have also discussed the dynamical problems for an internal penny shaped crack in an infinite thermoelastic solid. Recently, Sherief and El-Maghraby (2005) and Prasad and Mukhopadhyay (2013) have solved the mode-I crack problem of an infinite thermoelastic medium in the context of Lord-Shulman's theory (1967) and Green-Naghdi theory (1995), respectively.

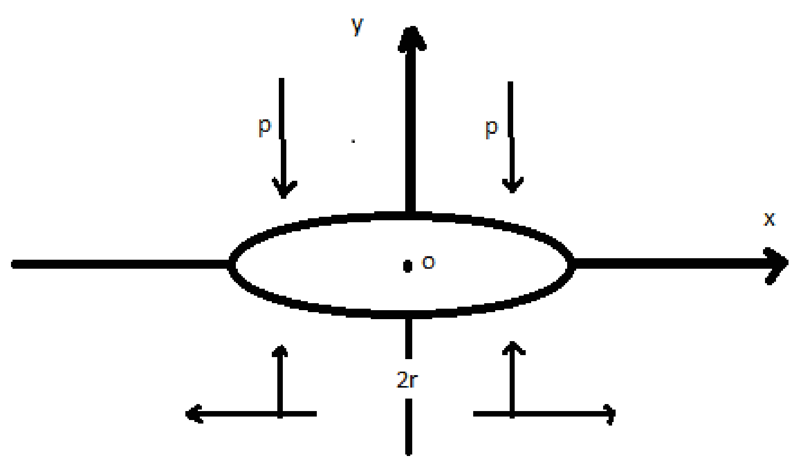

Fig. 6.1 Displacement of Mode-I crack

In this chapter, we consider a two dimensional dynamical problem of an infinite space with finite linear Mode-I crack and employ a recently proposed heat conduction model: an exact heat conduction with a single delay term. The thermoelastic medium is taken to be homogeneous and isotropic. However, the boundary of the crack is subjected to prescribed temperature and stress distributions. We have discussed the thermoelastic behavior inside the medium in the neighborhood of the crack in which we have used the thermoelasticity theory given by Quintanilla (2011), namely model of Quintanilla-I (new model-I) and model of Quintanilla-II (new model-II) and compared its results with the results of type-III thermoelasticity theory of Green and Naghdi which is discussed by Prasad and Mukhopadhyay (2013). We have formulated the problem in such way that all the three models (new model-I, II and GN-III model) can be written in a unified way, from which we can obtain results under every particular model. Two dimensional equations of motion along with the new heat conduction equation and constitutive relations are given to describe the present problem. Laplace and exponential Fourier transforms are used to solve the problem and we obtain the solution in the transformed domain. In section 6.4, the prescribed boundary temperature and stress distributions are used to derive four dual integral equations which are further reduced into two dual integral equations. These dual integral equations are solved by using regularization method. The method given by Bellman *et al.* (1966) is used to invert the Laplace transform numerically and obtain the final solution of the problem. We compute all the physical fields in the physical domain and represent them graphically. In section 6.6, we discuss and compare all the findings and highlight the specific behavior of different physical fields near the crack region.

6.2 Formulation of the problem

A two dimensional dynamical problem is considered in an infinite medium $-\infty < x < \infty$, $-\infty < y < \infty$ which has a Mode-I (opening mode) crack defined by $|x| \leq r,\ y = 0$. The crack surface is subjected to known temperature and normal stress distributions. We consider the basic governing equations of coupled thermoelasticity for isotropic and homogeneous medium as follows:

Equations of motion are given as

$$(\lambda+\mu)\frac{\partial e}{\partial x}+\mu\nabla^2 u-\gamma\frac{\partial T}{\partial x}=\rho\frac{\partial^2 u}{\partial t^2} \tag{6.1}$$

$$(\lambda+\mu)\frac{\partial e}{\partial y}+\mu\nabla^2 v-\gamma\frac{\partial T}{\partial y}=\rho\frac{\partial^2 v}{\partial t^2} \tag{6.2}$$

The heat conduction equation under theory of thermoelasticity of type-III due to Green and Naghdi (1995) is given by

$$(K^* + K\frac{\partial}{\partial t})\nabla^2 T = \frac{\partial^2}{\partial t^2}(\rho c_e T + \gamma T_0 e) \tag{6.3}$$

We consider **the heat conduction equation by Quintanilla (2011)** as

$$\left[K^*(1 + \tau_0\frac{\partial}{\partial t} + \frac{\tau_0^2}{2}\frac{\partial^2}{\partial t^2}) + K\frac{\partial}{\partial t}\right]\nabla^2 T = \frac{\partial^2}{\partial t^2}(\rho c_e T + \gamma T_0 e) \tag{6.4}$$

In the above equation, if we neglect the effect of higher order terms containing the delay time parameter τ_0**,** then we have

$$\left[K^*(1 + \tau_0\frac{\partial}{\partial t}) + K\frac{\partial}{\partial t}\right]\nabla^2 T = \frac{\partial^2}{\partial t^2}(\rho c_e T + \gamma T_0 e) \tag{6.5}$$

The **stress-strain-temperature relations** for the present case are given by

$$\sigma_{xx} = 2\mu\frac{\partial u}{\partial x} + \lambda e - \gamma(T - T_0) \tag{6.6}$$

$$\sigma_{yy} = 2\mu\frac{\partial v}{\partial y} + \lambda e - \gamma(T - T_0) \tag{6.7}$$

$$\sigma_{xy} = \mu(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) \tag{6.8}$$

Now, we aim to study the present problem by considering it as a problem of thermoelasticity in the contexts of three different forms of heat conduction equations as given by Eqs. $(6.3 - 6.5)$. Hence, we combine them in the following manner:

$$\left[K^*(1 + \tau_0\frac{\partial}{\partial t} + \tau_1\frac{\partial^2}{\partial t^2}) + K\frac{\partial}{\partial t}\right]\nabla^2 T = \frac{\partial^2}{\partial t^2}(\rho c_e T + \gamma T_0 e) \tag{6.9}$$

From Eq.(6.9), we can get the different heat conduction equations under different thermoelastic models in the following manner:

