# CHAPTER-4

# An investigation on responses of thermo-mechanical load inside an infinitely extended thick plate under thermoelasticity with memory dependent derivatives

## 4.1 Introduction

In last decades, Various applications in the area of physics and engineering are described by using fractional calculus. Recently, many researchers have tried to modify the classical Fourier law of heat conduction equation using the fractional calculus (see Povstenko (2008*a*, 2008*b*, 2009*a*, 2009*b*, 2010, 2011*a*, 2011*b*), Di Paola *et al.* (2009), Ezzat (2010, 2011), Ezzat and El-Karamany (2011*a,b*), Ezzat *et al.* (2012, 2014), Ezzat and El-Bary (2014), and Islam and Kanoria (2014)). Diethelm (2010) has developed the Caputo (1967) type fractional derivative. Recently, Ezzat *et al. (2015)* have proposed a new model of magneto thermoelasticity theory in the context of a new consideration of heat conduction with memory dependent derivative and compared it with the dynamical classical coupled thermoelasticity (see Biot (1956)). Subsequently, some researchers (see Ezzat *et al.* (2016*a,b*), Ezzat and El-bary (2015)) have considered some problems on the thermoelasticity theory using " memory dependent derivatives" and discussed the effects of memory dependent derivatives as compared to ordinary time derivatives. Shaw (2017) has given a note on the generalized thermoelasticity theory with memory dependent derivatives in which the discontinuity solutions of generalized thermoelasticity are discussed and he has also proposed a suitable Lyapunov function which is used as an important tool to show several qualitative properties. Tiwari and Mukhopadhyay (2017) have also used the new concept of a memory dependent derivative in a

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heat transfer process in a solid to investigate the problem of wave propagation in a homogeneous, isotropic and unbounded solid due to a continuous line heat source and they have attempted to exhibit the significance of a kernel function and a time-delay parameter that are characteristic of a memory dependent derivative heat transfer in the behavior of field variables.

In this Chapter, the work is carried out under generalized thermoelasticity with memory dependent derivatives. The main purpose of this work is to analyze the thermoelastic interactions inside an infinitely extended thick plate due to axi-symmetric temperature distribution applied at the lower and upper surface of the plate under the memory dependent generalized thermoelasticity. The formulation of the problem is done in the context of the theory of thermoelasticity under the memory dependent derivatives with inclusion of time delay parameter, $\tau$ and kernel functions that are defined in a slipping interval $[t-\tau, t)$. We aim to compare the results of this theory based on memory dependent derivative with the results of generalized thermoelasticity by Lord and Shulman (1967) as discussed in the Chapter-3. In section 4.2, the governing equations are formulated for the generalized thermoelasticity with one relaxation parameter and the heat conduction equation dependent on the memory dependent derivative is derived. We have considered four different kernel functions in which one kernel is non linear type. In section 4.5, the numerical results and discussion of the present work is explained in a detailed way and comparison of each physical field for different kernels and with different non dimensional delay time is discussed. In section 4.6, the conclusion of the present work is summarized to highlight the major findings.

## 4.2 Governing equations

In the case of generalized thermoelasticity with one relaxation time parameter in a homogeneous and isotropic elastic medium, the heat conduction equation in the absence of heat sources given by Lord and Shulman (1967) as follows:

**Heat conduction equation:**

$$K\nabla^2 T = (\frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2})(\rho\, c_e\, T + \gamma T_0\, e) \tag{4.1}$$

The governing equations involving the displacement, thermal and stress fields without any body force in an isotropic and homogeneous medium can be written as follows:

**Equation of motion:**

$$\sigma_{ij,j} = \rho \ddot{u}_i \tag{4.2}$$

**Strain-displacement relation:**

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{4.3}$$

**Stress-strain-temperature relation:**

$$\sigma_{ij} = \lambda\, e\, \delta_{ij} + 2\mu e_{ij} - \gamma(T - T_0)\delta_{ij} \tag{4.4}$$

In the above equations, $K$ is thermal conductivity and $\tau_0$ is the time relaxation parameter, $\delta_{ij}$ is the Kronecker delta function whereas, $T$ and $T_0$ are the reference and absolute temperatures. The $u_i$'s are the displacement components, while $\gamma = (3\lambda + 2\mu)\alpha_t$, where, $\alpha_t$ is the coefficient of linear thermal expansion parameter and $e$ is the dilatation which is expressed as

$$e = e_{ii} \tag{4.5}$$

**Derivation of the heat conduction equation with memory dependent derivative:**

By using the concept of generalized thermoelasticity, the relation between heat flux vector $\overrightarrow{q}(\overrightarrow{x}, t)$ and temperature gradient vector $\overrightarrow{\nabla} T$ can be written as

$$\overrightarrow{q}(\overrightarrow{x}, t + \tau) = -K\overrightarrow{\nabla}(T) \tag{4.6}$$

where, $\overrightarrow{x}$ is the position vector and $\tau$ is the time-delay parameter and $K$ is the thermal conductivity.

