CHAPTER-5

Effects of stochastic boundary conditions on wave propagation in thermoelastic medium

5.1 Investigation of a problem of an elastic half space subjected to stochastic temperature distribution at the boundary

5.1.1 Introduction

In recent years, the concept of stochastic simulation techniques are being used for the analysis of heat conduction and thermoelastic problem due to the fact that stochastic thermal stress analysis, instead of deterministic thermal stress analysis, is more necessary to maintain the reliability and safety gain in the design of high-temperature apparatuses or heat resistant structures. The stochastic thermal stress involves uncertainties in the apparatus structures and in thermal environments like, external temperatures, heat transfer coefficients etc. Therefore, it appears to be more realistic. In order to take into account of the noise or error in a system, it is suggested and as discussed in refs. (see Bellomo and Flandoli (1989), Omar (2009) and Sherief *et al.* (2013, 2016)), there are many reasons which allow to replace the deterministic cases with the stochastic simulations in which stationary stochastic precess are used. This is due to the fact that instead of dealing with only one possible reality of how the process might evolve under time, in a stochastic process there is some indeterminacy in its future evolution described by probability distributions. This means that even

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if the initial conditions are known, there are many possibilities that the process might go to, but some paths may be more probable than others.

Problems involving stochastic internal heat generation have been considered by Val'kovskaya and Lenyuk (1994). Chiba and Sugano (2007) investigated a stochastic thermoelastic problem of a functionally graded plate that is exposed to random external temperature load. Recently, Sherief *et al.* (2013) have discussed the effects of stochastic thermal shock at the boundary of an elastic medium and compared the results with deterministic thermal shock in the context of the theory of generalized themoelasticity with one relaxation time.

In this section of Chapter-5, we have investigated the responses of stochastic type temperature distribution applied at the boundary of an elastic medium in the context of thermoelasticity without energy dissipation. We have considered an one dimensional problem of half space and assume that the bounding surface of half space is traction free and is subjected to two types of time dependent temperature distributions which are of stochastic types. In order to compare the results predicted by stochastic temperature distributions with the results of deterministic type temperature distribution, the stochastic type temperature distributions applied at the boundary are taken in such a way that they reduce to the cases of deterministic types as special cases.

We aim to compare the results of deterministic cases of all the physical fields with the corresponding stochastic type solutions. The thermal shock problem in the context of thermoelasticity without energy dissipation has been solved by Li and Dhaliwal (1996) in which they have used the generalized thermoelasticity theory developed by Green and Naghdi to solve the boundary value problem of an isotropic elastic half space with its plane boundary either rigidly fixed or stress free and subjected to a sudden temperature increase. The deterministic solutions of all the physical field variables for this problem have been obtained. In the present work, we consider a half space problem of a thermoelastic material without energy dissipation subjected to stochastic temperature distribution at the boundary. This formulates many problems in geophysics and industry where a thermal shock is applied to a large block of material in a manufacturing process. We consider two different cases. In Case-I, the boundary is assumed to be stress free and is maintained at constant temperature, where as in Case-II, a ramp-type heating is applied at the stress free boundary of the half space. We assume the deterministic case as well as the realistic case when some noise exists, i.e., the boundary temperature distribution is of stochastic type. In sub-section 5.1.2, we write the basic governing equations in the context of Green-Naghdi model of type-II. In sub-section 5.1.3, we formulate the problem in context of this model and find the solution of all the physical fields in Laplace transform domain. In sub-section 5.1.4, firstly, by using short-time approximation and inversion of Laplace transform, we find the solution of thermal stress in deterministic case. In sub-section 5.1.4.2, we describe the theory as well as procedure to find the stochastic type solution of all the physical field variables. In sub-section 5.1.5, we find the deterministic as well as stochastic type solution for temperature distribution in Case-I and Case-II. In sub-section 5.1.6, we find the deterministic as well as stochastic type solution for displacement distribution for two cases. In sub-section 5.1.7, we carry out the numerical computation for which we take the copper materiel as a special case and use the MatLab code for computational work. We compute the deterministic as well as stochastic solutions of the physical fields in the both the cases and present them in different figures. Differences of stochastic and deterministic type solutions of all the physical fields are discussed. We have shown the variance of all the physical fields to highlight the differences of solutions.

5.1.2 Formulation of the problem

We consider a problem of an isotropic one dimensional half space $x \geq 0$. We assume the $x-$axis as a perpendicular to the upper plane pointing inwards. We formulate the problem in such a way that the surface of the bounding plane of the half-space is taken to be stress free and subjected to a time dependent temperature distribution. All the physical field variables are assumed to be bounded and vanish as $x \to \infty$. We can write the governing equations for thermoelasticity without energy dissipation in the absence of body forces and heat sources as given below:

Equation of motion:

$$\rho \ddot{u}_i = \mu u_{i,jj} + (\lambda+\mu) u_{j,ij} - \gamma T_{,i} \tag{5.1.1}$$

Heat conduction equation under Green-Naghdi-II model (1993):

$$K^* T_{,ii} = \rho c_e \ddot{T} + \gamma T_0 \ddot{e} \tag{5.1.2}$$

Stress-strain-temperature relation:

$$\sigma_{ij} = \lambda e \delta_{ij} + 2\mu e_{ij} - \gamma (T - T_0) \delta_{ij} \tag{5.1.3}$$

Strain-displacement relation:

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \qquad i,\, j = 1, 2, 3 \tag{5.1.4}$$

where, T is the absolute temperature and γ is a material constant given by $\gamma = (3\lambda + 2\mu)\alpha_t$, where, α_t is the coefficient of linear thermal expansion and K^* is the rate of thermal conductivity of the medium, c_e is the specific heat at constant strain, T_0 is the reference temperature assumed to be such that $|(T-T_0)/T_0| << 1$, and $e = u_{i,i}$ is the cubical dilatation.

Considering one dimensional problem, we can write the displacement vector in the form of $u = (u(x,t), 0, 0)$. Therefore, Eqs. (5.1.1-5.1.4) can be written as

$$\rho \ddot{u} = (\lambda + 2\mu) D^2 u - \gamma D T \tag{5.1.5}$$

$$K^* D^2 T = \rho c_e \ddot{T} + \gamma T_0 \ddot{e} \tag{5.1.6}$$

$$\sigma_{xx} = \sigma = (\lambda + 2\mu) D u - \gamma (T - T_0) \tag{5.1.7}$$

$$\sigma_{yy} = \sigma_{zz} = \lambda Du - \gamma(T - T_0) \tag{5.1.8}$$

$$\sigma_{xy} = \sigma_{zy} = \sigma_{xz} = 0 \tag{5.1.9}$$

where, $D = \frac{\partial}{\partial x}$

Now, for simplicity of the present problem, we use the following non-dimensional variables:

$$x^{'} = c_1 \xi x,\; t^{'} = c_1^2 \xi t,\, u^{'} = c_1 \xi u,\, \theta^{'} = \frac{\gamma(T - T_0)}{\lambda + 2\mu},\, \sigma_{ij}^{'} = \frac{\sigma_{ij}}{\lambda + 2\mu}$$

where, $\xi = \frac{\rho c_e}{K}$ and $c_1 = \sqrt{(\lambda + 2\mu)/\rho}$ is the speed of propagation of isothermal elastic waves. Using these non-dimensional variables and after dropping the dashed notations, Eqs. (5.1.5-5.1.7) reduce to the following forms:

$$\ddot{u} = D^2 u - D\theta \tag{5.1.10}$$

$$a_0 D^2 \theta = \ddot{\theta} + \varepsilon \ddot{e} \tag{5.1.11}$$

$$\sigma = e - \theta \tag{5.1.12}$$

where,

$$a_0 = \frac{K^*}{K c_1^2 \xi},\; \varepsilon = \frac{\gamma^2 T_0}{\rho c_e (\lambda + 2\mu)}$$

We assume the boundary conditions in the following forms:

$$\sigma(x,t)|_{x=0} = 0,\; \sigma(x,t)|_{x=\infty} = 0, \;\text{ for } t > 0 \tag{5.1.13}$$

$$\theta(x,t)|_{x=0} = \theta_0(t),\; \theta(x,t)|_{x=\infty} = 0, \;\text{ for } t > 0 \tag{5.1.14}$$

where, $\theta_0(t)$ is any function of t.

Initial conditions have been taken in homogeneous form.

5.1.3 Solution of the problem in the Laplace transform domain

Applying Laplace transform to Eqs. (5.1.10-5.1.12) along with the homogeneous initial conditions, we obtain

$$s^2\bar{u} = D^2\bar{u} - D\bar{\theta} \tag{5.1.15}$$

$$(a_0D^2 - s^2)\bar{\theta} - \varepsilon s^2\bar{e} = 0 \tag{5.1.16}$$

$$\bar{\sigma} = \bar{e} - \bar{\theta} \tag{5.1.17}$$

where, s is a Laplace parameter. Here, $\bar{f}(x,s) = L\{f(x,t)\}$.

After applying the divergence operator to (5.1.15), we obtain the following equation:

$$(D^2 - s^2)\bar{e} - D^2\bar{\theta} = 0 \tag{5.1.18}$$

Laplace transform of the boundary conditions (5.1.13-5.1.14) yields the following conditions in Laplace transform domain:

$$\bar{\sigma}(x,s)|_{x=0} = 0;\ \bar{\sigma}(x,s)|_{x=\infty} = 0;\ \bar{\theta}(x,s)|_{x=0} = \bar{\theta}_0(s);\ \bar{\theta}(x,s)|_{x=\infty} = 0 \tag{5.1.19}$$

Now, eliminating $\bar{e}$ between Eqs. (5.1.16) and (5.1.18), we obtain the fourth order differential equation in $\bar{\theta}$ as

$$(a_0D^4 - s^2(a_0 + (1+\varepsilon))D^2 + s^4)\bar{\theta} = 0 \tag{5.1.20}$$

Assuming that the solution is bounded at infinity, we can write the general solution of Eq. (5.1.20), in the following form:

$$\bar{\theta} = B e^{-k_1 x} + C e^{-k_2 x} \tag{5.1.21}$$

where, k_1 and k_2 are the roots with positive real parts of the characteristic equation

$$a_0 k^4 - s^2 (a_0 + (1+\varepsilon))\, k^2 + s^4 = 0 \tag{5.1.22}$$

and B, C are the parameters depending on Laplace transform parameter, s only.

Further, applying the boundary conditions (5.1.19), we obtain the parameters B and C in the following forms:

$B = \frac{k_1^2 - s^2}{k_1^2 - k_2^2} \bar{\theta}_0(s)$ and $C = -\frac{k_2^2 - s^2}{k_1^2 - k_2^2} \bar{\theta}_0(s)$.

Hence, we obtain $\bar{\theta}$ in the form

$$\bar{\theta}(x, s) = \frac{\bar{\theta}_0(s)}{(k_1^2 - k_2^2)} [(k_1^2 - s^2)\, e^{-k_1 x} - (k_2^2 - s^2)\, e^{-k_2 x}] \tag{5.1.23}$$

Now, in view of Eqs. (5.1.16) and (5.1.21), we obtain the solution for $\bar{e}$ as

$$\bar{e}(x, s) = \frac{\bar{\theta}_0(s)}{(k_1^2 - k_2^2)} [(k_1^2\, e^{-k_1 x} - k_2^2\, e^{-k_2 x}] \tag{5.1.24}$$

After substitution of these solutions into Eq. (5.1.17), we obtain the solution of stress field in Laplace transform domain as

$$\bar{\sigma}(x, s) = \frac{s^2 \bar{\theta}_0(s)}{(k_1^2 - k_2^2)} [e^{-k_1 x} - e^{-k_2 x}] \tag{5.1.25}$$

Using the relation $\bar{e} = D\bar{u}$ and integrating both sides of Eq. (5.1.24) from x to ∞, and assuming that u vanishes at infinity, we obtain the solution of displacement field in the Laplace transform domain in the following form:

$$\bar{u}(x,s) = \frac{\bar{\theta}_0(s)}{(k_1^2 - k_2^2)}[(-k_1\,e^{-k_1 x} + k_2\,e^{-k_2 x}] \tag{5.1.26}$$

Hence, we have completed the solution of the problem in Laplace transform domain.

5.1.4 Stress distributions in physical domain

5.1.4.1 Deterministic stress distribution

First, we consider two different types of deterministic temperature applied at the boundary of the half space. For this, we assume the boundary temperature $\theta_0(t)$ in equation (5.1.14) to be defined as follows:

Case-I: Constant temperature at the boundary

$$\theta_0(t) = \theta_1^*(t) = \theta_0 \tag{5.1.27}$$

where, θ_0 is a constant.

Case-II: Ramp type heating at the boundary

$$\theta_0(t) = \theta_2^*(t) = \theta_0\,h(t) \tag{5.1.28}$$

where, $h(t)$ is defined as

$$h(t) = \begin{cases} 0 & t \leq 0 \\ \frac{t}{t_0} & 0 < t \leq t_0 \\ 1 & t > t_0 \end{cases}$$

where, t_0 is a constant time that is the fixed time of rise of ramp-type heating.

After taking the Laplace transform of Eqs. (5.1.27) and (5.1.28) and substituting these into Eq. (5.1.25) and then taking Laplace inversion, we obtain the solution of stress distribution in the physical domain (x, t) in two cases of boundary temperature distributions as follows:

Case-I:

$$\sigma_1(x, t) = \frac{\theta_0}{a_3}[H(t - a_1 x) - H(t - a_2 x)] \tag{5.1.29}$$

Case-II:

$$\sigma_2(x, t) = \frac{\theta_0}{a_3 t_0}[(t - a_1 x)H(t - a_1 x) - (t - a_2 x)H(t - a_2 x) -$$
$$-(t - a_1 x - t_0)H(t - a_1 x - t_0) - (t - a_2 x - t_0)H(t - a_2 x - t_0)] \tag{5.1.30}$$

where, $H(.)$ is a Heaviside unit step function and $a_i, i = 1, 2, 3$ are the constants given in the Appendix-A3. We observe that the solutions given by Eq. (5.1.29) matches the corresponding solutions as reported in Li and Dhaliwal (1996).

5.1.4.2 Stochastic stress distribution

Now, we consider two types of boundary temperature distributions as

$$\theta_0(t) = \theta_i^*(t) + \psi_0(t), \quad i = 1, 2 \tag{5.1.31}$$

where, $\theta_i^*(t)$, $i = 1, 2$ are the same functions as we considered in deterministic cases, Case-I and Case-II. Here, $\psi_0(t)$ is a stochastic process characterized by the parameter, t satisfying the following condition:

$$E[\psi_0(t)] = 0 \tag{5.1.32}$$

In order to solve the present problem, we recall the following properties of Laplace transform (see Sherief *et al.* (2013), Nowinski (2000)) which are used in our work:

1. Stochastic process $x(t)$ satisfies the property

$$E[L\{x(t)\}] = L[E\{x(t)\}] \tag{5.1.33}$$

2. We have $\rho_{xx}(t_1, t_2) = E[x(t_1)\, x(t_2)]$ which represents the auto-correlation function of a stochastic process $x(t)$. Now, we define the following representation:

$$\tilde{\rho}_{xx}(s_1, s_2) = E\left[\bar{x}(s_1)\bar{x}(s_2)\right] \tag{5.1.34}$$

Then, we can write

$$\tilde{\rho}_{xx}(s_1, s_2) = L_2\left\{\rho_{xx}(t_1, t_2)\right\} \tag{5.1.35}$$

where, L_2 represents the double Laplace transform as defined by

$$L_2\left\{g(t_1, t_2)\right\} = \int_0^\infty e^{-s_1 t_1}\left(\int_0^\infty e^{-s_2 t_2} g(t_1, t_2) dt_2\right) dt_1$$

As we know that $\psi_0(t)$ is a stochastic process, then it is clear that each of the physical fields is also a stochastic process whose main randomness is due to the function $\psi_0(t)$ and from Eq. (5.1.25), we have

$$E[\bar{\sigma}(x, s)] = \frac{s^2 \bar{\theta}_i^*(s)}{(k_1^2 - k_2^2)}\left[e^{-k_1 x} - e^{-k_2 x}\right], \quad i = 1, 2 \tag{5.1.36}$$

In present work, the process is considered probabilistically stationary because it shows a form of probabilistic equilibrium in the sense that the particular instants, at which it is experimented, are not important.