1. **Quintanilla model (new model-I):** $\tau_1 = \frac{\tau_0^2}{2}$, $\tau_0 \neq 0$

2. **Quintanilla model (new model-II):** $\tau_1 = 0$, $\tau_0 \neq 0$

3. **GN-III model:** $\tau_1 = 0$, $\tau_0 = 0$

In above equations $(6.1-6.9)$, u and v are the displacement components along the x and y directions, respectively and t is the time. T is the absolute temperature, T_0 is the reference temperature, K is the thermal conductivity and K^* is the rate of the thermal conductivity. γ is the material constants given by $\gamma = (3\lambda + 2\mu)\alpha_t$, where, α_t is the coefficient of linear of thermal expansion. e is the dilatation given by

$$e = (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) \tag{6.10}$$

Now for simplicity, we use the following non-dimensional variables:

$x' = c\eta x$, $y' = c\eta y$,$r' = c\eta r$, $u' = c\eta u$, $v' = c\eta v$, $t' = c^2\eta t$, $\sigma'_{ij} = \frac{\sigma_{ij}}{\mu}$, $T' = \frac{T-T_0}{T_0}$, $\tau'_1 = c^4\eta^2\tau_1$, $\tau'_0 = c^2\eta\tau_0$

where, $\eta = \frac{\rho c_e}{K}$, and $c = \sqrt{\frac{\lambda+2\mu}{\rho}}$. Here, c, is the speed of the propagation of longitudinal elastic waves.

Now with the help of the above non dimensional quantities, after dropping the dashed notations for convenience, Eqs. $(6.1-6.2)$, (6.6) and $(6.7-6.9)$ can be reduced in the following non-dimensional forms:

$$(m^2-1)\frac{\partial e}{\partial x} + \nabla^2 u - a_2\frac{\partial T}{\partial x} = m^2\frac{\partial^2 u}{\partial t^2} \tag{6.11}$$

$$(m^2-1)\frac{\partial e}{\partial y} + \nabla^2 v - a_2\frac{\partial T}{\partial y} = m^2\frac{\partial^2 v}{\partial t^2} \tag{6.12}$$

$$\left[a_1(1+\tau\frac{\partial}{\partial t} + \tau_1\frac{\partial^2}{\partial t^2}) + \frac{\partial}{\partial t}\right]\nabla^2 T = \frac{\partial^2}{\partial t^2}(T + a_3 e) \tag{6.13}$$

$$\sigma_{xx} = 2\frac{\partial u}{\partial x} + (m^2 - 2)e - a_2 T \tag{6.14}$$

$$\sigma_{yy} = 2\frac{\partial v}{\partial y} + (m^2 - 2)e - a_2 T \tag{6.15}$$

$$\sigma_{xy} = \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \tag{6.16}$$

where, $a_1 = \frac{K*}{Kc^2\eta}$, $a_2 = \frac{\eta T_0}{\mu}$, $a_3 = \frac{\gamma}{K\eta}$, $m^2 = \frac{\lambda+2\mu}{\mu}$

Now, removing u and v from Eqs.$(6.11-6.12)$ using Eq.(6.10), we obtain

$$\left(\nabla^2 - \frac{\partial^2}{\partial t^2}\right) e = a_4 \nabla^2 T \tag{6.17}$$

where, $a_4 = \frac{a_2}{m^2}$

6.3 Solution in the Laplace and Fourier transform domain

After taking the Laplace transform to both the sides of Eqs. $(6.10-6.13)$ and (6.17), we obtain the following equations:

$$\bar{e} = \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} \tag{6.18}$$

$$(1-m^2)\frac{\partial \bar{e}}{\partial x} + a_2\frac{\partial \bar{T}}{\partial x} = (\nabla^2 - m^2 s^2)\bar{u} \tag{6.19}$$

$$(1-m^2)\frac{\partial \bar{e}}{\partial y} + a_2\frac{\partial \bar{T}}{\partial y} = (\nabla^2 - m^2 s^2)\bar{v} \tag{6.20}$$

$$\left[\left\{a_1\left(1+\tau_0 s + \tau_1 s^2\right) + s\right\}\nabla^2 - s^2\right]\bar{T} = s^2 a_3 \bar{e} \tag{6.21}$$

$$\left(\nabla^2 - s^2\right)\bar{e} = a_4 \nabla^2 \bar{T} \tag{6.22}$$

where, s, is the Laplace transform parameter.

Now, eliminating $\bar{e}$ from Eqs. $(6.21-6.22)$, we obtain the partial differential equation which is satisfied by $\bar{T}$ as

$$\left(\nabla^2 - k_1^2\right)\left(\nabla^2 - k_2^2\right)\bar{T} = 0 \tag{6.23}$$

where, k_1^2 and k_2^2 are the roots of the following characteristic equation:

$$k^4 - \frac{s^2\left\{1 + a_1\left(1 + \tau_0 s + \tau_1 s^2\right) + s + \varepsilon\right\}}{\left\{a_1\left(1 + \tau_0 s + \tau_1 s^2\right) + s\right\}}k^2 + \frac{s^4}{\left\{a_1\left(1 + \tau_0 s + \tau_1 s^2\right) + s\right\}} = 0 \tag{6.24}$$

where, $\varepsilon = a_3 a_4$

Now, we can obtain $\bar{T}$, the solution of Eq. (6.23) in the following form:

$$\bar{T} = \bar{T}_1 + \bar{T}_2$$

where, $\bar{T}_1$ and $\bar{T}_2$ are the solutions of the equations

$$\left(\nabla^2 - k_i^2\right)\bar{T}_i = 0, \quad i = 1, 2 \tag{6.25}$$

The exponential Fourier transform of a function $\bar{g}(x, y, s)$ cab be defined as

$$\bar{g}^*(q, y, s) = F\left[\bar{g}(x, y, s)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \bar{g}(x, y, s)\, e^{-iqx}\, dx$$

where, q is the Fourier transform parameter.