We know that the heat conduction equation due to Biot (1956) without any heat source is given by:

$$\frac{\partial}{\partial t}(\rho\, c_e T + \gamma T_0 e) = -K\overrightarrow{\nabla}.\overrightarrow{q}(\vec{x},t) \tag{4.7}$$

Ezzat *et al.* (2016*b*), Ezzat and El-bary (2015) have developed the new law of heat conduction with memory dependent derivative having the time-delay $\tau$ in the form as

$$\overrightarrow{q}(\overrightarrow{x},t) + \tau D_\tau \overrightarrow{q}(\overrightarrow{x},t) = -K\overrightarrow{\nabla}T \tag{4.8}$$

From Eq. (4.6) and Eq. (4.8), we have

$$\overrightarrow{q}(\overrightarrow{x},t+\tau) = \overrightarrow{q}(\overrightarrow{x},t) + \tau D_\tau \overrightarrow{q}(\overrightarrow{x},t) \tag{4.9}$$

Now, we apply the memory dependent derivative into Eq. (4.7) to get

$$\frac{\partial}{\partial t}D_\tau(\rho\, c_e T + \gamma T_0 e) = -K\,\overrightarrow{\nabla}.D_\tau\overrightarrow{q}(\vec{x},t) \tag{4.10}$$

Next, we multiply Eq. (4.10) with $\tau$ and add it to Eq. (4.7), to obtain

$$(1+\tau D_\tau)(\rho\, c_e\frac{\partial}{\partial t}T + \gamma T_0\frac{\partial}{\partial t}e) = -K\,\overrightarrow{\nabla}.(\overrightarrow{q}(\vec{x},t) + \tau D_\tau\overrightarrow{q}(\vec{x},t)) \tag{4.11}$$

From Eqs. (4.8) and (4.11), we have the following:

$$K\nabla^2 T = (1+\tau D_\tau)(\rho\, c_e\frac{\partial}{\partial t}T + \gamma T_0\frac{\partial}{\partial t}e) \tag{4.12}$$

where,

$$D_\tau g(t) = \frac{1}{\tau}\int_{t-\tau}^{t} K(t-s)\, g^{(1)}(s)\, ds$$

Therefore, Eq. (4.12) is the heat conduction equation with memory dependent derivative for the purpose of present study.

For the present problem, we assume the kernel function $K(t-s) = 1 - \frac{2a_{12}}{\tau}(t-s) + \frac{a_{11}^2}{\tau^2}(t-s)^2$
From above, we have four different kernel functions for different values of $a'_{ij}$s,where $i,\, j = 1, 2$ as given below

1. **Case-I:** $K(t-s) = 1;\ a_{11} = 0,\ a_{12} = 0$

2. **Case-II:** $K(t-s) = 1 - \frac{(t-s)}{\tau};\ a_{11} = 0,\ a_{12} = \frac{1}{2}$

3. **Case-III:** $K(t-s) = 1-(t-s);\ a_{11} = 0,\ a_{12} = \frac{\tau}{2}$

4. **Case-IV:** $K(t-s) = (1 - \frac{(t-s)}{\tau})^2;\ a_{11} = 1,\ a_{12} = 1$

## 4.3 Formulation of the problem

We consider a homogeneous, isotropic and infinitely extended thick plate of thickness $2l$ which has been taken in undisturbed state and initially at uniform reference temperature distribution $T_0$. The $z$-axis is assumed as the axis of symmetry and middle plane between lower and upper surfaces of the plate is taken to be the origin of the cylindrical polar co-ordinates $(r,\, \theta,\, z)$. Then the problem for the region $D$ can be defined as follows:

$$D = \{\, (r,\, \theta,\, z):\ 0 \le r \le \infty,\ 0 \le \theta \le 2\pi,\ -l \le z \le l\}$$

The Eq. (4.2) can be written as

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = \rho \frac{\partial^2 u}{\partial t^2} \tag{4.13}$$

$$\frac{\partial \sigma_{rz}}{\partial r}+\frac{\partial \sigma_{zz}}{\partial z}+\frac{\sigma_{rz}}{r}=\rho\frac{\partial^2 w}{\partial t^2} \tag{4.14}$$

In above equations, $u=u(r,z,t)$ and $w=w(r,z,t)$ are the displacement components along the $r$ and $z$ direction.

Now, the following non-dimensional quantities have been introduced for the simplification:

$c_1^2=\frac{\lambda+2\mu}{\rho},\ r'=c_1\eta\, r,\ u'=c_1\eta\, u,\ t'=c_1^2\eta\, t,\ \tau'=c_1^2\eta\,\tau, T'=\frac{T-T_0}{T},\ \sigma'_{rr}=\frac{\sigma_{rr}}{\mu},$

$\sigma'_{\theta\theta}=\frac{\sigma_{\theta\theta}}{\mu},\ \mu_1=\frac{\mu}{\lambda+2\mu}, w'=c_1\eta\, w,\ a_1=\frac{\gamma T_0}{(\lambda+2\mu)}, a_2=\frac{\gamma}{K\eta}\ \lambda_1=\frac{\lambda}{\mu},\ \beta=\frac{a_1}{\mu_1}$

Now, we drop the prime notations from all the quantities for the convenience.

Therefore, the dimensionless stress components obtained from Eq. (4.4) are given by

$$\sigma_{rr}=2e_{rr}+\lambda_1 e-\beta T \tag{4.15}$$

$$\sigma_{\theta\theta}=2e_{\theta\theta}+\lambda_1 e-\beta T \tag{4.16}$$

$$\sigma_{zz}=2e_{zz}+\lambda_1 e-\beta T \tag{4.17}$$

$$\sigma_{rz}=2e_{rz} \tag{4.18}$$

where, the non-dimensional strain components can be written as

$$e_{rr}=\frac{\partial u}{\partial r},\ e_{\theta\theta}=\frac{u}{r},\ e_{zz}=\frac{\partial w}{\partial z},\ e_{rz}=\frac{1}{2}(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r})$$

and Eqs. (4.12-4.14) can be written in non-dimensional form as follows:

$$\nabla^2 T=(1+\tau D_\tau)(\dot{T}+a_2\dot{e}] \tag{4.19}$$

$$\mu_1\nabla^2 u-\mu_1\frac{u}{r^2}+(1-\mu_1)\frac{\partial e}{\partial r}-a_1\frac{\partial T}{\partial r}=\frac{\partial^2 u}{\partial t^2} \tag{4.20}$$