Now, we define that a random process, $x(t)$ is said to be stationary if $\forall\, n$ and for every set of time instants $(t_i \in T,\, i = 1, 2, 3, \ldots, n)$:

$$g_x(x_1, \ldots, x_n;\, t_1, \ldots, t_n) = g_x\left(x_1, \ldots, x_n;\, t_1 + \tau, \ldots, t_n + \tau\right)$$

This implies that its joint probability density function does not vary with the shift of the parameter scale. Then, the auto-correlation function of a stationary process can be written in the following way:

$$E\left[x(t_1)\,x(t_2)\right] = \rho_{xx}(\tau),\ \ \tau = t_1 - t_2$$

Above expression clearly shows that auto-correlation function is the function of time interval τ only.

Now, we will evaluate variance, a measure for dispersion for the stresses as follows:

From (5.1.25) and (5.1.34), we have

$$\begin{aligned}
\tilde{\rho}_{\sigma\sigma}(x,\, s_1,\, s_2) &= E\left[\bar{\sigma}(x,\, s_1)\,\bar{\sigma}(x,\, s_2)\right] \\
&= E\left[\frac{s_1^2\bar{\theta}_0(s_1)}{\left(k_1^2(s_1)-k_2^2(s_1)\right)}\left[e^{-k_1(s_1)x}-e^{-k_2(s_1)x}\right]\frac{s_2^2\bar{\theta}_0(s_2)}{\left(k_1^2(s_2)-k_2^2(s_2)\right)}\left[e^{-k_1(s_2)x}-e^{-k_2(s_2)x}\right]\right] \\
&= \tilde{\rho}_{\psi_0\psi_0}(s_1,\, s_2)\,\bar{\Gamma}(x,\, s_1)\,\bar{\Gamma}(x,\, s_2)+\bar{\theta}_1^*(s_1)\bar{\Gamma}(x,\, s_1)\,\bar{\theta}_2^*(s_2)\bar{\Gamma}(x,\, s_2)
\end{aligned}$$

where,

$$\bar{\Gamma}(x,\, s_i) = \frac{s_i^2}{\left(k_1^2(s_i)-k_2^2(s_i)\right)}\left[e^{-k_1(s_i)x}-e^{-k_2(s_i)x}\right],\ \ i = 1,\, 2 \qquad (5.1.37)$$

and

$\tilde{\rho}_{\psi_0\psi_0}(s_1,\, s_2) = E\left[\bar{\psi}_0(s_1)\,\bar{\psi}_0(s_2)\right]$

From Eq. (5.1.36), we conclude that

$$\tilde{\rho}_{\sigma\sigma}(x,\, s_1, s_2) = \tilde{\rho}_{\psi_0\psi_0}(s_1,\, s_2)\,\bar{\Gamma}(x,\, s_1)\,\bar{\Gamma}(x,\, s_2)+E\left[\bar{\sigma}(x,\, s_1)\right]E\left[\bar{\sigma}(x,\, s_2)\right] \qquad (5.1.38)$$

We take the stochastic process, $\psi_0(t)$, included to the boundary condition to be of white noise type, which shows an extreme case in which there is no regularity. Then, from Eq. (5.1.33), we have

$$\rho_{\psi_0\psi_0}(t) = \delta(\tau) \qquad (5.1.39)$$

From Eq. (5.1.35), the double Laplace transform of Eq. (5.1.38) is given by

$$\tilde{\rho}_{\psi_0\psi_0}(s_1,s_2) = \frac{1}{s_1+s_2} \tag{5.1.40}$$

We recall the following formula for Laplace transform (see Oberhettinger and Badii (1973)):

1. $L_2\{g_1(x)\,g_2(y)\} = L\{g_1(x)\}\,L\{g_2(y)\}$

2. $\frac{1}{s_1+s_2}L_2\{g(x-\eta,y-\eta)\} = \begin{cases} L_2\left\{\int_0^x g(x-\eta,y-\eta)\,d\eta\right\}, & y>x \\ L_2\left\{\int_0^y g(x-\eta,y-\eta)\,d\eta\right\}, & y<x \end{cases}$

From (5.1.33) and (5.1.40), it is clear that Eq. (5.1.38) reduces to the following form:

$$\begin{aligned}\tilde{\rho}_{\sigma\sigma}(x,s_1,\,s_2) &= \tfrac{1}{s_1+s_2}L_2\{\Gamma(x,t_1)\Gamma(x,t_2)\} + E\left[\bar{\sigma}(x,\,s_1)\right]E\left[\bar{\sigma}(x,\,s_2)\right] \\ &= L_2\left\{\int_0^{t_1}\Gamma(x,\,t_1-\eta)\Gamma(x,\,t_2-\eta)\,d\eta\right\} + L_2\{E[\sigma(x,\,t_1)]\,E[\sigma(x,\,t_2)]\}, \qquad t_2>t_1\end{aligned}$$

After taking the inversion of double Laplace transform, we obtain the following:

$\rho_{\sigma\sigma}(x,\,t_1,\,t_2) = \int_0^{t_1}\Gamma(x,\,t_1-\eta)\Gamma(x,\,t_2-\eta)\,d\eta + E[\sigma(x,\,t_1)]\,E[\sigma(x,\,t_2)]$

Now, we assume $t_1 = t_2 = t$.

Hence, we can write the above equation in the following form:

$\rho_{\sigma\sigma}(x,\,t) = E[\sigma^2(x,\,t)] = \int_0^t \Gamma^2(x,t-\eta)\,d\eta + (E[\sigma(x,\,t)])^2$

Again we assume $t-\xi=\psi$, then we can write the above equation in the following form:

$\rho_{\sigma\sigma}(x,\,t) = \int_0^t \Gamma^2(x,\,\psi)\,d\psi + (E[\sigma(x,\,t)])^2$

Hence, we can write the variance of stress distribution function in the form

$Var[\sigma(x,\,t)] = E[\sigma^2(x,\,t)] - (E[\sigma(x,\,t)])^2 = \int_0^t \Gamma^2(x,\,\psi)\,d\psi$

From Eq. (5.1.37), we obtain

$$\bar{\Gamma}(x,\,s) = \frac{s^2}{\left(k_1^2(s)-k_2^2(s)\right)}\left[e^{-k_1(s)x} - e^{-k_2(s)x}\right] \tag{5.1.41}$$

Applying the asymptotic expansions in the similar way as in Li and Dhaliwal (1996), and taking the inversion of Laplace transform, we obtain

$$\Gamma(x,\,t) = \frac{1}{a_3}\left[\delta(t-a_1x) - \delta(t-a_2x)\right] \tag{5.1.42}$$

Now, from Eqs. (5.1.32 – 5.1.33) and (5.1.36), we obtain

$$L\{E[\sigma(x,t)]\} = \frac{s^2\bar{\theta}_i^*(s)}{(k_1^2-k_2^2)}\left[e^{-k_1x}-e^{-k_2x}\right], \qquad i=1,2$$

We observe that due to the randomness on the boundary, $E[\sigma(x,t)]$, the mean of all the sample paths of the stress field is similar to the solutions (5.1.29 – 5.1.30) for the deterministic case.

Now, in view of Eqs. (5.1.25) and (5.1.41), we obtain the stochastic stress distribution in the Laplace transform domain as

$$\bar{\sigma}(x,s) = \bar{\theta}_0(s)\bar{\Gamma}(x,s)$$

Then by using Eq. (5.1.31), we can write the above equation in the following form:

$$\bar{\sigma}(x,s) = \left[\bar{\theta}_i^*(s) + \bar{\psi}_0(s)\right]\bar{\Gamma}(x,s), \quad i=1,2$$

In view of Eqs. (5.1.25) and (5.1.29 – 5.1.30) and inverting the Laplace transform of the above equation by applying the convolution property, we find the stochastic stress distribution in the physical domain as

$$\sigma(x,t) = \sigma_i(x,t) + \int_0^t \psi_0(u)\,\Gamma(x,t-u)\,du, \;\; i=1,2$$

We can write above as

$$\sigma(x,t) = \sigma_i(x,t) + \int_0^t \Gamma(x,t-u)\,dW(u), \;\; i=1,2$$

where, $\sigma_i(x,t),\ i=1,2$ are the deterministic stress distributions as obtained in Case-I and Case-II given by Eqs. (5.1.29 – 5.1.30), respectively and $W(u)$ is the Wiener process.

Therefore, the stochastic stress distributions for Case-I and Case-II can be written as

Case-I:

$$\sigma(x,t) = \sigma_1(x,t) + \int_0^t \Gamma(x,t-u)\, dW(u)$$

Case-II:

$$\sigma(x,t) = \sigma_2(x,t) + \int_0^t \Gamma(x,t-u)\, dW(u)$$

where, $\Gamma(x,t)$ is given by Eq. (5.1.42) and $\sigma_1(x,t)$, $\sigma_2(x,t)$ are given by Eqs. (5.1.29 – 5.1.30), respectively.

5.1.5 Temperature distributions in physical domain

In the similar way as we have done in the previous section for stress distribution function, we can obtain the temperature distribution as described below.

5.1.5.1 Deterministic temperature distribution

By assuming the boundary temperature, $\theta_0(t)$ in Eq. (5.1.14) in the forms as in Case-I and Case-II given by Eqs. (5.1.27) and (5.1.28), we obtain deterministic temperature distribution functions, $\theta_1(x,t)$ and $\theta_2(x,t)$ corresponding to Case-I and Case-II, respectively as

Case-I:

$$\theta_1(x,t) = \theta_0 \left[a_4 H(t-a_1x) - a_5 H(t-a_2x)\right] \tag{5.1.43}$$

Case-II:

$$\theta_2(x,t) = \begin{array}{c} \frac{\theta_0}{t_0}\left[a_4\left\{(t-a_1x)H(t-a_1x) - (t-a_1x-t_0)H(t-a_1x-t_0)\right\} - \right. \\ \left. a_5\left\{(t-a_2x)H(t-a_2x) - (t-a_2x-t_0)H(t-a_2x-t_0)\right\}\right] \end{array} \tag{5.1.44}$$

where, $a_i, i = 1, 2, 3, 4, 5$ are the constants given in the Appendix-A3. Clearly, the solution in Case-I matches with the corresponding solution reported in Li and Dhaliwal (1996).

5.1.5.2 Stochastic temperature distribution

If the stochastic boundary condition is of white noise type, then as we derived the variance of stress in the previous section, the variance of the temperature distribution is given by

$$Var\left[\theta(x,t)\right] = \int_0^t \Theta^2(x,\psi)\,d\psi,$$

where,

$$\Theta(x,t) = [a_4\delta(t-a_1x) - a_5\delta(t-a_2x)] \tag{5.1.45}$$

Also, we can see that due to the randomness on the boundary, the mean ($E[\theta(x,t)]$) of all the sample paths of the temperature distribution is the same as the solutions given by Eqs. $(5.1.43 - 5.1.44)$.

Therefore, as we derived the stochastic stress in the previous section, we can rewrite the Eq. (5.1.23) in the following way

$$\bar{\theta}(x,s) = \bar{\theta}_0(s)\bar{\Theta}(x,s)$$

Now, using Eq. (5.1.31), we further get

$$\bar{\theta}(x,s) = \left[\bar{\theta}_i^*(s) + \bar{\psi}_0(s)\right]\bar{\Theta}(x,s), \quad i = 1, 2$$

Now, by inverting the Laplace transform of above equation and using the convolution property in second term, we obtain

$$\theta(x,t) = \theta_i(x,t) + \int_0^t \psi_0(u)\,\Theta(x,t-u)du, \;\; i = 1,2$$

We can write above as

$$\theta(x,t) = \theta_i(x,t) + \int_0^t \Theta(x,t-u)\,dW(u), \;\; i = 1,2$$

where, $\theta_i(x,t), i = 1, 2$, are given by Eqs. (5.1.43) and (5.1.44) which represent the deterministic temperature distributions in Case-I and Case-II, respectively. $W(u)$ is the Wiener process.
Hence, the stochastic temperature distributions for Case-I and Case-II can be written as

Case-I:

$$\theta(x,t) = \theta_1(x,t) + \int_0^t \Theta(x,t-u)\,dW(u)$$

Case-II:

$$\theta(x,t) = \theta_2(x,t) + \int_0^t \Theta(x,t-u)\,dW(u)$$

where, $\Theta(x,t)$ is given by Eq. (5.1.45).

5.1.6 Displacement distributions in physical domain

5.1.6.1 Deterministic displacement distribution

In view of prescribed boundary temperatures, given by equations (5.1.27) and (5.1.28) for Case-I and Case-II, respectively, the deterministic displacement distribution functions, $u_1(x,t)$ and $u_2(x,t)$ for two cases are evaluated as follows:

Case-I:

$$u_1(x,t) = \theta_0 \left[-a_6 (t - a_1 x) H(t - a_1 x) + a_7 (t - a_2 x) H(t - a_2 x)\right] \tag{5.1.46}$$

Case-II:

$$u_2(x,t) = \begin{array}{c} \frac{\theta_0}{2t_0}[a_6 \left\{(t - a_1 x - t_0)^2 H(t - a_1 x - t_0) - (t - a_1 x)^2 H(t - a_1 x)\right\} + \\ a_7 \left\{(t - a_2 x - t_0)^2 H(t - a_2 x - t_0) - (t - a_2 x)^2 H(t - a_2 x)\right\}] \end{array} \tag{5.1.47}$$

where, a_i, $i = 1, 2, 6, 7$ are the constants given in the Appendix-A3. The solution (5.1.46) in Case-I matches with the corresponding solution reported in Li and Dhaliwal (1996).

5.1.6.2 Stochastic displacement distribution

If the stochastic boundary condition is of white noise type, then by following way, like derivation of stress and temperature distributions, the variance of the displacement is derived as

$$Var[u(x,t)] = \int_0^t \Xi^2(x, \psi)\, d\psi$$

where,

$$\Xi(x,t) = [-a_6 H(t - a_1 x) + a_7 H(t - a_2 x)] \tag{5.1.48}$$

We note that $E[u(x,t)]$, the mean of all the sample paths of the displacement due to randomness on the boundary is the same as the solutions given by Eqs. $(5.1.46 - 5.1.47)$.

Therefore, as we derived the stochastic stress and temperature distributions, we can also rewrite the Eq. (5.1.26) in the following form:

$$\bar{u}(x,s) = \bar{\theta}_0(s)\bar{\Xi}(x,s)$$

Now, using Eq. (5.1.31), we further obtain

$$\bar{u}(x,s) = \left[\bar{\theta}_i^*(s) + \bar{\psi}_0(s)\right] \bar{\Xi}(x,s), \qquad i = 1,2$$

Hence, by inverting of Laplace transforms and applying the convolution property in second term, we obtain

$$u(x,t) = u_i(x,t) + \int_0^t \psi_0(u)\,\Xi(x,t-u)du, \;\; i = 1,2$$

$$u(x,t) = u_i(x,t) + \int_0^t \Xi(x,t-u)\,dW(u), \;\; i = 1,2$$

where, $u_i(x,t), i = 1,2$ are the deterministic displacement distributions in Case-I and Case-II as given by equations (5.1.46) and (5.1.47), respectively and $W(u)$ is the Wiener process.

Therefore, the stochastic displacement distributions for Case-I and Case-II can be derived as

Case-I:

$$u(x,t) = u_1(x,t) + \int_0^t \Xi(x,t-u)\,dW(u)$$

Case-II:

$$u(x,t) = u_2(x,t) + \int_0^t \Xi(x,t-u)\,dW(u)$$

where, $\Xi(x,t)$ is given by Eq. (5.1.48).