The inverse Fourier transform can be defined as

$$\bar{g}(x, y, s) = F^{-1}\left[\bar{g}^*(q, y, s)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \bar{g}^*(q, y, s)\, e^{iqx}\, dq$$

Now, we apply the exponential Fourier transform to both sides of Eq. (6.25) to get

$$\left(\frac{\partial^2}{\partial y^2} - k_i^2 + q^2\right)\bar{T}_i^* = 0, \quad i = 1, 2 \tag{6.26}$$

The solution of Eq.(6.26) which is bounded at infinity can be found in the following form:

$$\bar{T}_i^* = B_i(q,\,s)(k_i^2 - s^2)e^{-q_i|y|}, \quad i = 1,2$$

where, $q_i = \sqrt{q^2 + m_i^2}$ and $B_i(q,\,s)$ is the parameter which depends upon q and s only for $i = 1,2$. Due to symmetry of the problem, we take the case $y > 0$ only. Then the above equation can be written as

$$\bar{T}_i^* = B_i(q,\,s)(k_i^2 - s^2)e^{-q_i y}, \; i = 1,2 \tag{6.27}$$

In a similar manner, by eliminating $\bar{T}$ from Eqs. (6.21) and (6.22), we obtain $\bar{e}^* = \bar{e}_1^* + \bar{e}_2^*$, where, $\bar{e}_i^*$, $i = 1,2$ can be written as

$$\bar{e}_i^* = B_i'(q,\,s)(k_i^2 - s^2)e^{-q_i y}, \quad i = 1\,2 \tag{6.28}$$

where, $B_i'(q,\,s)$, $i = 1,2$ are also parameters which depend only on q and s.

Now, substituting Eqs. (6.27) and (6.28) into Eq. (6.22), we obtain the equation which relates the parameters $B_i(q,\,s)$ and $B_i'(q,\,s)$ for $i = 1,2$ in the following form:

$$B_i'(q,\,s) = a_4 k_i^2 B_i(q,\,s), \; i = 1,2 \tag{6.29}$$

Therefore, using Eq. (6.29) and (6.28), we find

$$\bar{e}_i^* = c a_4 k B_i(q,\,s)\,(k_i^2 - s^2)e^{-q_i y}, \quad i = 1,2 \tag{6.30}$$

Now, we take the exponential Fourier transform of Eqs. (6.19) and (6.20) to get

$$\left(\frac{\partial^2}{\partial y^2} - q^2 - m^2 s^2\right)\bar{u}^* = \left(1 - m^2\right) i\,q\,\bar{e}^* + i\,q\,a_2\,\bar{T}^* \tag{6.31}$$

$$\left(\frac{\partial^2}{\partial y^2} - q^2 - m^2 s^2\right)\bar{v}^* = \left(1 - m^2\right)\frac{\partial}{\partial y}\bar{e}^* + a_2\frac{\partial}{\partial y}\bar{T}^* \tag{6.32}$$

In view of Eqs. (6.27) and (6.30), Eqs. (6.31 – 6.32) can be rewritten as

$$\left(\frac{\partial^2}{\partial y^2} - q^2 - m^2 s^2\right)\bar{u}^* = i\,q\,a_4 \sum_{i=1}^{2}\left(k_i^2 - m^2 s^2\right) B_i(q,\,s)\,e^{-q_i s} \tag{6.33}$$

$$\left(\frac{\partial^2}{\partial y^2} - q^2 - m^2 s^2\right)\bar{v}^* = -a_4 \sum_{i=1}^{2}\left(k_i^2 - m^2 s^2\right) B_i(q,\,s)\,q_i\,e^{-q_i s} \tag{6.34}$$

The solution $\bar{u}^*$ of Eq. (6.33) can be written as

$$\bar{u}^* = i\,q\,a_4 \left(\sum_{i=1}^{2} B_i(q,\,s) e^{-q_i y} + H_1 e^{-\delta y}\right) \tag{6.35}$$

where, $\delta = \sqrt{q^2 + m^2 s^2}$ and $H_1 = H_1(q,\,s)$ is a parameter that depends on q and s only.

Taking the exponential Fourier transform of Eq. (6.18) with respect to x, we obtain

$$\frac{\partial \bar{v}^*}{\partial y} = \bar{e}^* - i\,q\,\bar{u}^* \tag{6.36}$$

Now, with the help of Eqs. (6.30) and (6.35) along with the integration with respect to y, Eq. (6.36) can be re-written as given below

$$\bar{v}^* = -a_4 \left(\sum_{i=1}^{2} B_i(q,\,s)\,q_i e^{-q_i s} + \frac{q^2 H_1(q,\,s)}{\delta} e^{-\delta y}\right) \tag{6.37}$$

Now, taking the Laplace and exponential Fourier transforms to the both sides of the Eqs. (6.14 – 6.16) and using the results of the Eqs. (6.27), (6.30), (6.35) and (6.37), we can write the components of the stress tensor in the Laplace and Fourier transform domain in the following form:

$$\bar{\sigma}_{xx}^* = a_4 \left[B_1 \left(m^2 s^2 - 2 q_1^2\right) e^{-q_1 y} + B_2 \left(m^2 s^2 - 2 q_2^2\right) e^{-q_2 y} - 2 H_1 q^2 e^{-\delta y}\right] \tag{6.38}$$

$$\bar{\sigma}_{yy}^* = a_4 \left[\left(m^2 s^2 + 2 q^2\right)\left(B_1 e^{-q_1 y} + B_2 e^{-q_2 y}\right) + 2 H_1 q^2 e^{-\delta y}\right] \tag{6.39}$$

$$\bar{\sigma}_{xy}^* = -i a_4 q \left[2\left(B_1 q_1 e^{-q_1 y} + B_2 q_2 e^{-q_2 y}\right) + \frac{q^2 + \delta^2}{\delta} H_1 e^{-\delta y}\right] \tag{6.40}$$