$$\mu_1\nabla^2 w+(1-\mu_1)\frac{\partial e}{\partial z}-a_1\frac{\partial T}{\partial z}=\frac{\partial^2 w}{\partial t^2} \tag{4.21}$$

where,

$$\nabla^2=(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2})$$

Using the Helmholtz decomposition, we get the displacement components $u$ and $w$ as follows

$$u=\frac{\partial \phi}{\partial r}+\frac{\partial^2 \psi}{\partial r\partial z} \tag{4.22}$$

$$w=\frac{\partial \phi}{\partial z}-\frac{\partial^2 \psi}{\partial r^2}-\frac{1}{r}\frac{\partial \psi}{\partial r} \tag{4.23}$$

where, $\phi$ and $\psi$ are the potential functions, showing the dilatation and rotational part of the displacement vector. Therefore, we obtain

$$e=\nabla^2\phi \tag{4.24}$$

From Eqs. (4.22-4.24) and (4.19-4.21), we get the system of equations containing the displacement potentials $\phi$, $\varphi$ and the non-dimensional temperature distribution $T$ given as

$$[\nabla^2-(1+\tau D_\tau)\frac{\partial}{\partial t}]T=a_2(1+\tau D_\tau)\frac{\partial}{\partial t}\nabla^2\phi \tag{4.25}$$

$$\nabla^2\phi-\frac{\partial^2\phi}{\partial t^2}=a_1 T \tag{4.26}$$

$$\nabla^2\psi - \frac{1}{\mu_1}\frac{\partial^2\psi}{\partial t^2} = 0 \tag{4.27}$$

From Eqs. (4.25-4.27), it is seen that $\phi$ and $T$ are coupled to each other, whereas $\psi$ is not coupled with $\phi$ and $T$ which imply that Eq. (4.27) shows the shear motion which is not affected by the thermal field and it is purely elastic in nature, while Eqs. (4.25-4.26) represent the field variables, $\phi$ and $T$ included in the coupled thermoelastic motion.

## 4.3.1 Boundary conditions

As per the consideration of the problem, it is takesn that both the upper and lower planes of the plate are traction free which imply that the mechanical boundary conditions are taken as stress free which means

$$\sigma_{zz}(r, \pm l, t) = 0, \quad \sigma_{rz}(r, \pm l, t) = 0 \tag{4.28}$$

However, both the upper and lower boundary planes of the plate are subjected to an axi-symmetric temperature distribution given as

$$T(r, \pm l, t) = C_0 H(a - r) H(t) \tag{4.29}$$

where, $H(.)$ is the Heaviside unit step function and it is assumed that the temperatures of the lower and upper planes suddenly increase at $t = 0$ such that the temperature of a circular region $r \leq a$ of both the upper and lower boundaries reach a fixed value $C_0$.

# 4.4 Solution of the problem

## 4.4.1 Laplace and Hankel transform

By assuming the homogeneous initial conditions and applying the Laplace transform and then Hankel transforms to Eqs. (4.25-4.27), we obtain

$$[(\frac{\partial^2}{\partial z^2}-\alpha^2)-(1+G)p]\bar{T}^* = a_2(1+G)p(\frac{\partial^2}{\partial z^2}-\alpha^2)\bar{\phi}^* \tag{4.30}$$

$$(\frac{\partial^2}{\partial z^2}-\alpha^2-p^2)\bar{\phi}^* = a_1\bar{T}^* \tag{4.31}$$

$$(\frac{\partial^2}{\partial z^2}-\alpha^2-\frac{1}{\mu_1}p^2)\bar{\psi}^* = 0 \tag{4.32}$$

where, $p$ and $\alpha$ denote the Laplace and Hankel transform parameters, respectively and $J_0(.)$ is the Bessel function of first kind of order zero. $\bar{g}(r,p)$ and $g^*(\alpha,t)$ are the Laplace and Hankel transform of $g(r,t)$ with respect to $r$ and $t$, respectively, whereas $\bar{g}^*(\alpha,p)$ represents the Hankel transform of $\bar{g}(r,p)$.

In above equations, we have

$G=(1-e^{-p\tau})(1-\frac{2a_{12}}{\tau p}+\frac{2a_{11}^2}{\tau^2p^2})-e^{-p\tau}(a_{11}^2-2a_{12}+\frac{2a_{11}^2}{\tau p})$ where, $a_{11}$, $a_{12}$ are the constants such that

$$G=L\{\tau D_\tau f(x,t)\}=F(x,p)\begin{cases}(1-e^{-p\tau}), & a_{11}=0,\ a_{12}=0\\ [1-\frac{1}{\tau p}(1-e^{-p\tau})], & a_{11}=0,\ a_{12}=\frac{1}{2}\\ [(1-e^{-p\tau})-\frac{1}{p}(1-e^{-p\tau})+\tau e^{-p\tau}], & a_{11}=0,\ a_{12}=\frac{\tau}{2}\\ [(1-\frac{2}{\tau p})+\frac{2}{\tau^2p^2}(1-e^{-p\tau})], & a_{11}=a_{12}=1\end{cases}$$

where, $F(x,p)$ is the Laplace transform of $f(x,t)$.