5.1.7 Numerical results and discussion

In this section, we solve the above problem numerically for the purpose of illustrating the analytical results obtained in previous sections and make attempt to compare the results of stochastic and deterministic solution of all the physical fields. We consider the data for copper material as given below:

$c_e = 383.1\,\mathrm{JKKg^{-1}}$, $\alpha_t = 1.78\times10^{-5}\mathrm{K^{-1}}$, $K = 386\,\mathrm{WmK^{-1}}$, $\lambda = 7.76\times10^{10}\mathrm{Kgm^{-1}s^{-2}}$, $\mu = 7.76\times10^{10}\mathrm{Kgm^{-1}s^{-2}}$, $\tau_0 = 0.05$, $\rho = 8954\,\mathrm{Kgm^{-3}}$

We do all the numerical computations at non-dimensional time, $t = 0.25$. To compute the stochastic type integration, we followed Hingham (2001), from which we take the concept of Brownian motion or standard Wiener processes. Two sample paths of stochastic process have been plotted to compare the results with deterministic cases of all the physical field variables.

Results of Case-I are shown in Figures *5.1.1 (a, b, c, d, e)*, *5.1.2 (a, b, c, d, e)* and *5.1.3 (a, b, c, d, e)*. For Case-I, deterministic stress distribution is shown in Fig. 5.1.1 (a) and stochastic stress distributions for sample path-I, II are shown in Figures 5.1.1 $(b,\ c)$. In order to show its comparison with deterministic stress distribution, we plot Fig. 5.1.1 (d) in which we take two sample paths of stochastic process. From Figure 5.1.1 (d), it is clear that the average of two sample paths coincide with deterministic stress distribution and verifies our analytical result in this respect. Fig. 5.1.1 (e) shows the variance of stress distribution. The deterministic temperature distribution is shown in Fig. 5.1.2 (a) and stochastic temperature distributions for sample path-I, II are shown in Figures 5.1.2 (b, c). Its comparison with deterministic temperature distribution is shown in Fig. 5.1.2 (d) in which we take two sample paths of stochastic process. From Fig. 5.1.2 (d), it is clear that the average of two sample paths coincide with deterministic temperature distribution. Fig. 5.1.2 (e) shows the variance of temperature distribution. Also, deterministic displacement distribution is shown in Fig. 5.1.3 (a) and stochastic displacement distributions for sample path-I, II are shown in Figures 5.1.3 (b, c), while its comparison with stochastic displacement distribution is shown in Fig. 5.1.3 (d) in which we take two sample paths of stochastic process. Fig. 5.1.3 (d) also indicates that the average of two sample paths match exactly with the deterministic displacement distribution. Fig. 5.1.3 (e) shows the variance of displacement distribution.

Results for Case-II are displayed in Figures *5.2.1(a, b, c, d, e)*, *5.2.2 (a, b, c, d, e)* and *5.2.3 (a, b, c, d, e)*. For the Case-II, the deterministic stress distribution is shown in Fig. 5.2.1 (a) and stochastic stress distributions for sample path-I, II are shown in Figures 5.2.1 $(b,\ c)$. Fig. 5.2.1 (d)

shows the comparison of stochastic stress distributions for two sample paths with the deterministic stress distribution and depicts that the average of two sample paths coincide with deterministic stress distribution. Fig. 5.2.1(e) shows the variance of stress distribution. Similarly, deterministic temperature distribution is shown in Fig. 5.2.2(a) and stochastic temperature distributions for sample paths I and II are shown in Figures 5.1.2(b, c) while its comparison with deterministic temperature distribution is shown in Fig. 5.2.2(d). From Fig. 5.2.2(d), it is clear that the average of two sample paths coincide with deterministic temperature distribution. Fig. 5.2.2(e) shows the variance of temperature distribution. The deterministic displacement distribution for Case-II is shown in Fig. 5.2.2(a) and stochastic displacement distributions for sample path-I, II are shown in Figures 5.1.3(b, c). Figure 5.2.2(d) displays the comparison of stochastic displacement distributions for two different sample paths with the deterministic displacement distribution and verifies the analytical results that the average of two sample paths coincide with deterministic displacement distribution. Fig. 5.2.2(e) shows the variance of displacement distribution. In case of stochastic distribution for any sample path, we observed fluctuation in the field variables near the boundary and it vanishes after some distance from the boundary. This implies that region of influence for the stochastic distribution of each field is finite.

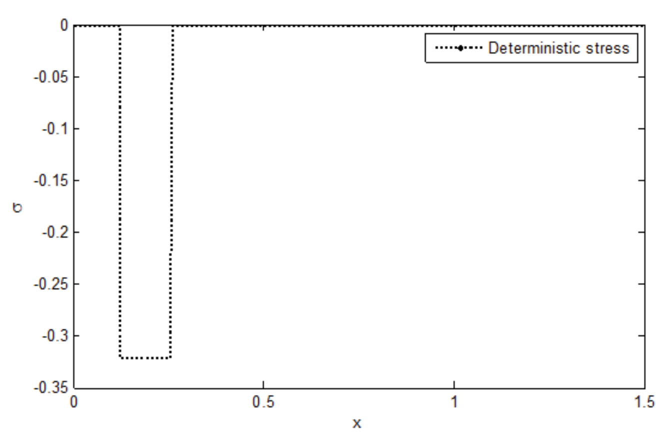

Fig. 5.1.1 (a) Deterministic stress distribution for case-I

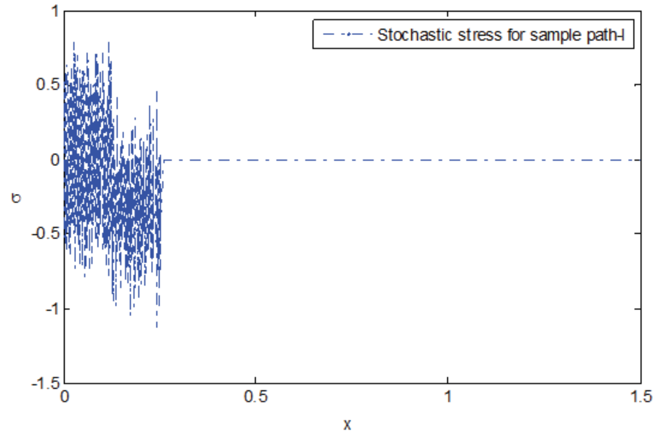

Fig. 5.1.1 (b) Stochastic stress distribution for sample path-I in case-I

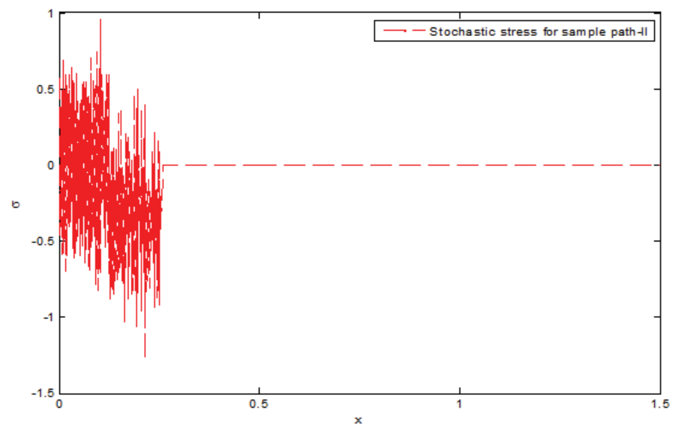

Fig. 5.1.1 (c) Stochastic stress distribution for sample path-II in case-I

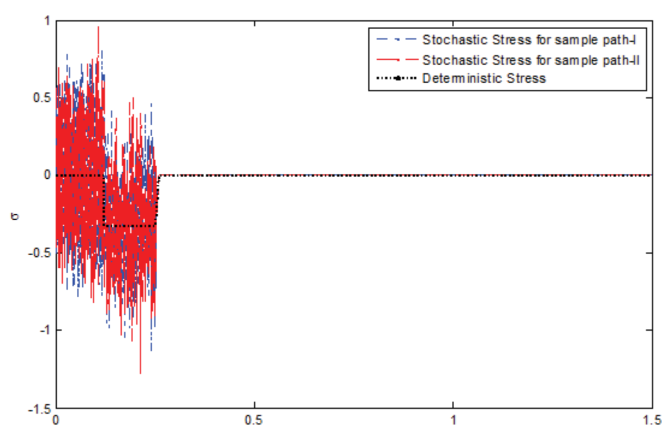

Fig. 5.1.1 (d) Comparison between deterministic and Stochastic stress distributions for case-I

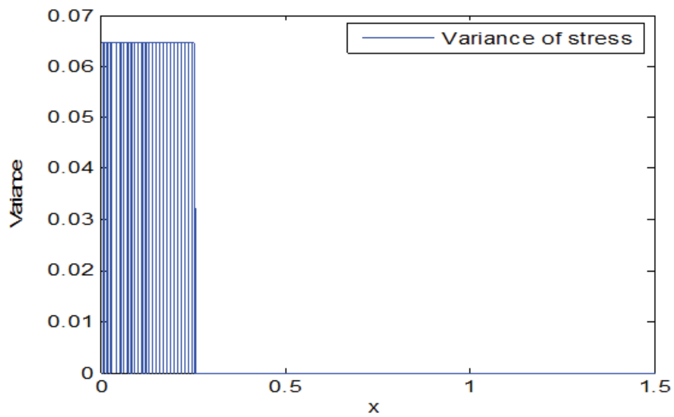

Fig. 5.1.1 (e) Variance of stress distribution for case-I

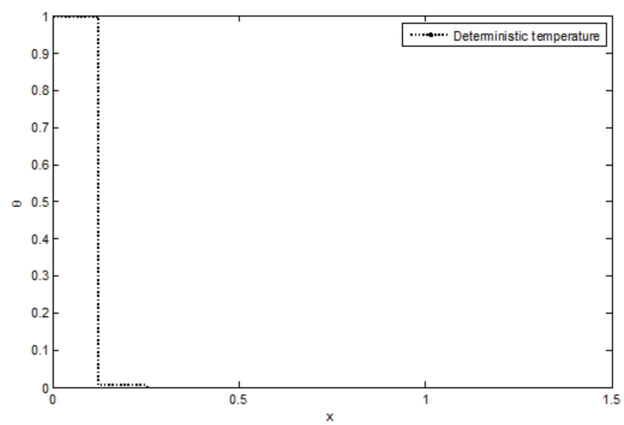

Fig. 5.1.2 (a) Deterministic temperaturedistribution for case-I

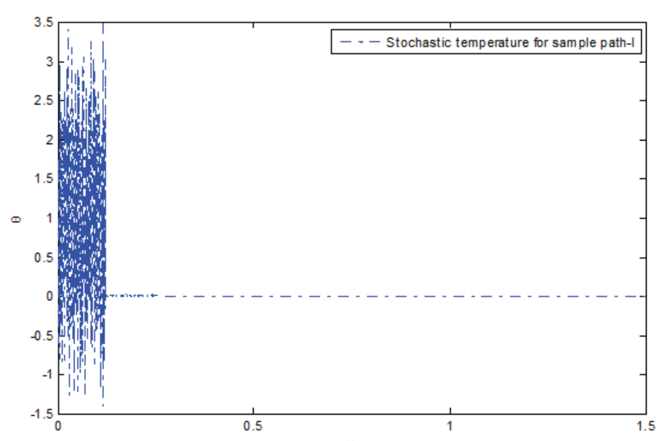

Fig. 5.1.2 (b) Stochastic temperature distribution for sample path-I in case-I

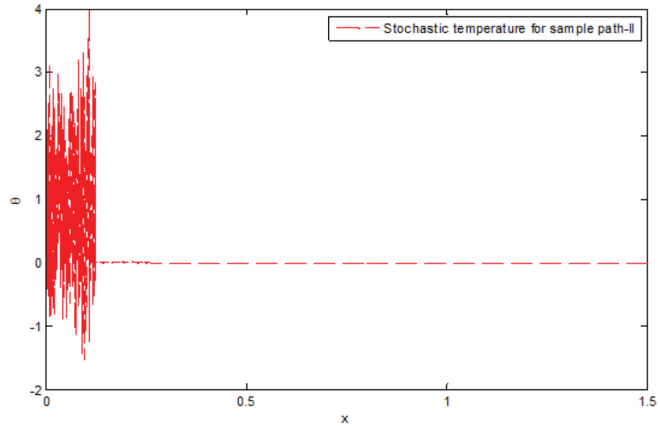

Fig. 5.1.2 (c) Stochastic temperature distribution for sample path-II in case-I

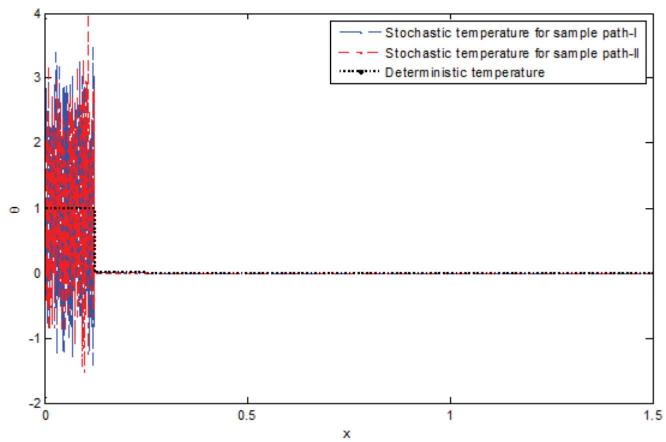

Fig. 5.1.2 (d) Comparison between deterministic and Stochastic temperaturedistributions for case-I