Now, taking the inverse Fourier transform of Eqs. (6.27), (6.30), (6.35), and (6.38 – 6.40), we find the solution in the Laplace transform domain for the present problem as given below

$$\bar{T} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[a_2(k_1^2 - s^2)e^{-q_1 y} + G_2(k_2^2 - s^2)e^{-q_2 y}\right] e^{iqx}\, dq \tag{6.41}$$

$$\bar{e} = \frac{a_4}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[B_1 k_1^2 e^{-q_1 y} + B_2 k_2^2 e^{-q_2 y}\right] e^{iqx}\, dq \tag{6.42}$$

$$\bar{u} = \frac{ia_4}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[B_1 e^{-q_1 y} + B_2 e^{-q_2 y} + H_1 e^{-\delta y}\right] q\, e^{iqx}\, dq \tag{6.43}$$

$$\bar{v} = \frac{-a_4}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[B_1 q_1 e^{-q_1 y} + B_2 q_2 e^{-q_2 y} + \frac{H_1 q^2}{\delta} e^{-\delta y}\right] e^{iqx}\, dq \tag{6.44}$$

$$\bar{\sigma}_{xx} = \frac{a_4}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[B_1\left(m^2 s^2 - 2q_1^2\right) e^{-q_1 y} + B_2\left(m^2 s^2 - 2q_2^2\right) e^{-q_2 y} - 2H_1 q^2 e^{-\delta y}\right] e^{iqx}\, dq \tag{6.45}$$

$$\bar{\sigma}_{yy} = \frac{a_4}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\left(m^2 s^2 + 2q^2\right)\left(B_1 e^{-q_1 y} + B_2 e^{-q_2 y}\right) + 2H_1 q^2 e^{-\delta y}\right] e^{iqx}\, dq \tag{6.46}$$

$$\bar{\sigma}_{xy} = \frac{-ia_4}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[2\left(B_1 q_1 e^{-q_1 y} + B_2 q_2 e^{-q_2 y}\right) + \frac{q^2 + \delta^2}{\delta} H_1 e^{-\delta y}\right] q e^{iqx}\, dq \tag{6.47}$$

6.4 Boundary conditions and dual integral equation formulation

For the present study, we assume the following boundary conditions at $y = 0$

$$\frac{\partial T}{\partial y} = 0, \qquad |x| > r \tag{6.48}$$

$$v = 0, \quad |x| > r \tag{6.49}$$

$$T = H(t), \quad |x| < r \tag{6.50}$$

$$\sigma_{yy} = -H(t), \quad |x| < r \tag{6.51}$$

$$\sigma_{xy} = 0, \quad -\infty < x < \infty \tag{6.52}$$

where, $H(.)$, is the Heaviside unit step function.

Now, using the boundary conditions given by Eqs. (6.48) and (6.50), Eq. (6.41) can be re-written as

$$\int_{-\infty}^{\infty} \left[B_1(k_1^2 - s^2) + B_2(k_2^2 - s^2)\right] e^{iqx}\, dq = \frac{\sqrt{2\pi}}{s}, \quad |x| < r \tag{6.53}$$

$$\int_{-\infty}^{\infty} \left[B_1 q_1(k_1^2 - s^2) + B_2 q_2(k_2^2 - s^2)\right] e^{iqx}\, dq = 0, \quad |x| > r \tag{6.54}$$

and using the boundary conditions given by Eqs. (6.49), (6.51) and (6.52), Eqs. (6.44), (6.46) and (6.47) can be written as

$$\int_{-\infty}^{\infty} \left[B_1 q_1 + B_2 q_2 + \frac{H_1 q^2}{\delta}\right] e^{iqx}\, dq = 0, \quad |x| > r \tag{6.55}$$

$$\int_{-\infty}^{\infty} \left[\left(m^2 s^2 + 2q^2\right)(B_1 + B_2) + 2H_1 q^2\right] e^{iqx}\, dq = -\frac{\sqrt{2\pi}}{s a_4}, \quad |x| < r \tag{6.56}$$

$$\int_{-\infty}^{\infty} \left[2\left(B_1 q_1 + B_2 q_2\right) + \frac{q^2 + \delta^2}{\delta} H_1\right] q e^{iqx}\, dq = 0, \quad -\infty < x < \infty \tag{6.57}$$

From Eq. (6.57), we have

$$H_1 = -\frac{2\delta\,(B_1 q_1 + B_2 q_2)}{q^2 + \delta^2} \tag{6.58}$$

Using Eq. (6.58) and the symmetry of the problem to consider x only in the intervals $[0,\ r]$ and $[r,\ \infty]$, we re-write Eqs. $(6.53-6.56)$ as

$$\sum_{i=1}^{2}(k_i^2 - s^2)\int_0^{\infty} B_i\, cos(qx)\, dq = \sqrt{\frac{\pi}{2}}\frac{1}{s},\ \ 0 < x < r \tag{6.59}$$

$$\sum_{i=1}^{2}(k_i^2 - s^2)\int_0^{\infty} B_i\, q\, cos(qx)\, dq = 0,\ \ x > r \tag{6.60}$$

$$\sum_{i=1}^{2}\int_0^{\infty} \frac{B_i q_i}{m^2 s^2 + 2q^2} cos(qx)\, dq = 0,\ \ \ x > r \tag{6.61}$$

$$\sum_{i=1}^{2}\int_0^{\infty} B_i \left[\frac{\left(m^2 s^2 + 2q^2\right)^2 - 4q^2 q_i \delta}{m^2 s^2 + 2q^2}\right] cos(qx)\, dq = -\sqrt{\frac{\pi}{2}}\frac{1}{s\, a_4},\ \ \ 0 < x < r \tag{6.62}$$

The Eqs. $(6.59-6.62)$ form a set of four dual integral equations. From these equations, we can obtain the unknown parameters B_1 and B_2 . Now, in order to solve these dual integral equations, we first assume the following:

$$B_i(q,\, s) = \int_0^{r} h_i(v,\, s)\, J_0(q\, v)\, dv \tag{6.63}$$

where, h_i, $i = 1, 2$ is the function of v and s only and $J_0(.)$ is the Bessel function of the first kind of order zero.