Now, eliminating $\overline{T}^*$ from Eqs. (4.30-4.31), we obtain $\overline{\phi}^*$ in the following form:

$$(\frac{\partial^2}{\partial z^2}-k_1^2)(\frac{\partial^2}{\partial z^2}-k_2^2)\bar{\phi}^*=0 \tag{4.33}$$

where, $\pm k_1$ and $\pm k_2$ are the roots of the following equation:

$$k^4-\{2\alpha^2+p^2+\varepsilon_1(1+G)p\}k^2+\alpha^2(\alpha^2+p^2)+p(p^2+\alpha^2\varepsilon_1)(1+G)=0 \tag{4.34}$$

where, $\varepsilon = a_1a_2 = \frac{\gamma^2 T_0}{\rho^2 c_e c_1^2}$, $\rho\, c_e = K\eta$ and $\varepsilon$ is the thermoelastic coupling constant.

The present problem is considered as a axi-symmetric problem in which we have assumed the $z$-axis as the axis of symmetry. Here, $w$ is the displacement component along $z-$direction and $u$ is the displacement component along $r$-direction. Hence, it is clear that due to symmetry of the problem, $u$ is the even function of $z$ and $w$ is the odd function of $z$. Further, we also assumed that temperature distribution is axi-symmetric about $z$- axis. Therefore, $T$ is also evidently an even function of $z$.

Therefore, from Eqs. (4.22-4.23), we conclude that $\phi$ and $\psi$ are also the even and odd functions of $z$, respectively. Hence, the expressions for $\bar{\phi}^*$, $\bar{\psi}^*$ and $\bar{T}^*$ are finally obtained as

$$\bar{\phi}^*=\sum_{i=1}^{2}A_i\, cosh(k_i z) \tag{4.35}$$

$$\bar{\psi}^*=C\, sinh(qz) \tag{4.36}$$

$$a_1\bar{T}^*=\sum_{i=1}^{2}(k_i^2-\alpha^2-p^2)\, A_i\, cosh(k_i z) \tag{4.37}$$

where, $q^2=\alpha^2+\frac{1}{\mu_1}p^2$

In Eqs. (4.35-4.37), $A_1, A_2$and $C$ are the arbitrary constants which are the function of $p$ and $\alpha$. However, they are independent of $z$.

Using the Eqs. (4.17-4.18), (4.22-4.23) and (4.35-4.37), we obtain the stress components $\bar{\sigma}^*_{zz}$ and $\bar{\sigma}^*_{rz}$ given as

$$\bar{\sigma}^*_{zz} = (\frac{1}{\mu_1}p^2 + 2\alpha^2)\bar{\phi}^* + 2\alpha^2\frac{\partial\bar{\psi}^*}{\partial z} \tag{4.38}$$

$$\bar{\sigma}^*_{rz} = \hbar(\frac{\partial}{\partial r}(2\frac{\partial\bar{\phi}}{\partial z} + (2\frac{\partial^2}{\partial z^2} - \frac{p^2}{\mu_1})\bar{\psi})) \tag{4.39}$$

Now, by using the boundary conditions given by Eqs. (4.28-4.29) along with Eqs. (4.35-4.39), we find

$$\sum_{i=1}^{2}(\frac{1}{\mu_1}p^2 + 2\alpha^2)A_i\, cosh\,(k_i l) = -2\alpha^2 q C\, cosh\,(ql) \tag{4.40}$$

$$\sum_{i=1}^{2} k_i A_i\, sinh\,(k_i l) = \frac{1}{2}(\frac{p^2}{\mu_1} - 2q^2)C\, sinh\,(ql) \tag{4.41}$$

$$\sum_{i=1}^{2}(k_i^2 - \alpha^2 - p^2)A_i\, cosh\,(k_i l) = \frac{a a_1 C_0}{\alpha p}J_1(\alpha a) \tag{4.42}$$

From Eqs. (4.40-4.42), we find the constants $A_1, A_2$ and $C$ in the following form:

$A_1 = -\frac{a_1(4\alpha^2 q k_2 \mu_1^2\, tanh\,(k_2 l) - (2\alpha^2\mu_1 + p^2)^2 tanh\,(ql))}{pX\, cosh\,(k_1 l)(2\alpha^2\mu_1 + p^2)^2\, tanh\,(ql)}\theta_0^*(\alpha)$,

$A_2 = \frac{a_1(4\alpha^2 q k_1 \mu_1^2\, tanh\,(k_1 l) - (2\alpha^2\mu_1 + p^2)^2 tanh\,(ql))}{pX\, cosh\,(k_2 l)(2\alpha^2\mu_1 + p^2)^2\, tanh\,(ql)}\theta_0^*(\alpha)$,

$C = \frac{2a_1\mu_1[k_2\, tanh\,(k_2 l) - k_1\, tanh\,(k_1 l)]}{pX\,(2\alpha^2\mu_1 + p^2)\, sinh\,(ql)}\theta_0^*(\alpha)$,

where, $\theta_0^*(\alpha) = \frac{aC_0}{\alpha}J_1(\alpha a)$ and

$X = k_1^2 - k_2^2 + \frac{4\alpha^2 q \mu_1^2}{(2\alpha^2\mu_1 + p^2)^2\, tanh\,(ql)}[k_1(k_2^2 - \alpha^2 - p^2)\, tanh\,(k_1 l) - k_2(k_1^2 - \alpha^2 - p^2)\, tanh\,(k_2 l)]$

Therefore, we have found the solution of the problem in the transformed domain.