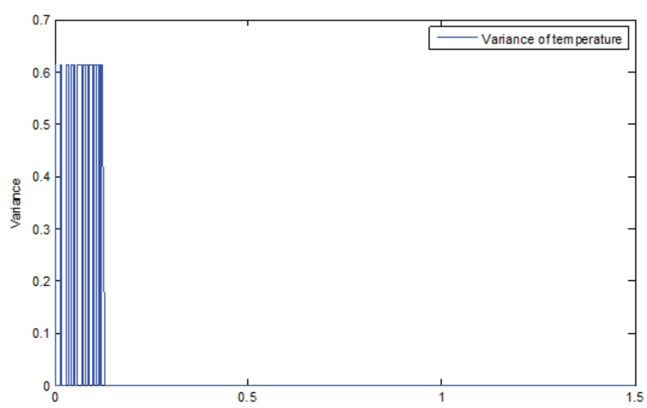

Fig. 5.1.2 (e) Variance of temperature distribution for case-I

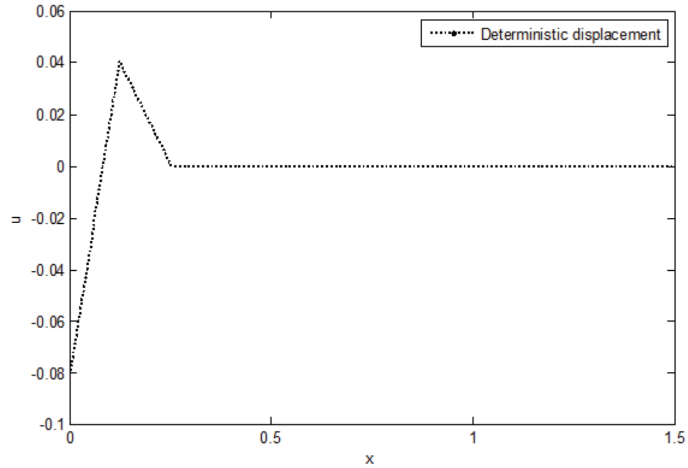

Fig. 5.1.3 (a) Deterministic displacementdistribution for case-I

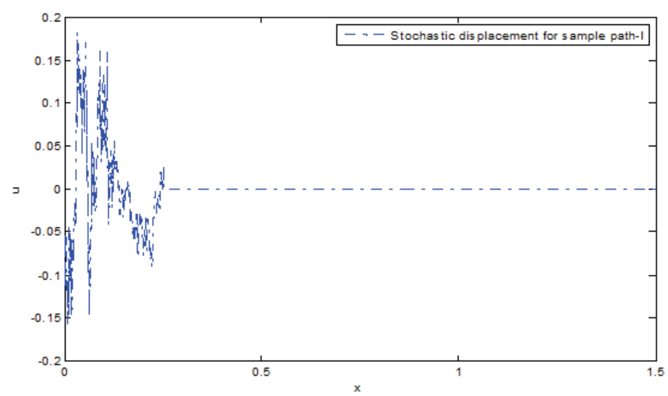

Fig. 5.1.3 (b) Stochastic displacementdistribution for sample path-I in case-I

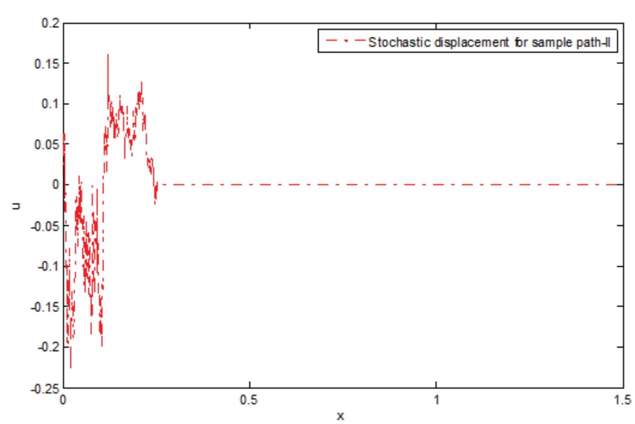

Fig. 5.1.3 (c) Stochastic displacementdistribution for sample path-II in case-I

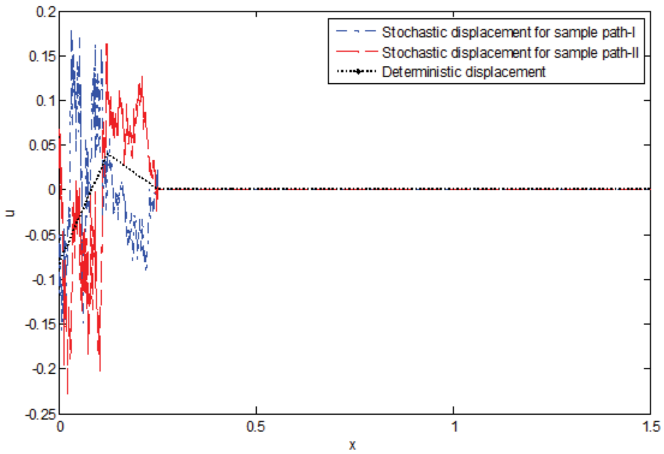

Fig. 5.1.3 (d) Comparison between deterministic and Stochastic displacementdistributions for case-I

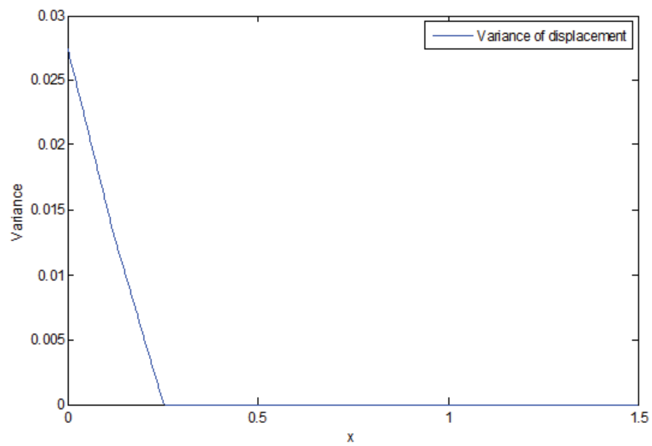

Fig. 5.1.3 (e) Variance of displacementdistribution for case-I

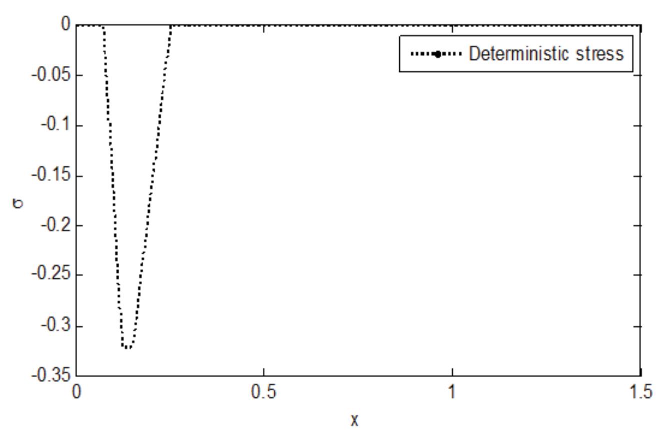

Fig. 5.2.1 (a) Deterministic stress distribution for case-II

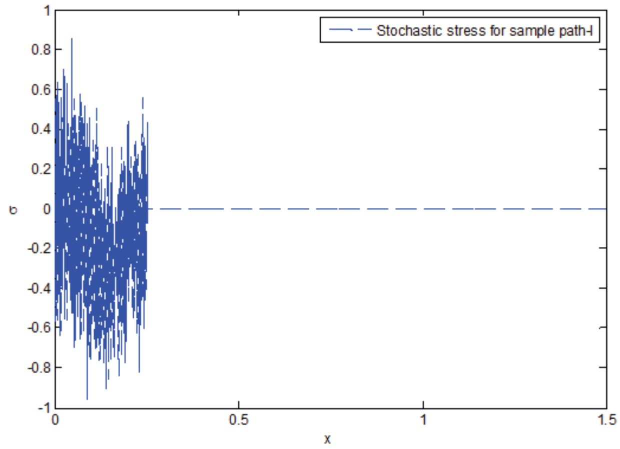

Fig. 5.2.1 (b) Stochastic stress distribution for sample path-I in case-II

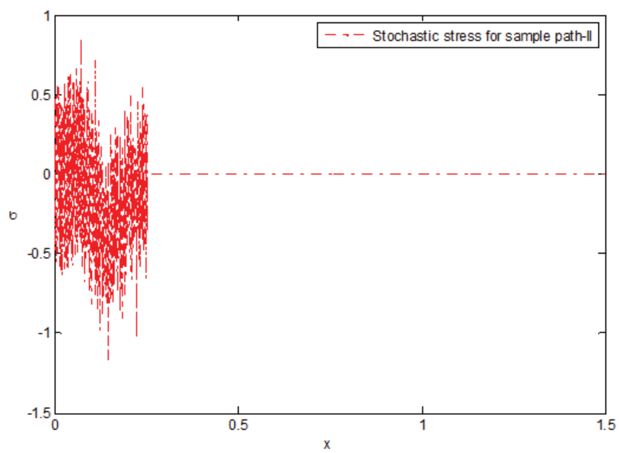

Fig. 5.2.1 (c) Stochastic stress distribution for sample path-II in case-II

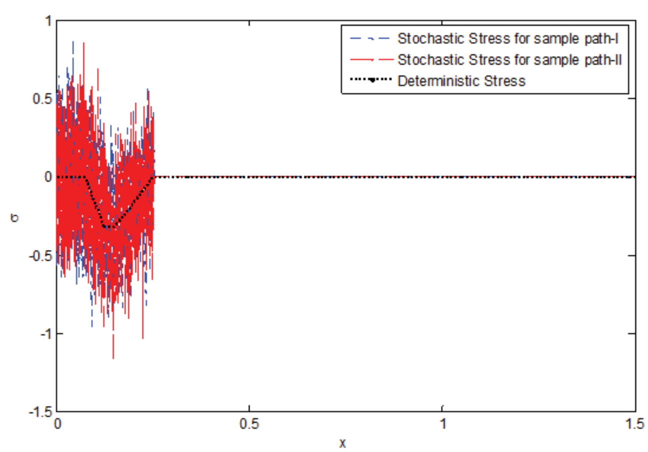

Fig. 5.2.1 (d) Comparison between deterministic and stochastic stress distributions for case-II

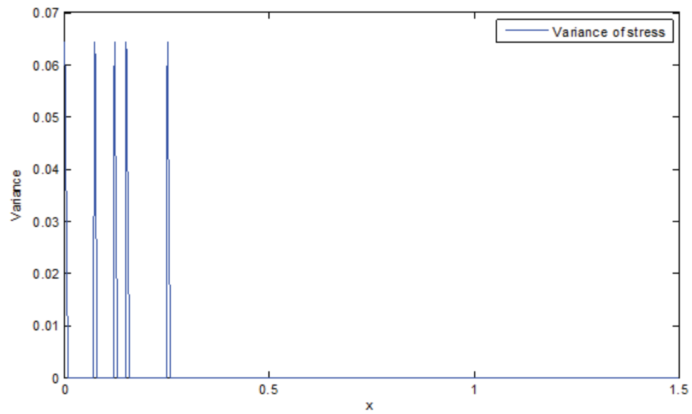

Fig. 5.2.1 (e) Variance of stressdistribution for case-II

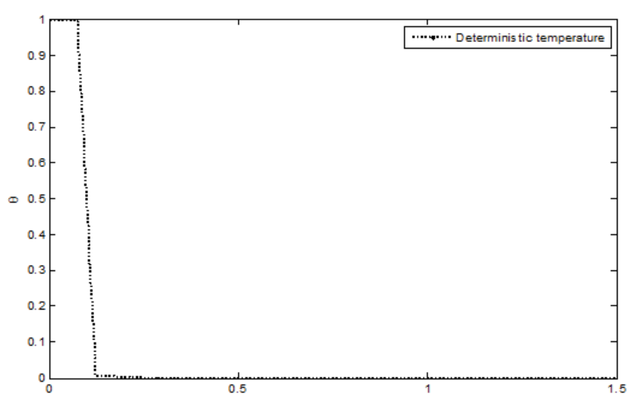

Fig. 5.2.2 (a) Deterministic temperature distribution for case-II

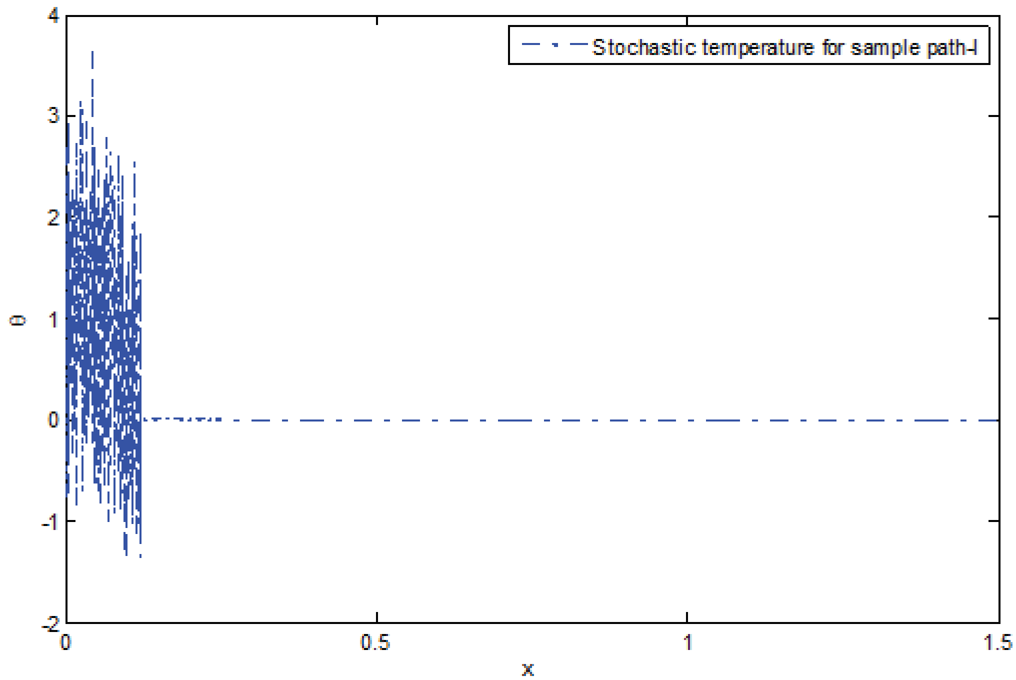

Fig. 5.2.2 (b) Stochastic temperature distribution for sample path-I in case-II

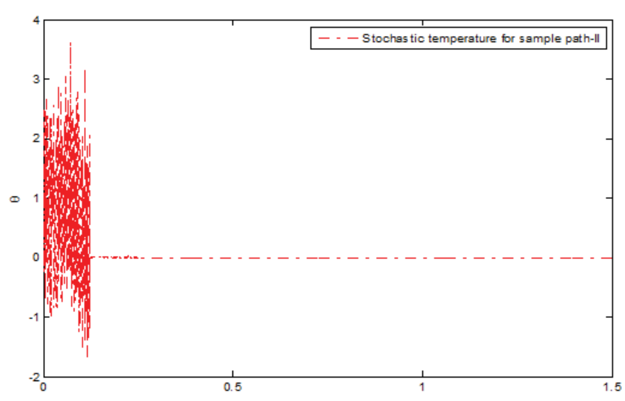

Fig. 5.2.2 (c) Stochastic temperature distribution for sample path-II in case-II

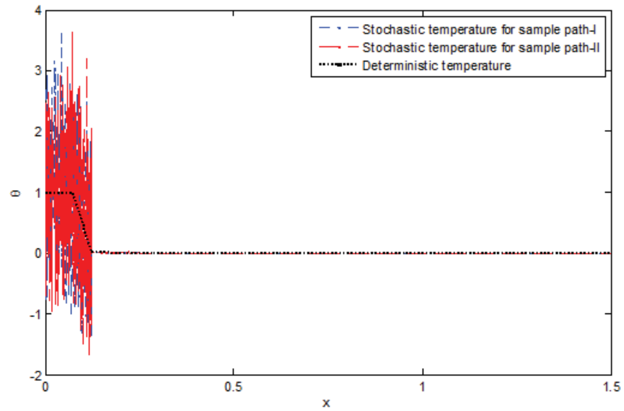

Fig. 5.2.2 (d) Comparison between deterministic and stochastic temperature distributions for case-II