Now, substituting the value of B_i from Eq. (6.63) into the Eq. (6.59) and after changing the order of integration, we obtain the following relation:

$$\sum_{i=1}^{2}(k_i^2 - s^2)\int_0^{a} h_i(v,\, s)\, dv \int_0^{\infty} cos(qx)\, J_0(qv)\, dq = \sqrt{\frac{\pi}{2}}\frac{1}{s},\ \ 0 < x < r \tag{6.64}$$

We further have the integral relation of Bessel function (see Watson (1996), Mandal and Mandal

(1999)) as given below

$$\int_0^\infty cos(qx)J_0(qv)\,dq = \begin{cases} \frac{1}{\sqrt{v^2-x^2}}, & x < v \\ 0, & x > v \end{cases} \tag{6.65}$$

Hence, using above substitution, we can write Eq.(6.64) as

$$\sum_{i=1}^{2}(k_i^2 - s^2)\int_x^\infty \frac{h_i(u,s)}{\sqrt{u^2-s^2}}\,du = \sqrt{\frac{\pi}{2}}\frac{1}{s}, \;\; 0 < x < r$$

Now, multiplying the above equation with $\frac{x}{\sqrt{x^2-v^2}}$ and integrating with respect to x from v to r after changing the order of integration and differentiating the resultant equation, we have the following:

$$(k_1^2 - s^2)h_1(v,s) + (k_2^2 - s^2)h_2(v,s) = -\frac{N(v)}{s}, \; 0 < x < r \tag{6.66}$$

where,

$$N(v) = -\sqrt{\frac{2}{\pi}}\frac{v}{\sqrt{r^2-v^2}} \tag{6.67}$$

Now, multiplying both sides of the Eq. (6.66) by $J_0(qv)$ and integrating with respect to v from 0 to r, we obtain

$$B_2 = -\frac{1}{k_2^2 - s^2}\left[\frac{J(q)}{s} + \left(k_1^2 - s^2\right)B_1\right], \qquad 0 < x < r \tag{6.68}$$

where,

$$J(q) = \int_0^r N(v)J_0(qv)\,dv \tag{6.69}$$

Next, for obtaining the similar relation to Eq. (6.68) between B_1 and B_2 for the case $x > r$, we take the following:

$$B_i(q,s) = \frac{1}{q_i}\int_r^\infty h_i(v,s)J_0(qv)\,dv, \quad x > r, \; 1 = 1,2 \tag{6.70}$$

Using the relation (6.65) into Eq. (6.60) and after changing the order of integration, we obtain the following:

$$\sum_{i=1}^{2}(k_i^2-s^2)\int_x^{\infty}\frac{h_i(u,s)}{\sqrt{u^2-x^2}}du=0,\;\; x>r$$

Now, multiplying both sides of the above relation by $\frac{x}{\sqrt{x^2-v^2}}$ and integrating with respect to x from v to ∞, after changing the order of integration with the help of relation (6.70), we have the following:

$$B_2=-\frac{\left(k_1^2-s^2\right)q_1}{\left(k_2^2-s^2\right)q_2}B_1,\quad x>r \tag{6.71}$$

Now, substituting from Eq.(6.68) into the Eq. (6.62), we have the following:

$$\int_0^{\infty}\frac{B_1q_1L_1(q,s)}{m^2s^2+2q^2}cos(qx)\,dq=\bar{L}_2(x,s),\quad x<r \tag{6.72}$$

where,

$$L_1(q,s)=-\frac{\left(k_2^2-k_1^2\right)\left(m^2s^2+2q^2\right)^2-4q^2\delta\left[q_1\left(k_2^2-s^2\right)-q_2\left(k_1^2-s^2\right)\right]}{q_1}$$

$$\bar{L}_2(x,s)=-\sqrt{\frac{2}{\pi}}\frac{\left(k_2^2-s^2\right)}{s\,a_4}+\frac{1}{s}\int_0^{\infty}J(q)\left[\frac{\left(m^2s^2+2q^2\right)^2-4q^2\delta\,q_2}{(m^2s^2+2q^2)}\right]cos(qx)\,dq,\quad x<r \tag{6.73}$$

Hence, substituting from Eq. (6.71) into Eq. (6.61), we obtain

$$\int_0^{\infty}\frac{B_1q_1cos(qx)}{(m^2s^2+2q^2)}dq=0,\;\; x>r \tag{6.74}$$

In this way the original four dual integral Eqs. $(6.59-6.62)$ having the parameters B_1 and B_2 are now changed into two dual integral Eqs.(6.72) and (6.74) in the single parameter B_1only.

6.5 Solution of the dual integral equations

For solving the above two dual integral Eqs. (6.72) and (6.74), we assume the following substitution:

$$B_1(q,\, s) = \frac{\left(m^2 s^2 + 2q^2\right)}{q_1} \phi(q,\, s) \tag{6.75}$$

Therefore, Eqs. (6.72) and (6.74) are reduced into the following forms:

$$\int_0^\infty L_1(q,\, s)\, \phi(q,\, s)\, cos(qx)\, dq = \bar{L}_2(x,\, s), \;\; 0 < x < r \tag{6.76}$$

$$\int_0^\infty \phi(q,\, s)\, cos(qx)\, dq = 0, \;\; x > r \tag{6.77}$$

In order to define above for all the values of x, we are now extending the definition of the integral which is given in Eq. (6.77) in the following manner:

$$\int_0^\infty \phi(q,\, s)\, cos(qx)\, dq = \begin{cases} \sqrt{2\pi}\frac{d}{dx}\left[x\int_x^r \frac{\varphi(z,\, s)\, dz}{\sqrt{z^2 - x^2}}\right], & 0 < x < r \\ 0, & x > r \end{cases} \tag{6.78}$$

where, $\varphi(z,\, s)$ is a function which has to be determined.

We see that the left hand side of the Eq. (6.78) is just the Fourier cosine transform of $\phi(q,\, s)$. Therefore, by using the inverse Fourier cosine transform (Sherief and El-Maghraby (2005), Sneddon (1995), Churchil (1972)), we find the following:

$$\phi(q,\, s) = \int_0^r \frac{d}{dx}\left[x\int_x^r \frac{\varphi(z,\, s)\, dz}{\sqrt{z^2 - x^2}}\right] cos(qx)\, dx \tag{6.79}$$

Now, using integration by parts followed by the changing the order of integration to solve the above equation, we have

$$\varphi(q,\, s) = q\int_0^r \phi(z,\, s)\, dz \int_0^z \frac{x\, sin(qx)\, dx}{\sqrt{z^2 - x^2}} \tag{6.80}$$