### 4.4.2 Inversion of the Hankel transform

Now, the inversion of Hankel transform of $\bar{f}^*(\alpha, p)$ can be defined as

$$\bar{f}(r,p) = \hbar^{-1}[\bar{f}^*(\alpha,p)] = \int_0^\infty \bar{f}^*(\alpha,p)\alpha J_0(\alpha r)\,d\alpha \tag{4.43}$$

Hence, by using the inverse Hankel transform to Eqs. (4.35-4.37), we find

$$a_1\bar{T}(r,z,p) = \int_0^\infty \alpha J_0(\alpha r) \sum_{i=1}^{2} A_i(k_i^2 - \alpha^2 - p^2)\,cosh\,(k_i z)\,d\alpha \tag{4.44}$$

$$\bar{\phi}\,(r,z,p) = \int_0^\infty \alpha J_0(\alpha r) \sum_{i=1}^{2} A_i\,cosh\,(k_i z)\,d\alpha \tag{4.45}$$

$$\bar{\psi}\,(r,z,p) = \int_0^\infty \alpha J_0(\alpha r)\,C\,sinh\,(qz)\,d \tag{4.46}$$

Further, by putting Eqs. (4.44-4.46) into the Eqs. (4.20-4.23), we obtain the solutions for the displacement components in the Laplace transform domain in the terms given as

$$\bar{u}(r,z,p) = -\int_0^\infty \alpha^2 J_1(\alpha r)\,[\sum_{i=0}^{2} A_i\,cosh\,(k_i z) + C\,q\,cosh\,(qz)]\,d\alpha \tag{4.47}$$

$$\bar{w}(r,z,p) = \int_0^\infty \alpha J_0(\alpha r)\,[\sum_{i=1}^{2} k_i A_i\,sinh\,(k_i z) + C\,\alpha^2\,sinh\,(qz)]\,d\alpha \tag{4.48}$$

Now, using the Laplace transform to both sides of Eqs. (4.15-4.17) and with the help of the solutions given by Eqs. (4.44-4.48), the stress components are obtained in the Laplace transform domain as

$$\begin{aligned}\bar{\sigma}_{rr} = \textstyle\int_0^\infty \sum_{i=1}^{2}\{\frac{2\alpha^2}{r}J_1(\alpha r) + \alpha J_0(\alpha r)\,(\frac{p^2}{\mu_1} - 2k_i^2)\}A_i\,cosh\,(k_i z)\,d\alpha \\ + \textstyle\int_0^\infty 2\alpha^3[\frac{1}{\alpha r}J_1(\alpha r) - J_0(\alpha r)]\,C\,q\,cosh\,(qz)\,d\alpha\end{aligned} \tag{4.49}$$

$$\begin{aligned}\bar{\sigma}_{\theta\theta} = \textstyle\int_0^\infty \sum_{i=1}^{2}[\alpha J_0(\alpha r)\,(\frac{p^2}{\mu_1} + 2\alpha^2 - 2k_i^2) - \frac{2}{r}\alpha^2 J_1(\alpha r)]A_i\,cosh\,(k_i z)\,d\alpha \\ -\frac{2}{r}\textstyle\int Cq\,\alpha^2 J_1(\alpha r)\,cosh\,(qz)\,d\alpha\end{aligned} \tag{4.50}$$

$$\bar{\sigma}_{zz} = \int_0^\infty \alpha J_0(\alpha r)\{(\frac{p^2}{\mu_1}+2\alpha^2)[\sum_{i=1}^2 A_i cosh(k_i z)] \\ +2\alpha^2 qC\, cosh(qz)\}\, d\alpha \tag{4.51}$$

The Eqs. (4.44-4.51) are the solutions of physical field variables in Laplace transform domain.

## 4.5 Numerical results and discussion

In this section, we obtain the solution of the problem in physical domain in order to understand the effects of the memory dependent derivative in the physical field variables namely, temperature distribution, displacement distribution, stress distribution. For this we compute numerical values of all the physical field variables by programming using the MatLab software. We carry out our computational work for four different kernel functions at three different values of time delay parameter $\tau$ . We also compare the results in the present context with the results discussed in Chapter-3 in the context of generalized thermoelasticity with one relaxation parameter, $\tau_0$. The copper material is considered for the purpose of the numerical computation using the following data given by Chandrashekharaiah and Srinath (1998).

$\lambda = 7.76\times10^{10}\text{Nm}^{-2}$, $\mu = 3.38\times10^{10}\text{Nm}^{-2}$, $\alpha_t = 1.78\times10^{-5}\text{K}^{-1}$, $d = \frac{K}{\rho c_e} = 0.000113\text{m}^2\text{s}^{-1}$, $c_e = 383.1\text{J KKg}^{-1}$, $\rho = 8954\text{Kgm}^{-3}$, $T_0 = 293\text{K}$, $\tau_0 = 0.2$, $l = 1$, $\tau = 0.2,\ 0.1,\ 0.01$, $\eta = \frac{1}{d}$, $C_0 = 1$

The inversion of the Laplace transform is carried out by using the method given by Bellman *et al.* (1966) as given in Appendix-A2 and the Gauss-Laguerre quadrature integration method is used to invert the Hankel transform. We have computed the values of all the physical field variables in the physical domain $(r,t)$ at the middle plane of the plate $z = 0$ using the above non dimensional values with the help of the solutions in the transformed domain given by the Eqs. (4.44-4.51). We find the values of physical fields at different times and plot them to show the time variation at two different non-dimensional times 0.35 and 0.69 . We also show the variations of the fields for three different values of the time delay parameter $\tau = 0.2,\ 0.1$, and 0.01. In each figure, one particular physical field variable is shown for four different kernel functions, $K(t-s) = 1$,

$K(t-s) = 1 - \frac{(t-s)}{\tau}$ , $K(t-s) = 1-(t-s)$ and $K(t-s) = [1 - \frac{(t-s)}{\tau}]^2$ along with the comparison of the physical fields in absence of the "memory dependent derivatives", i.e., the case as discussed in Chapter-3. The detailed comparative discussion on each physical field variable is given in the following manner.