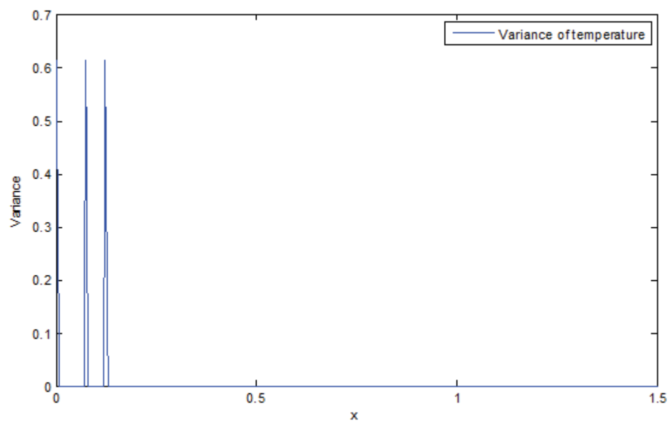

Fig. 5.2.2 (e) Variance of temperaturedistribution for case-II

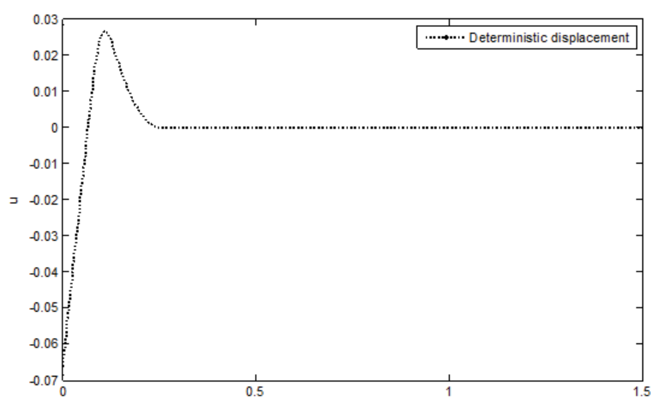

Fig. 5.2.3 (a) Deterministic displacement distribution for case-II

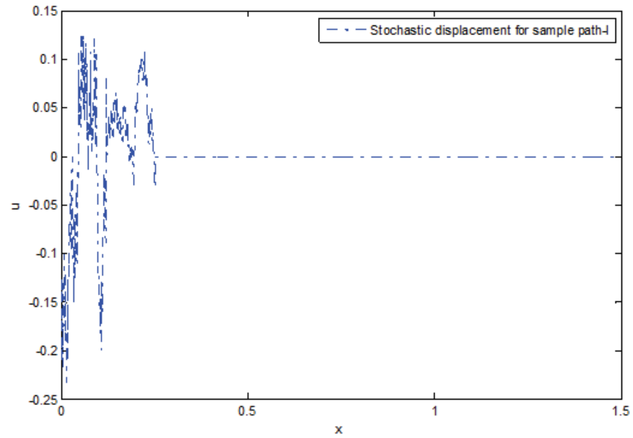

Fig. 5.2.3 (b) Stochastic displacement distribution for sample path-I in case-II

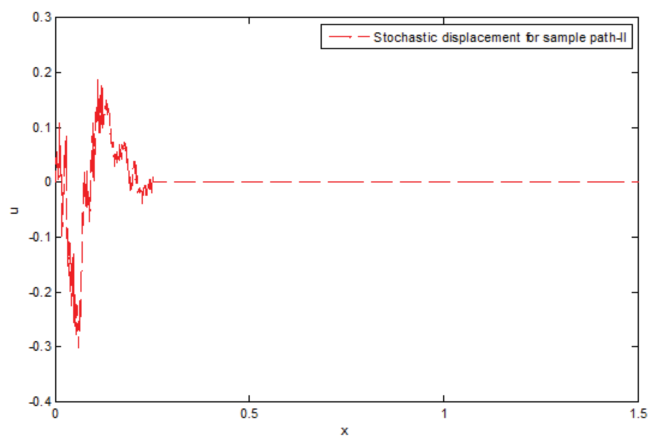

Fig. 5.2.3 (c) Stochastic displacement distribution for sample path-II in case-II

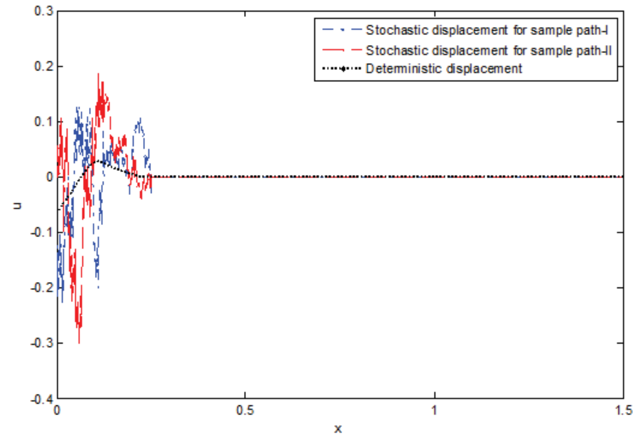

Fig. 5.2.3 (d) Comparison between deterministic and stochastic displacement distributions for case-II

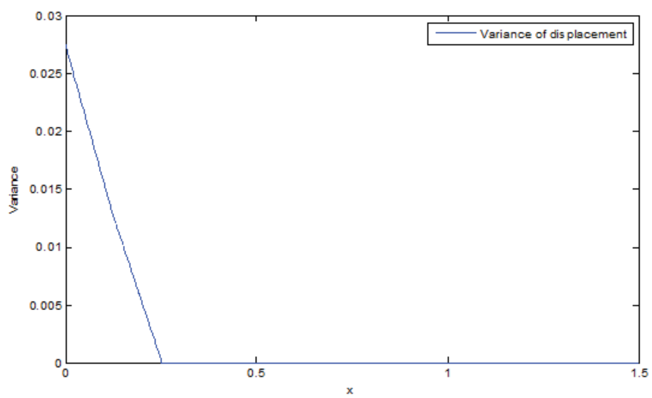

Fig. 5.2.3 (e) Variance of displacement distribution for case-II

5.1.8 Conclusions

The half space problem has been solved for two types of stochastic white noise type thermal boundary conditions when the boundary is assumed to be stress free. The white noise stochastic process has been chosen as it is the most common type. The corresponding deterministic type thermal boundary conditions are also considered in order to investigate the effects of stochastic conditions in more detailed way. We have shown both analytically and numerically that mean of the stochastic solution of all the physical fields coincides with the respective deterministic solution.

It is found that the deviation of the stochastic solution from its mean decreases with the distance from the bounding plane which is the source of noise. We find that fluctuations on the solution due to the noise on the boundary travels inside the medium with finite speed as we have seen it is also observed in the case of deterministic boundary conditions. Our analytical results as well as numerical results clearly indicate that all the physical fields vanish after a certain distance from the boundary in the stochastic case as well as in the deterministic case. This ascertains the fact that the thermoelasticity without energy dissipation theory predicts finite speed of thermal waves.

By comparing our results in Case-I with the corresponding results as reported in Sherief *et al.* (2013), we note that the heat conduction equations of both the theories (thermoelasticity theory with one relaxation time and thermoelasticity without energy dissipation) are of wave types. Therefore, they automatically ensure the finite speed of propagation for heat and elastic waves. However, an important feature of the thermoelasticity theory under GN-II model is that waves propagate without any attenuation, which is unlike the case of thermoelasticity theory with one relaxation time as obtained by Sherief *et al.* (2013) and we find that wave propagate with attenuation in this case. Furthermore, in Case-I of our present study and in case of thermoelasticity with one relaxation time, stress and temperature distributions show the finite jump discontinuity at both the elastic and thermal wave fronts and displacement is continuous in nature. However, in case of ramp-type heating (Case-II), all the field variables are continuous in nature. We observe that the region of influence is less in the case of all the physical field variables under the thermoelasicity theory without energy dissipation as compared to generalized thermoelasticity theory with one relaxation time parameter. Furthermore, it is noted that the fluctuations of the field variables due to presence of noise in the case of stochastic distribution are more concentrated near the boundary of the half space in case of thermoelasicity theory without energy dissipation as compared to generalized thermoelasticity theory with one relaxation time parameter. Under thermoelasticity with one relaxation time, the fluctuations in the field variables continue upto a longer distance as compared to the case of GN-II model.

5.2 Investigation on effects of stochastic loading at the boundary under thermoelasticity with two relaxation parameters

5.2.1 Introduction

In the previous section (First section) of Chapter-5, we have investigated the effects of stochastic temperature distribution at the boundary of an elastic half space under the thermoelasticity theory without energy dissipation. However, in the present section, the main objective is to study the responses of stochastic type mechanical distribution at the boundary of an elastic half space in the context of generalized thermoelasticity with two relaxation times. Here, we have added the effects of the presence of some noise or uncertainty in the applied load in the boundary of elastic medium and represented it as a stochastic load. We have compared all the stochastic physical field variables to the respective deterministic fields which are discussed by Sherief (1994). In order to compare the results under the stochastic mechanical distribution, we have also considered the case of deterministic mechanical distribution prescribed at the boundary. Hence, the stochastic mechanical distribution is considered in such a way that it reduces to a deterministic type distribution as a special case. We compute the deterministic as well as stochastic solutions of the physical fields and present them in different figures. Differences of stochastic and deterministic type solutions of all the physical field variables are discussed. The overall conclusion regarding the responses of stochastic type mechanical distribution in the present context is presented in a detailed way.

The work of this section is published in ***Applied Mathematical Modelling (Elsevier)***, 648-669, 54 (2018).

5.2.2 Formulation of the problem

An infinite, homogeneous and isotropic thermoelastic medium is considered in the half space $x \geq 0$ in which $x-$ axis has been taken as a perpendicular to the bounding plane directing inwards. However, the formulation of the problem is carried out in such a way that the surface of the bounding plane of the half space is considered to a time dependent stress distribution. We consider the following governing equations for the thermoelasticity with two relaxation time parameters as introduced by Green and Lindsay (1972):

Equation of motion:

$$\rho \ddot{u} = (\lambda + 2\mu)\frac{\partial^2 u}{\partial x^2} - \gamma \frac{\partial}{\partial x}\left[T - T_0 + \tau_1 \dot{T}\right] \tag{5.2.1}$$

Heat conduction equation:

$$K\frac{\partial^2 T}{\partial x^2} = \rho\, c_e \left[\dot{T} + \tau_2 \ddot{T}\right] + \gamma T_0 \frac{\partial \dot{u}}{\partial x} \tag{5.2.2}$$

Constitutive equation:

$$\sigma = (\lambda + 2\mu)\frac{\partial u}{\partial x} - \gamma\left[T - T_0 + \tau_1 \dot{T}\right] \tag{5.2.3}$$

where, T is the absolute temperature and T_0 is a reference temperature such that $\mid (T - T_0)/T_0 \mid \ll 1$. Further, we assume that displacement and temperature are dependent on space co-ordinate, x and time t only. u is the displacement component in the $x-$direction, t is the time variable. τ_1 and τ_2 are the relaxation time parameters with dimension of time. K is the thermal conductivity and $\gamma = (3\lambda + 2\mu)\alpha_t$, α_t being the coefficient of linear thermal expansion. $\sigma = \sigma_{xx}$ is the component of stress tensor in the $x-$direction.

Above governing Eqs. $(5.2.1 - 5.2.3)$ are considered only in the absence of body forces and heat sources. However, all the physical distributions are taken to be bounded which vanish as $x \to \infty$.

We now use the following non-dimensional variables:

$x' = c_1 \eta x$, $t' = c_1^2 \eta t$, $\tau_1' = c_1^2 \eta \tau_1$, $u' = c_1 \eta u$, $\tau_2' = c_1^2 \eta \tau_2$, $\sigma' = \frac{\sigma}{\mu}$, $\theta = (T - T_0)/T_0$
where, $\eta = \rho c_e / K$ and $c_1 = \sqrt{\frac{(\lambda+2\mu)}{\rho}}$ is the velocity of longitudinal waves. For convenience, we now remove the dashed notation from above variables. Hence, finally the Eqs. $(5.2.1-5.2.3)$ take the following forms:

$$\beta^2 \ddot{u} = \beta^2 \frac{\partial^2 u}{\partial x^2} - B \frac{\partial}{\partial x} \left[\theta + \tau_1 \dot{\theta}\right] \tag{5.2.4}$$

$$\frac{\partial^2 \theta}{\partial x^2} = \dot{\theta} + \tau_2 \ddot{\theta} + C \frac{\partial \dot{u}}{\partial x} \tag{5.2.5}$$

$$\sigma = \beta^2 \frac{\partial u}{\partial x} - B \left[\theta + \tau_1 \dot{\theta}\right] \tag{5.2.6}$$

where, $B = \gamma T_0/\mu$, $C = \gamma / K\eta$ and $\beta^2 = (\lambda + 2\mu)/\mu$
We now assume the boundary conditions of the medium for $t > 0$ in the following forms:

$$\sigma(0,t) = F(t), \quad \sigma(\infty,t) = 0 \tag{5.2.7}$$

$$\theta(0,t) = 0, \quad \theta(\infty,t) = 0 \tag{5.2.8}$$

where, $F(t)$ is the mechanical load acting at the boundary of the plane. However, the initial conditions are considered to be homogeneous.

5.2.3 Solution of the problem in Laplace transform domain

After taking the Laplace transform with the homogeneous initial conditions, the Eqs.$(5.2.4-5.2.6)$ are transformed to the forms:

$$\frac{\partial^2 \bar{u}}{\partial x^2} - s^2 \bar{u} = C_1 (1 + \tau_1 s) \frac{\partial \bar{\theta}}{\partial x} \tag{5.2.9}$$

$$\frac{\partial^2\bar{\theta}}{\partial x^2} - s(1+\tau_2 s)\bar{\theta} = C s\frac{\partial \bar{u}}{\partial x} \tag{5.2.10}$$

$$\bar{\sigma} = \beta^2\frac{\partial \bar{u}}{\partial x} - B(1+\tau_1 s)\bar{\theta} \tag{5.2.11}$$

where, $C_1 = B/\beta^2$ and s is the Laplace transform parameter. Here, $\bar{f}(x,s) = L\{f(x,t)\}$.

Now for simplicity, we define the potential function , φ by the relation:

$$u = \frac{\partial \varphi}{\partial x} \tag{5.2.12}$$

Hence, by substituting u from Eq. $(5.2.12)$ into the Eqs. $(5.2.9-5.2.11)$ and after integrating the first resulting equation with respect to x, Eqs. $(5.2.9-5.2.11)$ can be written as

$$(\frac{\partial^2}{\partial x^2} - s^2)\bar{\varphi} = C_1(1+\tau_1\, s)\bar{\theta} \tag{5.2.13}$$

$$(\frac{\partial^2}{\partial x^2} - s - \tau_2\, s^2)\bar{\theta} = C\, s\,\frac{\partial^2}{\partial x^2}\bar{\varphi} \tag{5.2.14}$$

$$\bar{\sigma} = \beta^2 s^2\bar{\varphi} \tag{5.2.15}$$

With the help of potential function, φ the boundary conditions can be rewritten as given below:

$$\bar{\varphi}(0,s) = \frac{\bar{F}(s)}{\beta^2 s^2} \tag{5.2.16}$$

$$\beta^2\frac{\partial^2\bar{\varphi}(0,s)}{\partial x^2} = \bar{F}(s) \tag{5.2.17}$$

$$\bar{\varphi}(\infty,s) = \frac{\partial}{\partial x}\bar{\varphi}(\infty,s) = \frac{\partial^2\bar{\varphi}(\infty,s)}{\partial x^2} = 0 \tag{5.2.18}$$

After eliminating $\bar{\theta}$ from Eqs. (5.2.13-5.2.14), we obtain the following equation:

$$\left\{\frac{\partial^4}{\partial x^4} - \left[(1+\varepsilon)s + (1+\tau_2+\varepsilon\tau_1)s^2\right]\frac{\partial^2}{\partial x^2} + s^3(1+\tau_2 s)\right\}\bar{\varphi} = 0 \tag{5.2.19}$$

where, $\varepsilon = C_1 C$

It is clear that the Eq. (5.2.19) is a fourth order differential equation with constant coefficient. Therefore, characteristic equation of Eq. (5.2.19) can be written as

$$m^4 - \left[(1+\varepsilon)s + (1+\tau_2+\varepsilon\tau_1)s^2\right]m^2 + s^3(1+\tau_2 s) = 0$$

We find that above equation have four complex roots in which two of them are having positive real parts and other two have negative real parts. Therefore, for the stability of the solution, we will consider only the roots which have positive real parts. Hence, we can express the solution of the above equation (5.2.19) satisfying the boundary conditions given in Eqs. $(5.2.16 - 5.2.18)$ in the following form:

$$\bar{\varphi}(x,\, s) = B_1(s)\, e^{-m_1 x} + B_2(s) e^{-m_2 x} \tag{5.2.20}$$

where, $B_1(s)$ and $B_2(s)$ are the parameters which depend on s only where as m_1 and m_2 are the roots with positive real parts of the characteristic equation.

Further, we obtain the expressions for $B_1(s)$ and $B_2(s)$ as

$$B_1(s) = -\frac{(m_2^2 - s^2)\bar{F}(s)}{\beta^2 s^2 (m_1^2 - m_2^2)}, \;\; B_2(s) = \frac{(m_1^2 - s^2)\bar{F}(s)}{\beta^2 s^2 (m_1^2 - m_2^2)}$$

Now, using above expressions for $B_1(s)$ and $B_2(s)$ into the Eq. (5.2.20), we get

$$\bar{\varphi}(x,\, s) = -\frac{\bar{F}(s)}{\beta^2 s^2 (m_1^2 - m_2^2)}\left[(m_2^2 - s^2)e^{-m_1 x} - (m_1^2 - s^2)e^{-m_2 x}\right] \tag{5.2.21}$$

The function, $\bar{\varphi}(.)$ given by Eq. (5.2.21) represents the responses of mechanical shock as we have assumed in Eq. (5.2.7). Here, $\bar{F}(s)$ is the Laplace transform of $F(t)$, which is the mechanical load applied at the boundary.