By using the formula from Watson (1996), Mandal and Mandal (1999), we have

$$\int_0^z \frac{x\,sin(qx)\,dx}{\sqrt{z^2-x^2}} = \frac{\pi}{2} z J_1(qz)$$

Therefore, Eq. (6.80) can be re-write in the following form:

$$\phi(q,\,s) = \frac{\pi}{2} q \int_0^r z\,\varphi(z,\,s) J_1(qz) dz, \tag{6.81}$$

Now, putting the value of $\phi(q,\,s)$ from Eq. (6.81) into the Eq. (6.76), we have the following relation:

$$\int_0^r \bar{L}_1(z,\,x,\,s)\,\varphi(z,\,s)\,dz = \bar{L}_2(x,\,s),\ \ 0 < x < r \tag{6.82}$$

where,

$$\bar{L}_1(z,\,x,\,s) = \frac{\pi z}{2} \int_0^\infty q\,L_1(q,\,s)\,J_1(qz)\,cos(qx)\,dq$$

We see that the Eq. (6.82) is the Fredholm's integral equation of the first kind in the unknown parameter function $\varphi(z,\,s)$ which can be obtained by solving numerically and then $\phi(q,\,s)$ can be obtained from Eq. (6.81). Therefore, by using the value of $\phi(q,\,s)$ into Eq. (6.75), we can get the value of B_1. In this way, the expression for B_2 can be obtained using the value of B_1 for the case $x < r$ and $x > r$ with the help of the Eqs. (6.68) and (6.71), respectively.

6.6 Numerical results and discussions

In order to obtain the final solution of the present problem in space-time domain, we proceed as follows. The method (see Delves and Mohammed (1985)) described in the Appendix-A5 is used to solve the dual integral equations. However, the inversion of Laplace transform is carried out using the Bellman *et al.* (1966) which is described in the Appendix-A2. In order to see the behavior of all the physical fields near the crack region, we have considered the copper material having the Mode-I crack with unit length. The material constants are taken as follows (see Sherief and

El-Maghraby (2005)):

$m = 2$, $\alpha_t = 1.78(10^{-5})\text{K}^{-1}$, $c = 4.158(10^3)\text{ms}^{-1}$, $a_4 = 0.01$, $\rho = 8954\text{Kgm}^{-3}$, $\eta = 8886\text{sm}^{-2}$, $r = 1\text{m}$, $c_e = 383.1\,\text{JKKg}^{-1}$, $\lambda = 7.76(10^{10})\text{Nm}^{-2}$, $\mu = 3.86(10^{10})\text{Nm}^{-2}$, $T_0 = 293\text{K}$, $a_2 = 0.042$, $\tau_0 = 0.02$, $\tau_1 = \frac{\tau_0^2}{2}$.

We carry out programming by using the software Mathematica-7 to find out the non-dimensional numerical values of all the different physical fields like temperature, vertical and horizontal stresses, vertical and horizontal displacements for different values of vertical distance y. We make an attempt to compare the predictions by all three models namely, new model-I, new model-II and GN-III model and the graphical representation of our results is carried out for each physical field with respect to the horizontal distance, x. Due to symmetricity of the problem, we show the results for half length ($x \geq 0$) only. We specially observe the behavior of the physical fields in the vicinity of crack. Each physical field under all models is plotted for different values of y at non-dimensional time 1.2. Figs. (6.2), (6.4), (6.6), (6.8), and (6.10) show the nature of the different physical fields at the non-dimensional vertical distance 0.2, whereas Figs. (6.3), (6.5), (6.7), (6.9), and (6.11) are showing the nature of different physical fields under all three models for non-dimensional vertical distance 0.3.

The temperature distribution is shown in the Figs. (6.2) and (6.3) for non-dimensional vertical distances, 0.2 and 0.3, respectively. From Figs. (6.2) and (6.3), we note that the maximum value of temperature distribution under all three models is occurred at the beginning of the crack for both the vertical distances 0.2 and 0.3 and it decreases very slowly up to the middle of the crack region. Thereafter the decreasing rate increases. Further, at the end of the crack edge, it suddenly decreases which becomes zero after some distance. We further observe that the value of the temperature for lower vertical distance 0.2 at the beginning of the crack is more as compared to the values for the non-dimensional distance 0.3 under all three models. However, the values under GN-III model and new model-II are almost the same for both the vertical distances 0.2 and 0.3 which is significantly different from the values under new model-I.

The vertical stress distribution is shown in Figs. (6.4) and (6.5) for different non-dimensional vertical distances. We find that at the start of the crack, the values under all three models are maximum which are started to decrease up to middle of the crack. Then two local minima and one local maximum are occurred before the end of the crack edge. From there, it increases which becomes zero after some distance. We also find that the values under all three models for the non-dimensional vertical distance 0.2 is greater than the values for non dimensional vertical distance 0.3. However, the values under GN-III model and new model-II are almost the same which are significantly different from the values obtained under the new model-I. Furthermore, we note that the value under new model-I is larger up to the middle of the crack edge as compared to the values under other two models for both the vertical distances 0.2 and 0.3. The nature of the vertical stress is oscillatory in nature near the end of the crack and it is more pronounced for lower vertical distance and for the GN-III model and new model-II.

The Figs. (6.6) and (6.7) are representing the behavior of the horizontal stress for the non-dimensional vertical distances 0.2 and 0.3, respectively. It is indicated that the value under the new model-I is significantly different from the values under new model-II and GN-III model at the beginning of the crack edge. However, under all models the horizontal stress increases when we move towards the end of the middle edge and this field yields a maximum value at a point near the middle of the crack edge which is the same under all three models. Further, the values decrease up to the end of the crack edge and from there, it again starts to increase which finally become zero after some distance. Therefore, one local maximum and one local minimum are occurred for this field inside the crack edge. Furthermore, it is noted that near the middle of the crack edge, the value under the new model-I is significantly different with the values occurred under the GN-III model and new model-II. We further observe that the values under all three models are more for the vertical distance 0.2 as compared to the values found for the vertical distance 0.3. This implies that the horizontal stress decreases with the increase of vertical distance.