The nature of temperature distribution is shown in Figs. $(4.1-4.6)$ in which Figs. $(4.1)$ and $(4.2)$ are showing the nature of the temperature distribution at two non-dimensional times, $t$ = 0.69 and 0.35, respectively for the non-dimensional delay time 0.2 and Figs. $(4.3)$ and $(4.4)$ are drawn at the non-dimensional times $t$= 0.69 and 0.35, respectively in the case when the non dimensional delay time is equal to 0.1. However, the nature for non-dimensional delay time $\tau = 0.01$ is shown in Figs. $(4.5)$ and $(4.6)$ at the non-dimensional times, $t$ = 0.69 and 0.35, respectively. From these Figs. $(4.1-4.6)$, it is clear that the value of temperature at the boundary $r = 0$, decreases with the decrease of time delay parameter $\tau$ for all the cases of the kernels $K(t-s) = 1 - \frac{(t-s)}{\tau}$, $K(t-s) = [1 - \frac{(t-s)}{\tau}]^2$ and the difference between the values for different kernel functions also decreases with the decrease of time delay parameter, $\tau$. However, there is no prominent difference in the plots for different kernel function for the case when $\tau = 0.01$. We also note that the difference between values for different kernels is more prominent at lower time ($t = 0.35$) as compared to higher time, i.e., at non-dimensional time 0.69. However, the temperature distribution corresponding to the generalized theory without memory dependent derivative has the prominent difference with the values corresponding to the theory with presence of different kernels. We further observe that at the first local minima, its value is low at non-dimensional time 0.69 however it is large at non-dimensional time 0.35 in the case of the generalized thermoelasticity without memory dependent derivative as compared to the case of theory with memory dependent derivatives. It is also clear that the nature of the temperature distribution for all four different kernels is similar with the case of without memory dependent derivative effect. Further, it is seen that the value at the non-dimensional delay time $\tau = 0.01$ for all four different kernels is matched which has the prominent difference with the values in case of the theory without memory dependent derivative.

The comparison of the displacement distribution is shown in Figs. $(4.7-4.12)$ for three non-dimensional delay times $\tau = 0.2,\ 0.1,\ 0.01$ at two different non-dimensional time $t = 0.69$ and 0.35 and for four different kernels and for the case of thermoelasticity with one relaxation parameter without memory dependent derivative. It is clear that the nature of the displacement distribution is similar in each case. Figs. (4.7), (4.9) and (4.11) are drawn at non-dimensional time 0.69 using three non-dimensional delay time 0.2, 0.1 and 0.01, respectively while Figs. (4.8), (4.10) and (4.12) are drawn at non-dimensional time 0.35. We find that displacement field is zero at the boundary and there are the local maxima and minima after some distance from the boundary at which its value has a significant difference for different profiles. When we move from the boundary, firstly the local maxima is occurred at non-dimensional time 0.69. However, the local minima is appeared at non-dimensional time 0.35. In the case of displacement distribution, we have also seen that the difference in the profiles decreases with the decrease of the value of the non-dimensional delay time for different kernel functions. However, it is more prominent in the case of generalized thermoelasticity with one relaxation time parameter without memory dependent derivative. It is further seen that this difference is more prominent at lower time. The value of the displacement distribution is matched for four different kernels at non-dimensional delay time $\tau = 0.01$ in the case of both the non-dimensional times and it has a prominent difference with the values obtained in the case of generalized thermoelasticity with one relaxation time parameter without memory dependent derivative.

Similarly, the comparison of the stress distribution is represented in the Figs. $(4.13-4.18)$ for four different kernels and without using kernel function with three different non-dimensional delay time $\tau = 0.2,\ 0.1,\ 0.01$ at two non-dimensional times, $t = 0.69$ and 0.35. The Figs. (4.13) and (4.14) are showing the nature of the stress distribution for delay time $\tau = 0.2$ at non-dimensional time 0.69 and 0.35, respectively and the distribution with delay time $\tau = 0.1$ is shown in Figs. (4.15) and (4.16) at time 0.69 and 0.35 respectively. However, the remaining Figs. (4.17) and (4.18) are representing the behavior of stress field with non-dimensional delay time $\tau = 0.01$ at non-dimensional time 0.69 and 0.35. From Figs. $(4.13-4.18)$, it is clear that the value of stress is minimum near the boundary $r = 0$. We have also seen that the difference in the values of the displacement distribution for four different kernels with the values corresponding to the thermoelasticity without memory dependent derivative are more prominent for all three different values of non-dimensional delay time at both the non-dimensional time. However the difference in the values corresponding to four different kernels is not significant and this difference decreases with the decrease of the non-dimensional delay time and its values are matched in the case when we take $\tau = 0.01$. Further, we have also seen that the difference in the values of the stress distribution decreases with the increase of non-dimensional time. Thus, it is more prominent at lower time for all non-dimensional delay time parameters and also in the cases of four different kernels.

From above discussion, it is clear that the nature of all the physical field variables corresponding to all four different kernels is similar in nature with the physical field variables corresponding to generalized thermoelasticity with one relaxation parameter without memory dependent derivative. However, there is significant effect of using the memory dependent derivatives. It can be observed that our results for all the physical fields corresponding to the generalized thermoelasticity with one relaxation parameter without memory dependent derivative is in agreement with the results of the corresponding physical field variables for the case of Lord and Shulman's theory which has been obtained in Chapter-3.