Using Eqs. (5.2.13) and (5.2.21), we have

$$\bar{\theta}(x, s) = \frac{\varepsilon s \bar{F}(s)}{B(m_1^2 - m_2^2)} \left[e^{-m_1 x} - e^{-m_2 x}\right] \tag{5.2.22}$$

Further, using Eqs. (5.2.15) and (5.2.22), we find

$$\bar{\sigma}(x, s) = -\frac{\bar{F}(s)}{(m_1^2 - m_2^2)} \left[(m_2^2 - s^2)e^{-m_1 x} - (m_1^2 - s^2)e^{-m_2 x}\right] \tag{5.2.23}$$

These are the solutions of the physical field variables in the Laplace transform domain.

5.2.4 Temperature distributions in physical domain

5.2.4.1 Deterministic temperature distribution

Now, we assume the deterministic type mechanical load applied at the boundary of half space in the following form:

$$F(t) = \sigma_0 h_l(t) \tag{5.2.24}$$

where,

$$h_l(t) = \begin{cases} 1, & \text{if } 0 \leq t \leq l \\ 0 & \text{otherwise} \end{cases}$$

and σ_0 is a constant mechanical load, whereas the function $h_l(t)$ is a pulse function which is used to define the constant deterministic mechanical load applied at the boundary. We assume that at the boundary of the medium a constant load σ_0 is maintained for certain period of non dimensional time, $0 \leq t \leq l$ and then after time, $t > l$, the load is zero.

Now, we take the Laplace transform of Eq. (5.2.24) and putting it into Eq. (5.2.22), and then we take the Laplace inversion of the resulting equation in a similar way as given in Sherief (1994).

Then, we obtain the temperature distribution in the physical domain (x, t) as follows:

$$\bar{\theta}(x,t) = \frac{\sigma_0 \varepsilon}{B}[e^{-b_{11}x} \sum_{j=0}^{3} \frac{b_j}{(2b_{12})^{j+1}} \left\{T_1^{j+1} J_{j+1}(T_1) - M_1^{j+1} J_{j+1}(M_1)\right\} H(t - b_{10}x)$$
$$-e^{-b_{21}x} \sum_{j=0}^{3} \frac{b_j}{(2b_{22})^{j+1}} \left\{T_2^{j+1} I_{j+1}(T_2) - M_2^{j+1} I_{j+1}(M_2)\right\} H(t - b_{20}x)] \qquad (5.2.25)$$

where, b_{10}, b_{11}, b_{12}, b_{21}, b_{20}, b_{22}, b_0, b_1, b_2, b_3, T_1, T_2, M_1, and M_2 are the constants which have been given in the Appendix-A4. J_j and I_j, $j = 0, 1, 2, 3$ are the j^{th} order Bessel and modified Bessel functions of first kind, whereas $H(.)$ is the Heaviside unit-step function. This solution is clearly matched with corresponding temperature distribution which is given in Sherief (1994) with some changes in notations.

5.2.4.2 Stochastic temperature distribution

In this section, we are going to evaluate the stochastic temperature distribution. By considering the presence of white noise in the load distribution in the boundary, we assume the mechanical distribution at the boundary of the plane in the following form:

$$F(t) = f_1(t) + \phi_0(t) \qquad (5.2.26)$$

where, $f_1(t) = \sigma_0 h_l(t)$ and $\phi_0(t)$ is a stochastic process which is described by the parameter, t, has the property given below:

$$E[\phi_0(t)] = 0 \qquad (5.2.27)$$

As described in refs. (Sherief *et al.* (2013, 2016) and Nowinski (2000)), a stochastic process $r(t)$ satisfies the following property:

$$E\left[L\{r(t)\}\right] = L\left[E\{r(t)\}\right] \tag{5.2.28}$$

Also, the auto-correlation function of a stochastic process $r(t)$ can be written as $R_{rr}(t_{1,}t_2) = E\left[r(t_1)\, r(t_2)\right]$, where t_1 and t_2 are two successive time instants. Then, the Laplace transform of $R_{rr}(t_{1,}t_2)$ can be written as

$$\hat{R}_{rr}(s_1,\, s_2) = E\left[\bar{r}(s_1)\bar{r}(s_2)\right] \tag{5.2.29}$$

which can be re-written as

$$\hat{R}_{rr}(s_{1,}s_2) = L_2\left\{R_{rr}(t_{1,}t_2)\right\} \tag{5.2.30}$$

In above equation, $L_2(.)$ represents the double Laplace transform which can be defined in the following form:

$$L_2\left\{f(t_1,\, t_2)\right\} = \int\limits_0^\infty e^{-s_1 t_1}\left(\int\limits_0^\infty e^{-s_2 t_2} f(t_1,\, t_2)\, dt_2\right) dt_1$$

After introducing the stochastic mechanical distribution, $\phi_0(t)$ at the boundary of the plane, all the physical field variables behave like a stochastic process in which the main randomness is present due to the stochastic process $\phi_0(t)$.

Therefore, from Eq. (5.2.22), we have

$$E\left[\bar{\theta}(x,\, s)\right] = \frac{\varepsilon s \bar{f}_1(s)}{B(m_1^2 - m_2^2)}\left[e^{-m_1 x} - e^{-m_2 x}\right] \tag{5.2.31}$$

We have considered here the stochastic process which is the probabilistically stationary. It represents a form of probabilistic equilibrium in the sense that the particular instant is not important at which it is observed.

We would also like to define that a stochastic process, $r(t)$ to be stationary if for all n and for every set of time instants ($t_i \in T,\ i = 1, 2, 3, ..., n$), its joint probability density function $G_r(.)$ does not change with a shift of the time parameter scale which can be expressed as

$$G_r(r_1, r_2, ..., r_{n;} t_1, t_2, ..., t_n) = G_r(r_1, r_2, ..., r_{n;} t_1 + \varsigma, t_2 + \varsigma, ..., t_n + \varsigma)$$

It implies that the auto-correlation function, $R_{rr}(\varsigma)$ of a stationary process, $r(t)$ is a function of the time interval ς alone which can be defined as

$$E\left[r(t_1)r(t_2)\right] = R_{rr}(\varsigma)$$

where, $\varsigma = t_1 - t_2$

The measure for dispersion of the temperature field can be defined by using the Eqs. (5.2.22) and (5.2.29) in the following way:

$$\begin{aligned}
\hat{R}_{\theta\theta}(x, s_1, s_2) &= E\left[\bar{\theta}(x, s_1)\bar{\theta}(x, s_2)\right] \\
&= E\left[\frac{\varepsilon\, s_1 \bar{f}_1(s_1)}{B(m_1^2(s_1) - m_2^2(s_1))}\left[e^{-m_1(s_1)x} - e^{-m_2(s_1)x}\right]\frac{\varepsilon\, s_2 \bar{f}_1(s_2)}{B(m_1^2(s_2) - m_2^2(s_2))}\left[e^{-m_1(s_2)x} - e^{-m_2(s_2)x}\right]\right] \\
&= \hat{R}_{\phi_0\phi_0}(s_1, s_2)\bar{\Gamma}(x, s_1)\bar{\Gamma}(x, s_2) + \bar{f}_1(s_1)\bar{\Gamma}(x, s_1)\,\bar{f}_1(s_2)\bar{\Gamma}(x, s_2)
\end{aligned}$$

where,

$$\bar{\Gamma}(x, s_i) = \frac{\varepsilon\ s_i}{B(m_1^2(s_i) - m_2^2(s_i))}\left[e^{-m_1(s_i)x} - e^{-m_2(s_i)x}\right], \qquad i = 1, 2 \tag{5.2.32}$$

and

$\hat{R}_{\phi_0\phi_0}(s_1, s_2) = E\left[\bar{\phi}_0(s_1)\bar{\phi}_0(s_2)\right]$

From Eq.(5.2.31), we have

$$\hat{R}_{\theta\theta}(x, s_1, s_2) = \hat{R}_{\phi_0\phi_0}(s_1, s_2)\bar{\Gamma}(x, s_1)\bar{\Gamma}(x, s_2) + E\left[\bar{\theta}(x, s_1)\right]E\left[\bar{\theta}(x, s_2)\right] \tag{5.2.33}$$

Now, we see that the stochastic process, $\phi_0(t)$ shows an extreme case in which there is no regularity found. From Refs. (Sherief *et al.* (2013, 2016) and Nowinski (2000)), we have

$$R_{\phi_0\phi_0}(t_1,t_2)=\delta(\varsigma) \tag{5.2.34}$$

By using Eq. (5.2.30), the double Laplace transform of the Eq. (5.2.34) can be written as given below

$$\hat{R}_{\phi_0\phi_0}(s_1,s_2)=\frac{1}{s_1+s_2} \tag{5.2.35}$$

From Oberhettinger and Badii (1973), we have used the following properties of double Laplace transform:

- $L_2\{G_1(x)\,G_2(y)\}=L\{G_1(x)\}\,L\{G_2(y)\}$
- $\frac{1}{s_1+s_2}L_2\{G(x-z,y-z)\}=\begin{cases} L_2\left\{\int_0^x G(x-z,y-z)\,dz\right\}, & y>x \\ L_2\left\{\int_0^y G(x-z,y-z)\,dz\right\}, & y<x \end{cases}$

Using Eqs. (5.2.28) and (5.2.34), the Eq. (5.2.33) can be written in the following form:

$$\hat{R}_{\theta\theta}(x,s_1,s_2)=\frac{1}{s_1+s_2}L_2\{\Gamma(x,t_1)\,\Gamma(x,t_2)\}+E\left[\bar{\theta}(x,s_1)\,\bar{\theta}(x,s_2)\right]$$

$$=L_2\left\{\int_0^{t_1}\Gamma(x,t_1-z)\Gamma(x,t_2-z)\,dz\right\}+L_2\{E[\theta(x,t_1)]\,E[\theta(x,t_2)]\},\qquad t_2>t_1$$

After taking the inversion of the double Laplace transform, the above equation can be written as

$$R_{\theta\theta}(x,t_1,t_2)=\int_0^{t_1}\Gamma(x,t_1-z)\Gamma(x,t_2-z)\,dz+E[\theta(x,t_1)]\,E[\theta(x,t_2)]$$

Now, we take $t_1=t_2=t$ in the above equation. Then, we have

$$R_{\theta\theta}(x,t)=\int_0^{t}\Gamma^2(x,t-z)\,dz+(E[\theta(x,t)])^2$$

Again if we assume, $t-z=\xi$ to solve the integration in the above equation, then we obtain

$$R_{\theta\theta}(x,t)=\int_0^t \Gamma^2(x,\xi)\,d\xi+(E[\theta(x,t)])^2$$

We can also re-write the above equation in the following form:

$$E[\theta^2(x,t)]-(E[\theta(x,t)])^2=\int_0^t \Gamma^2(x,\xi)\,d\xi$$

which further can be re-written in the following form:

$$Var[\theta(x,t)]=E\left[\theta^2(x,t)\right]-(E[\theta(x,t)])^2=\int_0^t \Gamma^2(x,\xi)\,d\xi$$

From Eq. (5.2.32), we obtain

$$\bar{\Gamma}(x,s)=\frac{\varepsilon s}{B(m_1^2(s)-m_2^2(s))}\left[e^{-m_1(s)x}-e^{-m_2(s)x}\right] \tag{5.2.36}$$

Now, we apply the asymptotic expansion to invert the Laplace transform in the above equation in a similar way as given in Sherief (1994). Therefore, we find

$$\Gamma(x,t)=\frac{\varepsilon}{B}[e^{-b_{11}x}\left\{\sum_{j=0}^{3}b_j\left(\frac{t-b_{10}x}{b_{12}x}\right)^{\frac{j}{2}}J_j\left(2\sqrt{b_{12}x(t-b_{10}x)}\right)\right\}H(t-b_{10}x)$$

$$-e^{-b_{21}x}\left\{\sum_{j=0}^{3}b_j\left(\frac{t-b_{20}x}{-b_{22}x}\right)^{\frac{j}{2}}I_j\left(2\sqrt{-b_{22}x(t-b_{20}x)}\right)\right\}H(t-b_{20}x)] \tag{5.2.37}$$

By using Eqs. $(5.2.26-5.2.27)$ and (5.2.22), we have

$$L\{E[\theta(x,t)]\}=\frac{\varepsilon s\bar{f}_1(s)}{B(m_1^2(s)-m_2^2(s))}\left[e^{-m_1(s)x}-e^{-m_2(s)x}\right]$$

We have seen that the deterministic temperature distribution given by Eq. (5.2.25) is similar to the mean of all the sample paths of the temperature field, $E[\theta(x,t)]$.

Also in the view of Eqs. (5.2.22) and (5.2.26), we can write the stochastic temperature distribution in Laplace transform domain in the the following form:

$$\bar{\theta}(x, s) = \bar{F}(s)\bar{\Gamma}(x, s)$$

$$= \left[\bar{f}_1(s) + \bar{\phi}_0(s)\right]\bar{\Gamma}(x, s)$$

Now, by inverting the Laplace transform in the above equation using Eqs. (5.2.22) and (5.2.25) with the help of convolution property, we obtain the stochastic temperature distribution in the physical domain as

$$\theta(x, t) = \theta_1(x, t) + \int_0^t \phi_0(u)\,\mathit{\Gamma}(x, t-u)\,du$$

By using the concept of Wiener process, $W(u)$, we can write the above equation in the following form (see Sherief *et al.* (2016)):

$$\theta(x, t) = \theta_1(x, t) + \int_0^t \mathit{\Gamma}(x, t-u)\,dW(u)$$

where, $\theta_1(x, t)$ is the deterministic temperature distribution given by Eq. (5.2.25) and $\Gamma(x, t)$ is given by Eq. (5.2.37). In the above equation, second term is a stochastic integral which can be computed numerically by using the Wiener process.

5.2.5 Stress distributions in physical domain

5.2.5.1 Deterministic stress distribution

Now, by using Eqs. (5.2.7) and (5.2.23) along with the short time approximation to invert the Laplace transform in a similar manner which have been done in the previous section, we obtain the deterministic stress distribution in the physical domain in the following form:

$$\sigma(x,t) = -\sigma_0 e^{-b_{11}x}\left\{c_{20}\left[h_l(t-b_{10}x)+J_0(T_1)-J_0(M_1)\right]+\sum_{j=0}^{3}\frac{c_{2(j+1)}}{(2b_{12}x)^{j+1}}\left[T_1^{j+1}J_{j+1}(T_1)-M_1^{j+1}J_{j+1}(M_1)\right]\right\}H(t-b_{10}x)$$
$$+\sigma_0 e^{-b_{21}x}\left\{c_{10}\left[h_l(t-b_{20}x)+I_0(T_2)-I_0(M_2)\right]+\sum_{j=0}^{3}\frac{c_{1(j+1)}}{(2b_{22}x)^{j+1}}\left[T_2^{j+1}I_{j+1}(T_2)-M_2^{j+1}I_{j+1}(M_2)\right]\right\}H(t-b_{20}x) \quad (5.2.38)$$

where, b_{10}, b_{11}, b_{12}, b_{20}, b_{21}, b_{22}, c_{10}, c_{11}, c_{12}, c_{13}, c_{14}, c_{20}, c_{21}, c_{22}, c_{23}, c_{24} are the constants which are given in the Appendix-A4. J_j and I_j, $j = 0, 1, 2, 3$ are the j^{th} order Bessel and modified Bessel functions of first kind whereas $H(.)$ is the Heaviside unit step function. The solution given in Eq. (5.2.38) is also matched with the corresponding solution given in Sherief (1994).