The nature of the vertical displacement distribution near the crack edge for the vertical distance 0.2 and 0.3 can be seen from the Figs. (6.8) and (6.9), respectively. We observe that the values are significantly different under all three models near the crack edge for both the vertical distances which is maintained up to the end of the crack edge. Further, we have also seen that one local maximum is occurred near the middle of the crack edge and showing a decreasing trend thereafter upto the end of crack edge. However, it again starts increasing after the end of the crack edge which finally vanishes after some distance. The values for the vertical distance 0.3 are smaller as compared to the values for the vertical distance 0.2 under all three models. The behavior of this physical field is same under all three models: new model-I, new model-II, GN-III model in the case of both the vertical distances 0.2 and 0.3.

The horizontal displacement distribution is shown in Figs. (6.10) and (6.11). We see that the values are same at the beginning of the crack edge under all three models and decrease with vertical distance. There is a significant difference up to the middle of the crack edge for both the vertical distances. The value upto end of the crack edge under the new model-I show a prominent difference with the values predicted by other two models. After the end of the crack, the horizontal displacement suddenly decreases to a local minimum value and increases thereafter. Finally, it becomes zero after some distance. It is also seen that within the crack edge, two local minima and one local maximum are occurred for both the vertical distances. It is also observed that the values of horizontal displacement for the vertical distance 0.2 are larger than the values obtained for the vertical distance 0.3 under all three models.

Therefore, it is clear from above discussion that all the physical fields under all three models: new model-I, new model-II and GN-III model vanish after some distance from the end of the crack edge. There is a prominent difference in the predictions of different models for each field and it is more prominent in the crack region. The vertical distance also plays a role in the behavior of each physical field. The value of each physical field decreases with the increase of the vertical distance in the crack region under each thermoelasticity theory.