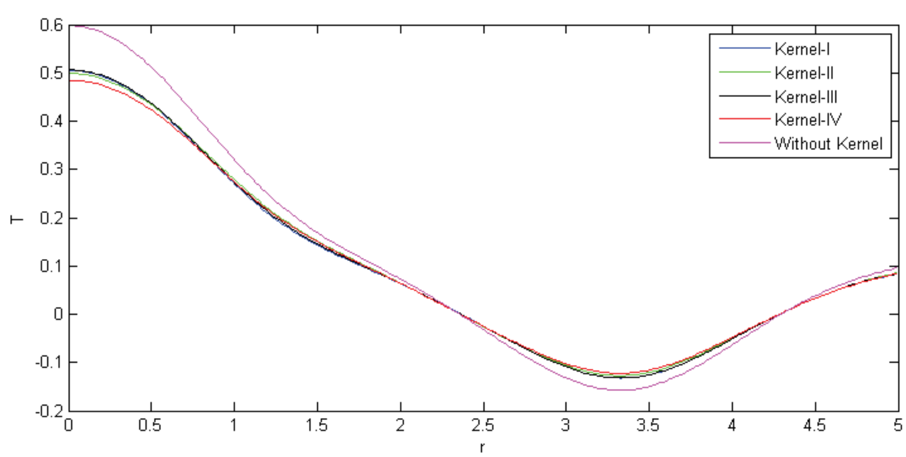


**Fig. 4.1** Temperature distributions with delay time 0.2 and non dimensional time 0.69

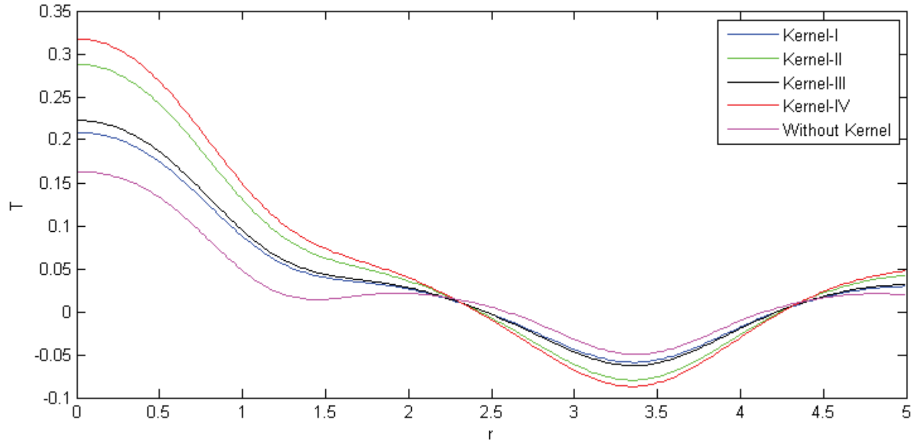


**Fig. 4.2** Temperature distributions with delay time 0.2 and non dimensional time 0.35

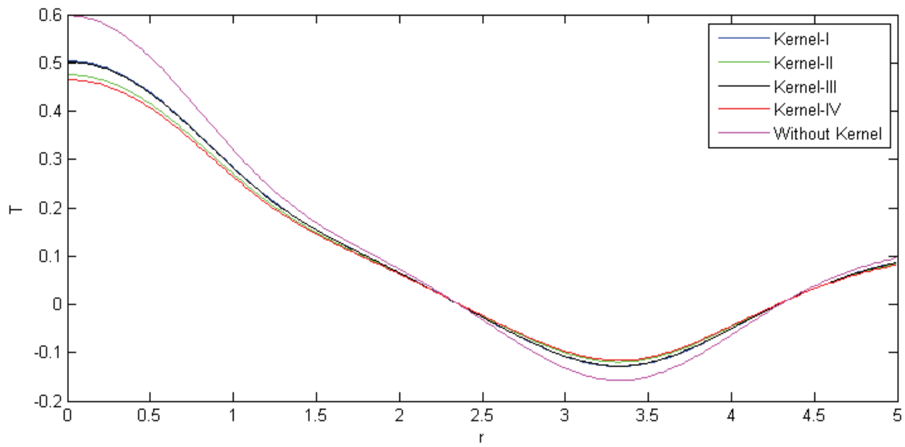


**Fig. 4.3** Temperature distributions with delay time 0.1 and non dimensional time 0.69

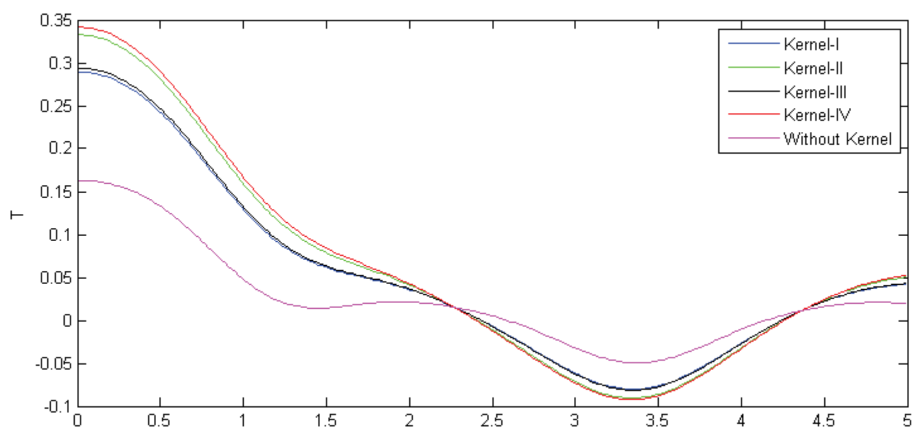


**Fig. 4.4** Temperature distributions with delay time 0.1 and non dimensional time 0.35

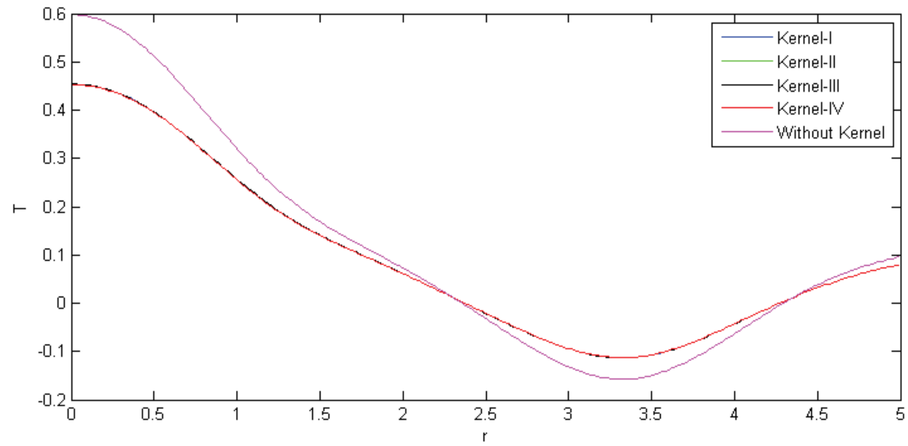