5.2.5.2 Stochastic stress distribution

As we assume the boundary condition is of white noise type, we find the variance of the stochastic stress distribution in a similar manner as we have done in the previous section, in the following form:

$$Var(\sigma(x,t) = \int_0^t \Xi^2(x,t)\, dt$$

where, $E(x,t)$ has been obtained in a similar way as we obtain $\Gamma(x,t)$ in the previous section. Therefore, $E(x,t)$, can be written in the following form:

$$\Xi(x,t) = -e^{-b_{11}x}\left\{c_{20}\left[\delta(t-b_{10}x)-\sqrt{\frac{b_{12}x}{t-b_{10}x}}J_1\left(2\sqrt{b_{12}x(t-b_{10}x)}\right)\right]+\sum_{j=0}^{3}c_{2(j+1)}\left(\frac{t-b_{10}x}{b_{12}x}\right)^{\frac{j}{2}}J_j\left(2\sqrt{b_{12}x(t-b_{10}x)}\right)\right\}H(t-b_{10}x)$$
$$+e^{-b_{21}x}\left\{c_{10}\left[\delta(t-b_{20}x)+\sqrt{\frac{-b_{22}x}{t-b_{20}x}}J_1\left(2\sqrt{-b_{22}x(t-b_{20}x)}\right)\right]+\sum_{j=0}^{3}c_{1(j+1)}\left(\frac{t-b_{20}x}{-b_{22}x}\right)^{\frac{j}{2}}J_j\left(2\sqrt{-b_{22}x(t-b_{20}x)}\right)\right\}H(t-b_{20}x) \quad (5.2.39)$$

We observe that $E[\sigma(x,t)]$ the mean of all the sample paths of stress distribution due to randomness at the boundary of an elastic medium is same as the solution given by Eq. (5.2.38).

Therefore, we can re-write the Eq. (5.2.23) using Eq. (5.2.26) in the following form:

$$\bar{\sigma}(x,s) = \bar{F}(s)\,\bar{\Xi}(x,s),$$

which can also be written in the following form:

$$\bar{\sigma}(x,s) = \left[\bar{f}_1(s) + \bar{\phi}_0(s)\right] \bar{\Xi}(x,s)$$

Now, inverting the Laplace transform of the above equation using Eqs. (5.2.23) and (5.2.38) with the help of convolution property, we obtain the stochastic stress distribution in the physical domain given as

$$\sigma(x,t) = \sigma_1(x,t) + \int_0^t \phi_0(u)\, \Xi(x,t-u)\, du$$

By using the concept of Wiener process, $W(u)$, we can write the above equation in the following form (see Sherief *et al.* (2016)):

$$\sigma(x,t) = \sigma_1(x,t) + \int_0^t \Xi(x,t-u)\, dW(u)$$

where, $\sigma_1(x,t)$, is the deterministic stress distribution given by Eq. (5.2.38) and $\Xi(x,t)$ is given in Eq. (5.2.39). In the above equation, the second term is a stochastic integral which has been solved using Wiener process.

5.2.6 Numerical results

In this section, the numerical simulation is carried out in order to illustrate the theoretical results obtained in previous section. We carry out stochastic simulation and show the variations of stochastic field distributions inside the medium and compare the results with the deterministic cases. Stochastic simulation is done by using Wiener process. We have considered the aluminum material for the purpose of numerical simulation for which the constants are given as follows:
$\varepsilon = 0.0286$, $B = 0.0549$, $\tau_1 = 0.02$, $\tau_2 = 0.02$, $\sigma_0 = 1$, $l = 0.1$, $t = 0.15,\ 0.10,\ 0.05$

The value of the mechanical shock parameter, σ_0, is given above and the computation is carried out for three non-dimensional times namely, $t = 0.05$, 0.10, and 0.15 to represent the time evaluation of the temperature and stress distributions. From Hingham (2001), the concept of Brownian motion or standard Wiener processes is used to compute the stochastic integrals involved in the solution of physical field variables. Two sample paths of stochastic process are considered to compare the results with deterministic cases of all the physical field variables. The numerical results are displayed in Figs. $(5.2.1-5.2.24)$ for different physical fields at three different times. Figures $(5.2.1-5.2.8)$ show the nature of variation of different physical field variables at time $t=0.15$ in which the temperature and stress distributions are plotted in Figs. $(5.2.1-5.2.2)$ and Figs. $(5.2.3-5.2.4)$, respectively. Fig. $(5.2.1)$ depicts the deterministic temperature distribution. The comparison of deterministic temperature distribution with stochastic temperature distribution is displayed in Fig. $(5.2.2)$ in which we note clearly that the mean of all the sample paths of the stochastic temperature distribution coincides with the deterministic temperature distribution. Similarly in Fig. $(5.2.3)$, the deterministic stress distribution is shown and its comparison with stochastic stress distribution is depicted in Fig. $(5.2.4)$ in which the deterministic stress distribution coincides with the mean of all the sample paths of the stochastic stress distribution. However, the variance of temperature and stress distributions are shown in Fig. $(5.2.5)$ and Fig. $(5.2.6)$, respectively from which we observe that it increases near the boundary of the plane and after some distance from the boundary it starts to decrease and finally becomes zero for both the physical field variables. From Figs. $(5.2.3-5.2.4)$, we find that the stress field is compressive near the boundary of the half space, although the values of stress below x-axis is very very small compared to its value above the axis after some distance from the boundary. We have drawn two more Figs. $(5.2.7-5.2.8)$ to show the compressive nature of stress field and to show that stress after some distance from the boundary becomes zero. The distinction of stochastic distribution and deterministic distribution is significant in this reason. Similarly, the nature of different physical field variables at two different times $t=0.10$ and $t=0.05$ are shown by the Figs. $(5.2.9-16)$ and Figs. $(5.2.17-5.2.24)$, respectively. From Figs. $(5.2.1-5.2.2)$, $(5.2.9-5.2.10)$ and $(5.2.17-5.2.19)$, it is clear that the temperature distribution is continuous at all three times $t=0.15$, 0.10 and 0.05. However, from Figs. $(5.2.2-5.2.4)$, $(5.2.11-5.2.12)$ and $(5.2.19-5.2.20)$, it is clear that stress distribution has finite jump discontinuity at two different points at two times $t = 0.10$ and 0.05, while for time $t = 0.15$, the finite jump discontinuity at four different points is appeared. Hence as the time increases, the discontinuity in the stress is shifted to outward from the boundary of the plane. From numerical results, it is clear that all the deterministic field variables are matched with the corresponding results reported by Sherief (1994) and this validates the results of present work. We have also observed that all the physical field variables become zero after some distance from the boundary of the plane in both the cases: deterministic and stochastic physical field variables, which justify that our results are matched with the assumptions of the problem.

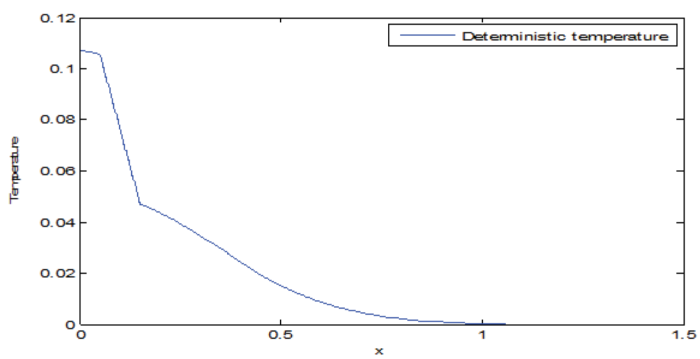

Fig. 5.2.1 Deterministic temperature distribution at time *t=0.15*

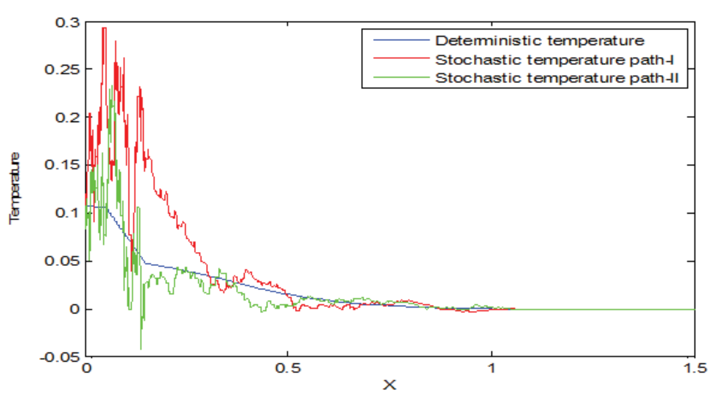

Fig. 5.2.2 Deterministic and stochastic temperature distributions at time *t=0.15*

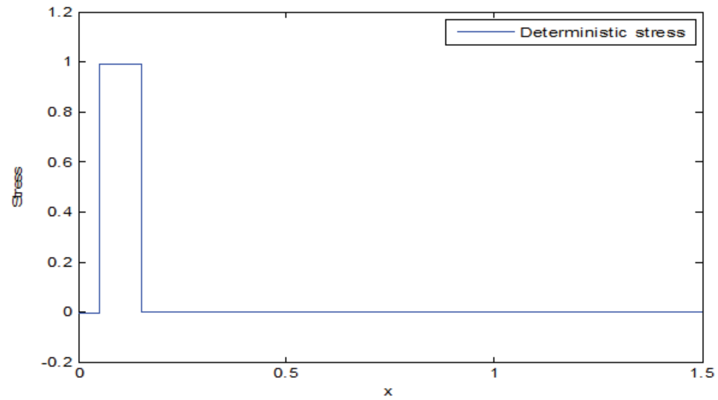

Fig. 5.2.3 Deterministic stress distribution at time *t=0.15*

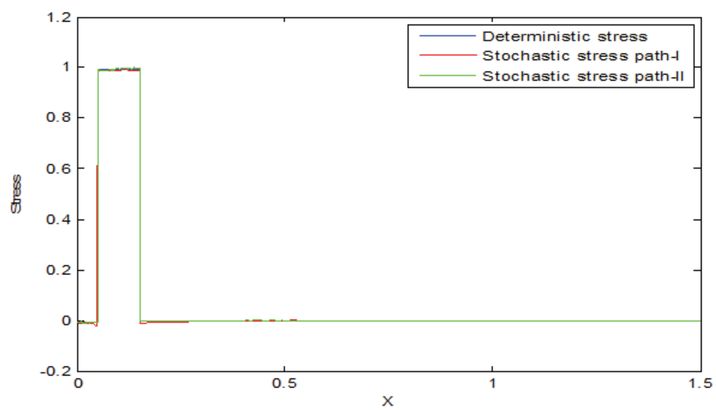

Fig. 5.2.4 Deterministic and stochastic stress distributions at time *t=0.15*

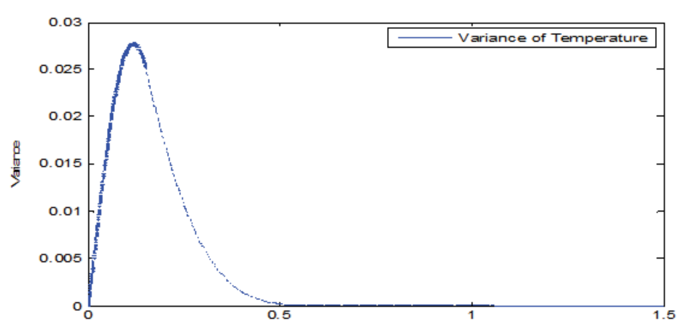

Fig. 5.2.5 Variance of temperature distribution at time *t=0.15*

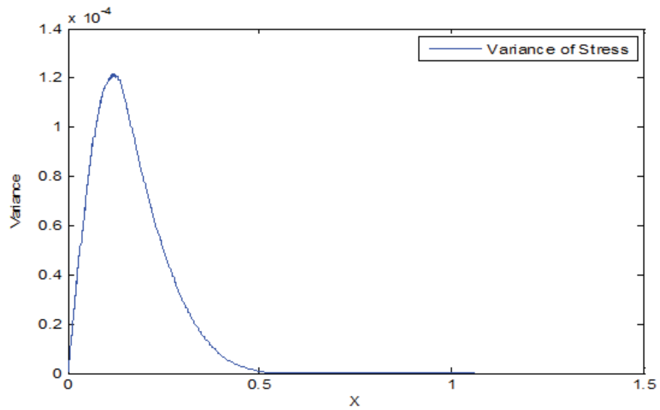

Fig. 5.2.6 Variance of stress distribution at time *t=0.15*

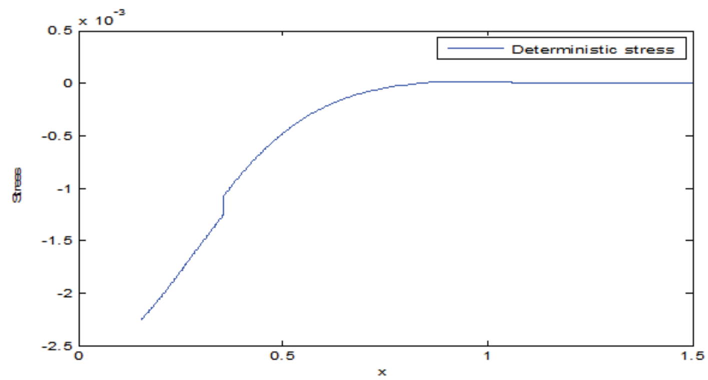

Fig. 5.2.7 Deterministic stress distribution at time *t=0.15*

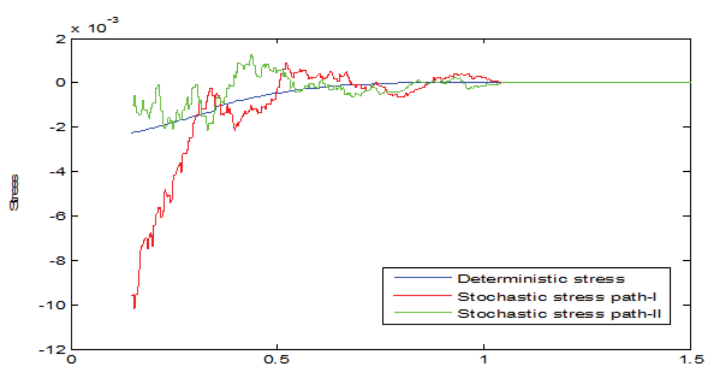

Fig. 5.2.8 Deterministic and stochastic stress distributions at time *t=0.15*

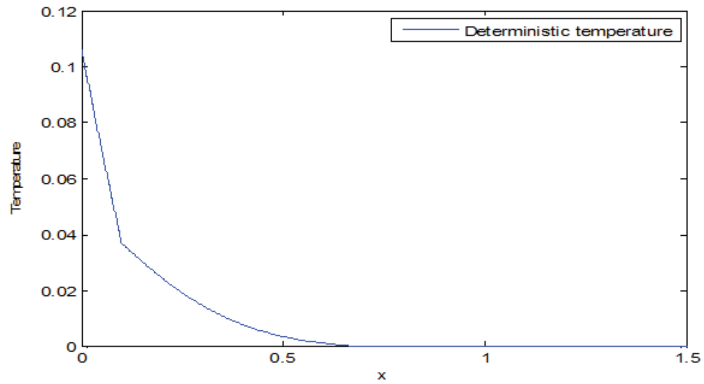

Fig. 5.2.9 Deterministic temperature distribution at time *t=0.10*

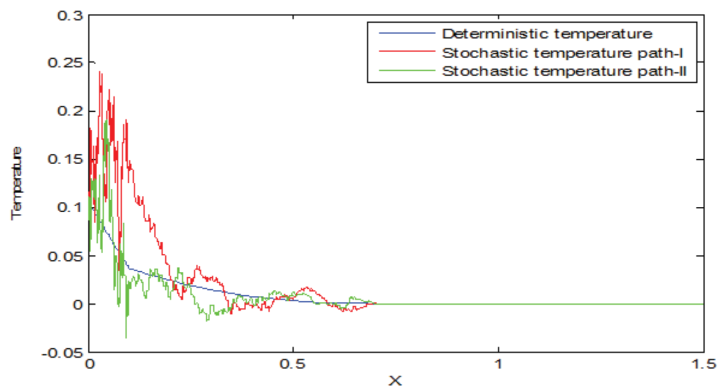

Fig. 5.2.10 Deterministic and stochastic temperature distributions at time *t=0.10*