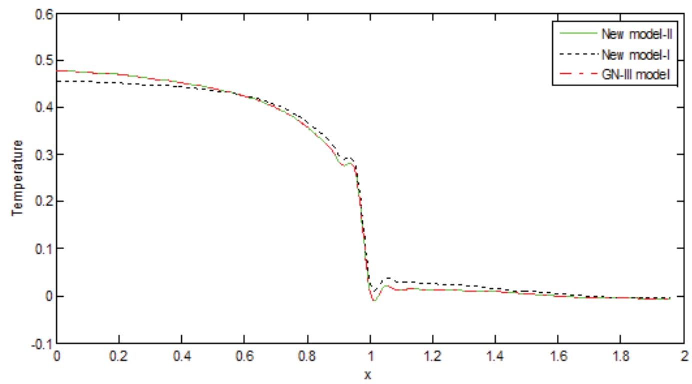

Fig. 6.2 Temperature distributions at the vertical distance 0.2

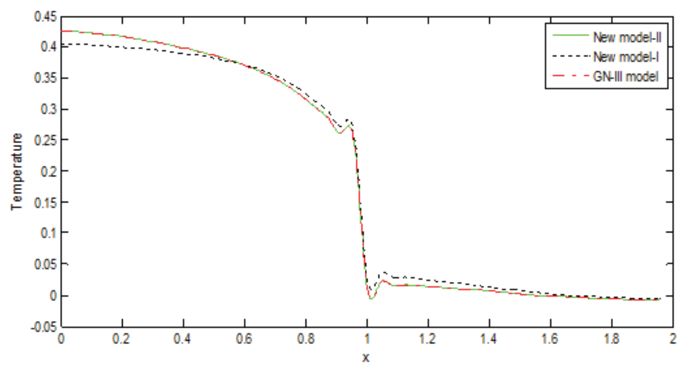

Fig. 6.3 Temperature distributions at the vertical distance 0.3

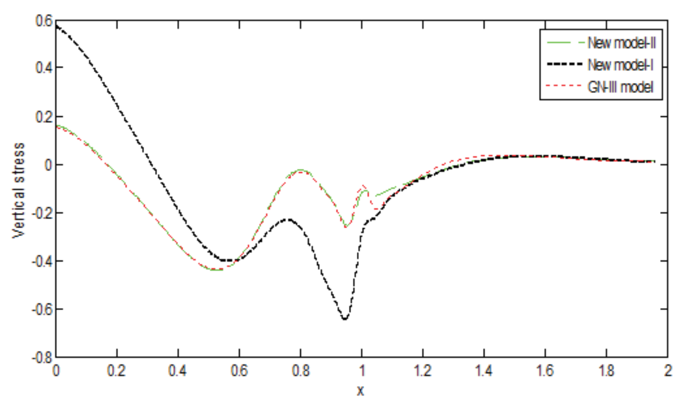

Fig. 6.4 Vertical stress distributions at the vertical distance 0.2

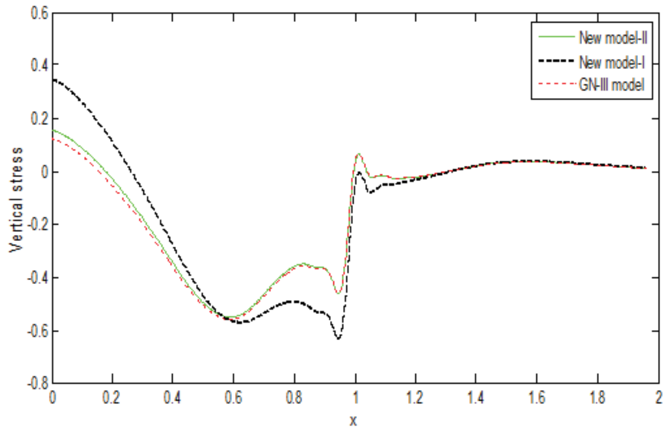

Fig. 6.5 Vertical stress distributions at the vertical distance 0.3

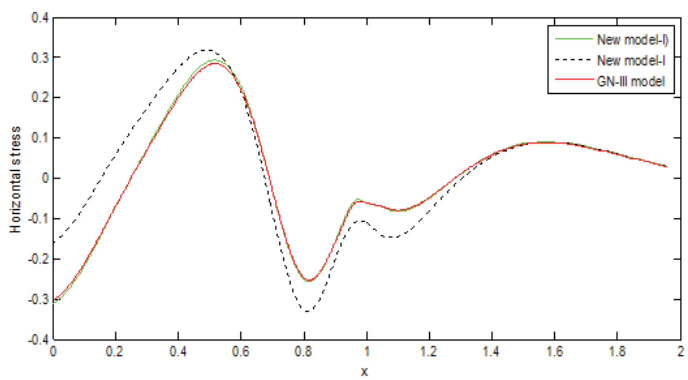

Fig. 6.6 Horizontal stress distributions at the vertical distance 0.2

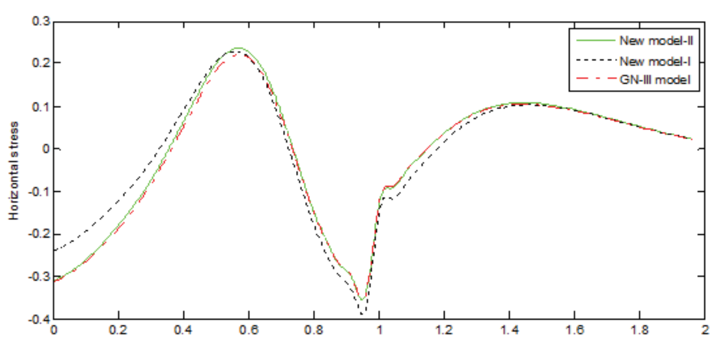

Fig. 6.7 Horizontal stress distributions at the vertical distance 0.3

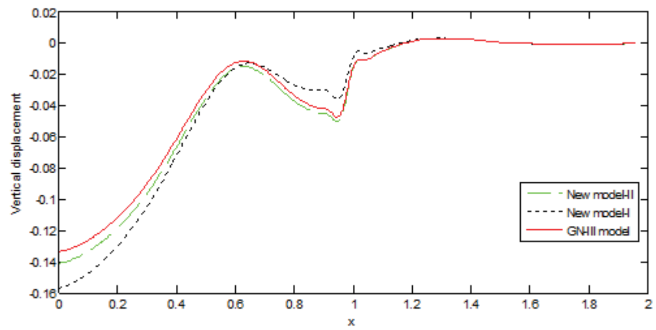

Fig. 6.8 Vertical displacement distributions at the vertical distance 0.2

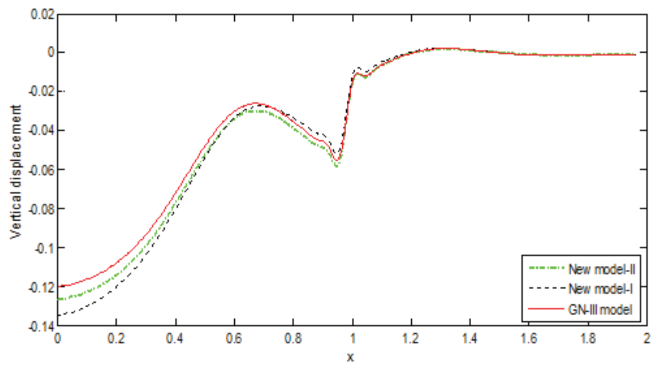

Fig. 6.9 Vertical displacement distributions at the vertical distance 0.3

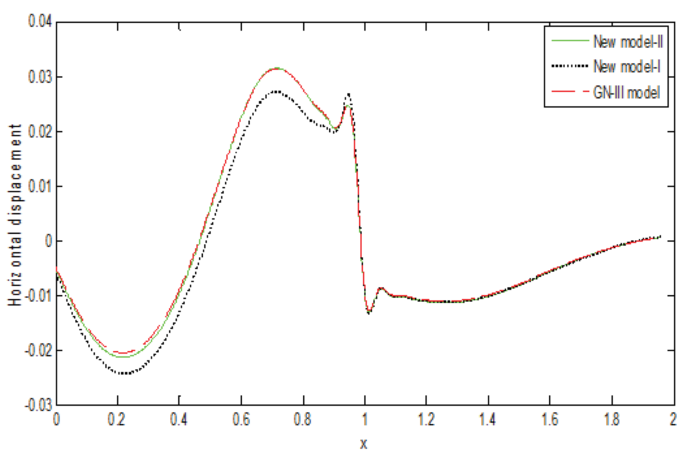

Fig. 6.10 Horizontal displacement distributions at the vertical distance 0.2

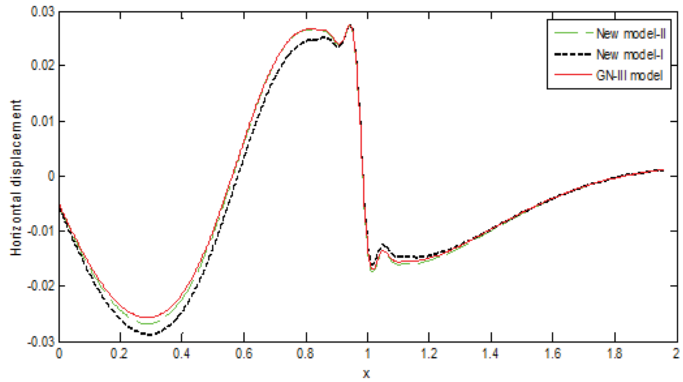

Fig. 6.11 Horizontal displacement distributions at the vertical distance 0.3

6.7 Conclusion

In this chapter, we have investigated a dynamical problem of an infinite two dimensional elastic medium with a crack of Mode-I type in the contexts of thermoelasticity theories, namely, Quintanilla's theory (2011) and Green-Naghdi theory (1995). The temperature and impact loading are considered at the boundary of the crack inside the medium. Laplace and exponential Fourier transform techniques are used to solve the problem. We obtain four dual integral equations which are further reduced into two dual integral equations. These dual integral equations are solved by using regularization method and a numerical method is used to invert the Laplace transform numerically to obtain the final solution of the problem. In order to compare the results under different models, we carry out computational work for finding the numerical values of all the physical field variables for different vertical distances. We observe the behavior of all the physical fields in the vicinity of the crack and concluded that under all models, each physical field shows the same nature throughout the domain which vanishes after some distance from the end edge of the crack. However, the value of each physical field decreases with the increase of the vertical distance from the end of the crack region under each thermoelasticity theory. The results under different models differ significantly, although the new model-II and GN-III model predict more similar results as compared to new model-I. This implies that there is a significant effect of single delay time parameter for the present crack problem.