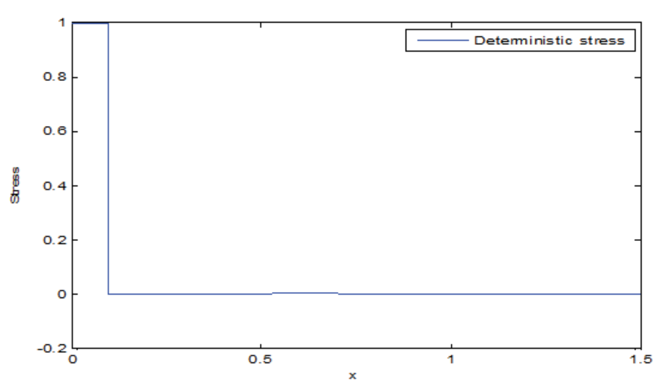

Fig. 5.2.11 Deterministic stress distribution at time *t=0.10*

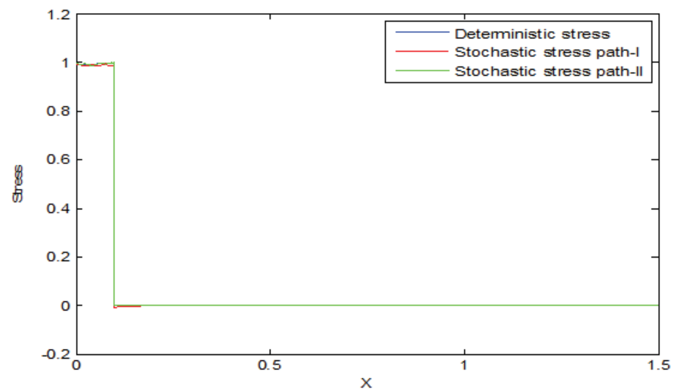

Fig. 5.2.12 Deterministic and stochastic stress distributions at time *t=0.10*

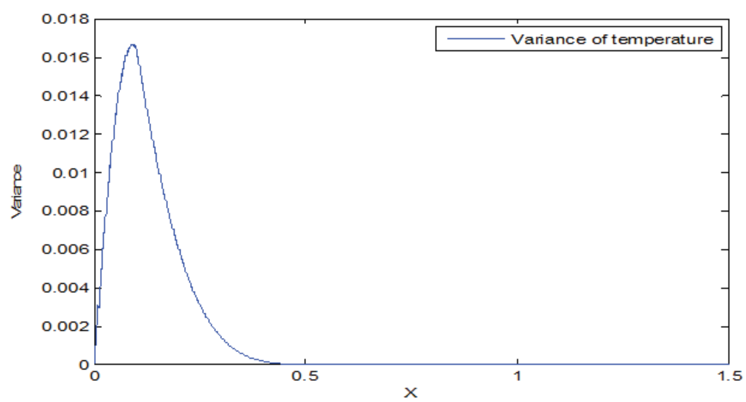

Fig. 5.2.13 Variance of temperature distribution at time *t=0.10*

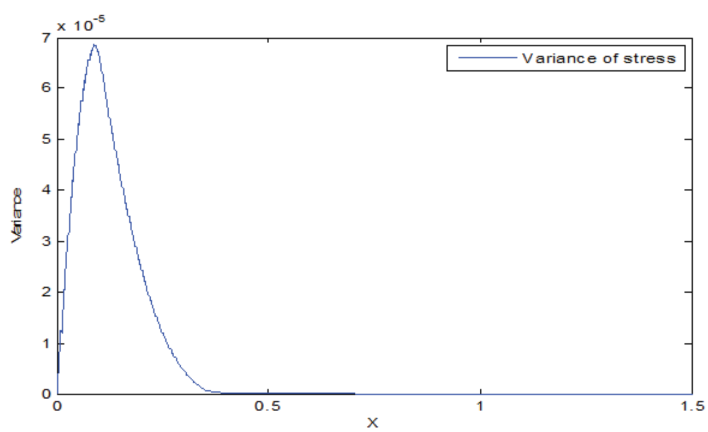

Fig. 5.2.14 Variance of stress distribution at time *t=0.10*

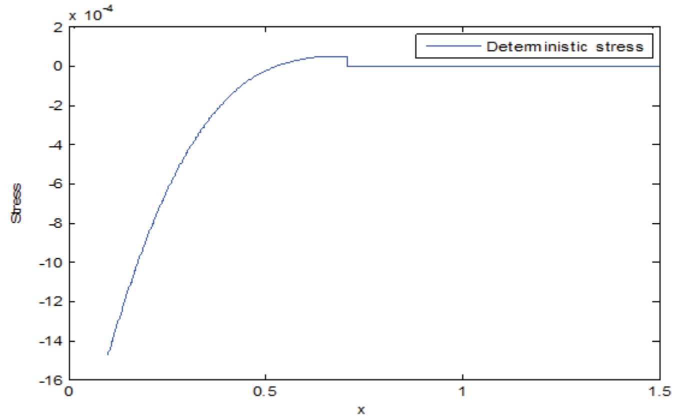

Fig. 5.2.15 Deterministic stress distribution at time *t=0.10*

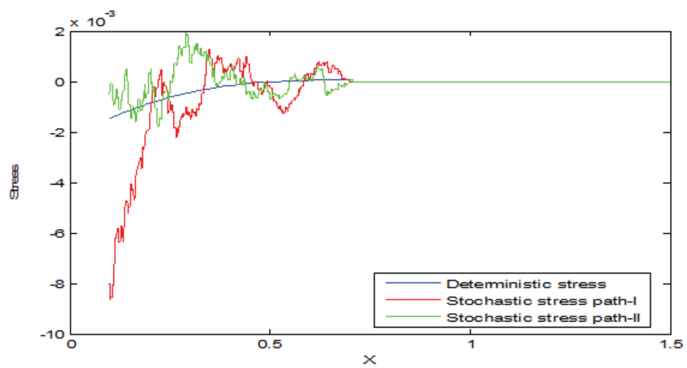

Fig. 5.2.16 Deterministic and stochastic stress distributions at time *t=0.10*

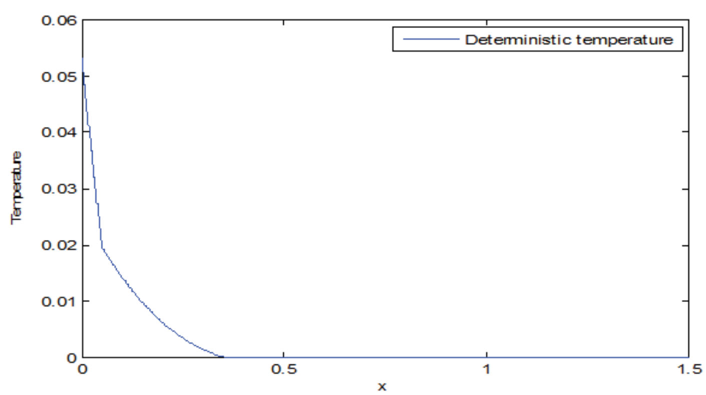

Fig. 5.2.17 Deterministic temperature distribution at time *t=0.05*

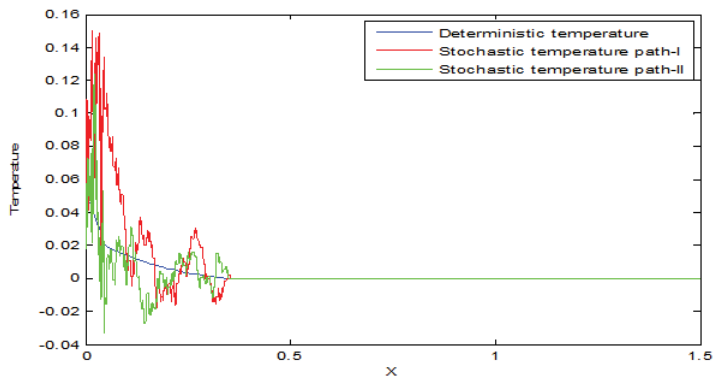

Fig. 5.2.18 Deterministic and stochastic temperature distributions at time *t=0.05*

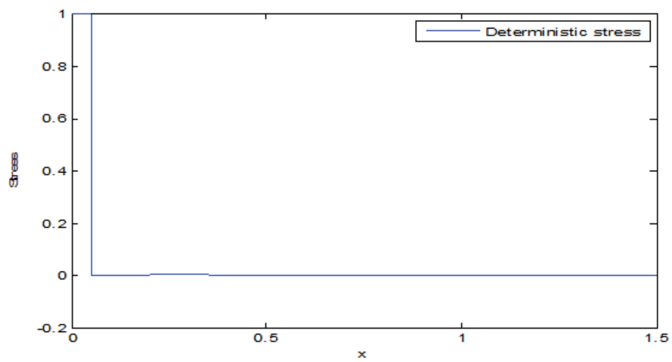

Fig. 5.2.19 Deterministic stress distribution at time *t=0.05*

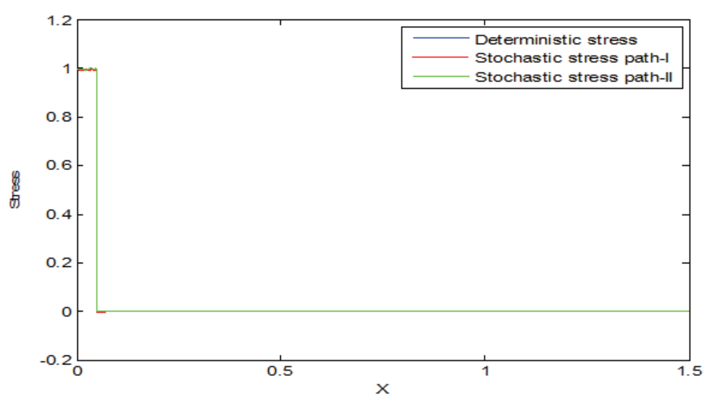

Fig. 5.2.20 Deterministic and stochastic stress distributions at time *t=0.05*

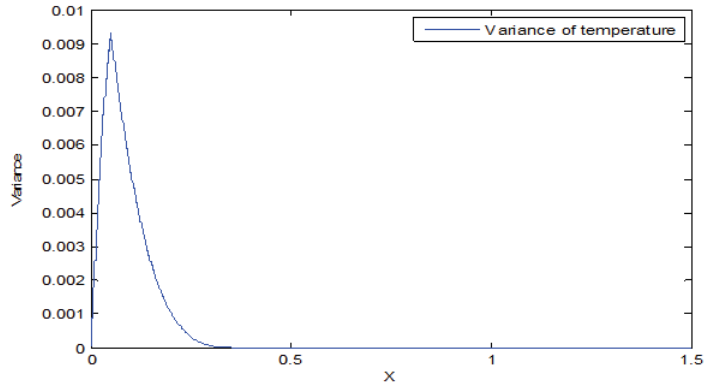

Fig. 5.2.21 Variance of temperature distribution at time *t=0.05*

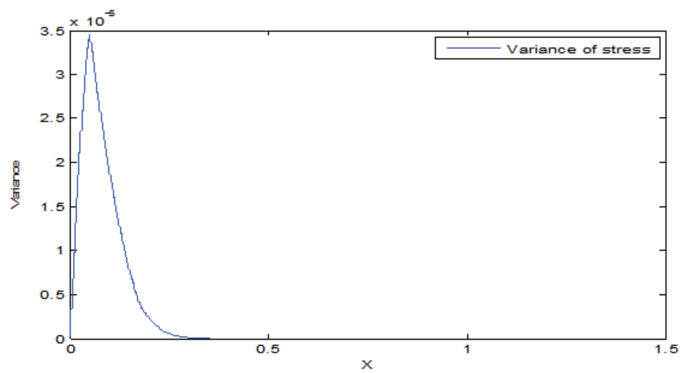

Fig. 5.2.22 Variance of stress distribution at time *t=0.05*

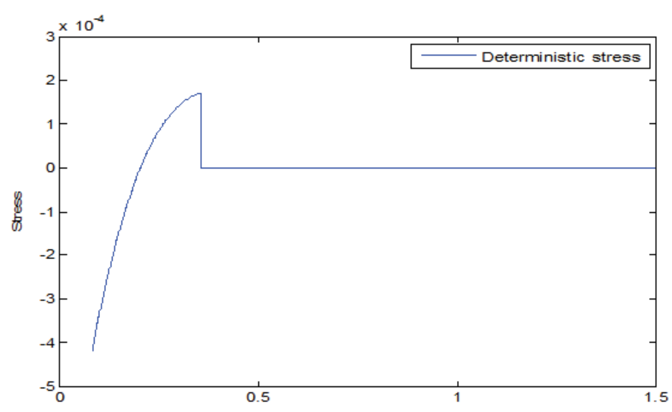

Fig. 5.2.23 Deterministic stress distribution at time *t=0.05*

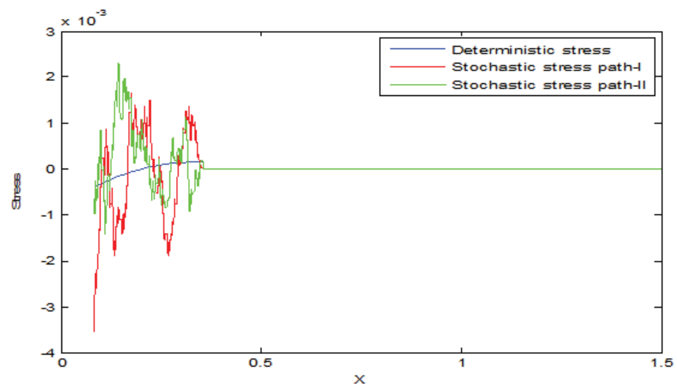

Fig. 5.2.24 Deterministic and stochastic stress distributions at time *t=0.05*

5.2.7 Discussion and conclusions

Present work is based on the generalized thermoelasticity with two relaxation time parameters for two types of mechanical boundary conditions: deterministic and stochastic types. The white noise stochastic process has been considered for the stochastic type boundary conditions. The mechanical load is prescribed at the boundary of the medium to study the nature of different physical fields with its stochastic counterparts. In order to take into account the presence of noise or error in the system, it is suggested that the deterministic model can be replaced with the stochastic one and the deterministic models that represent idealized situations are often improved by including stochastic effects. Here, we have added the effect of presence of some noise in the boundary load of elastic medium and represented it as a stochastic load. We have compared our results with the corresponding results of deterministic case. From numerical results, we observe that all the physical field variables for the deterministic case are matched with the corresponding physical fields reported by Sherief (1994). We also note that the mean of all the sample paths of the stochastic physical fields coincides with its deterministic part for each field variable at all the three times $t = 0.15, 0.10$ and 0.05. It is further observed that both the numerical and analytical results indicate that the physical fields vanish after some distance from the boundary in the stochastic as well as deterministic type distributions. We observe a significant change in the variations of deterministic and stochastic fields at all time. The fluctuation in stochastic distributions of the fields increases with the increase of time. However, the reasons of influence is almost the same for deterministic and stochastic fields at all time. The temperature distribution is observed to be continuous throughout the physical domain. However, the stress field has the finite jump discontinuity in which at lower times at $t = 0.10$ and 0.05, it is occurred at two different points of the domain while at higher time, i.e. at time $t = 0.15$, the finite jump discontinuity is appeared at four different points of the domain. It is also indicated that the deviation of all the stochastic distributions from its mean decreases with the distance from the bounding plane which is the source of the noise. The fluctuation on the solution due to the noise on the boundary travels inside the medium with a finite speed as it is also seen for the deterministic waves which move inside the medium with a finite speed.