**Fig. 4.5** Temperature distributions with delay time 0.01 and non dimensional time 0.69

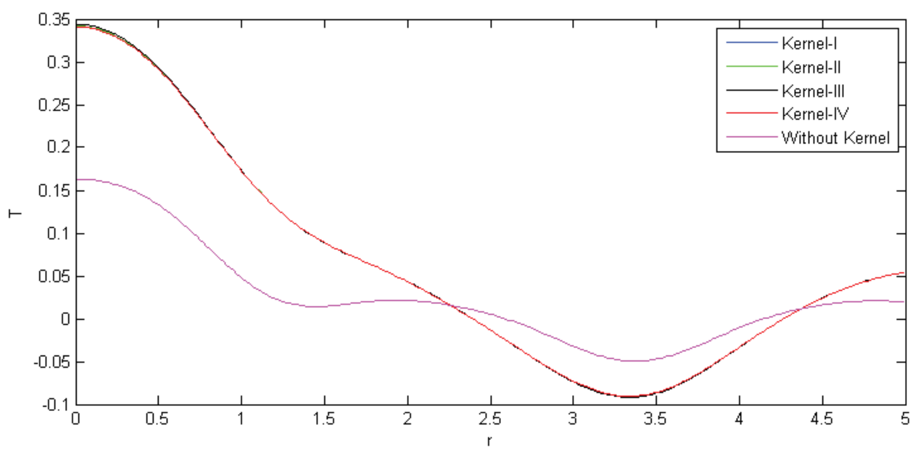


**Fig. 4.6** Temperature distributions with delay time 0.01 and non dimensional time 0.35

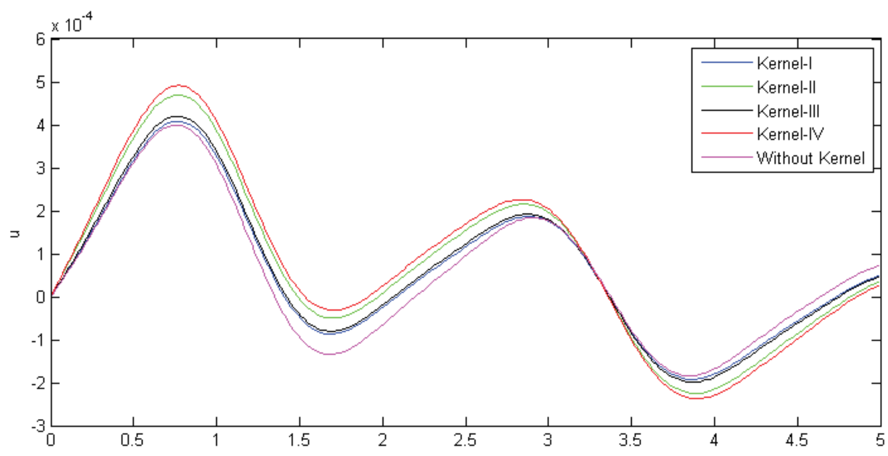


**Fig. 4.7** Displacement distributions with delay time 0.2 and non dimensional time 0.69

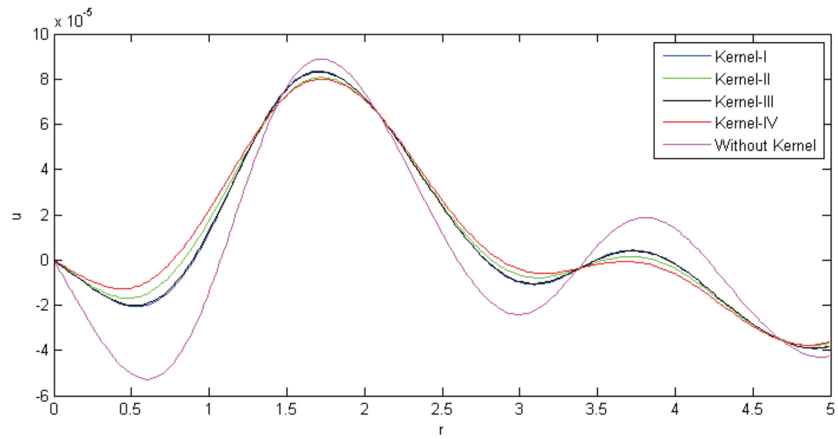


**Fig. 4.8** Displacement distributions with delay time 0.2 and non dimensional time 0.35

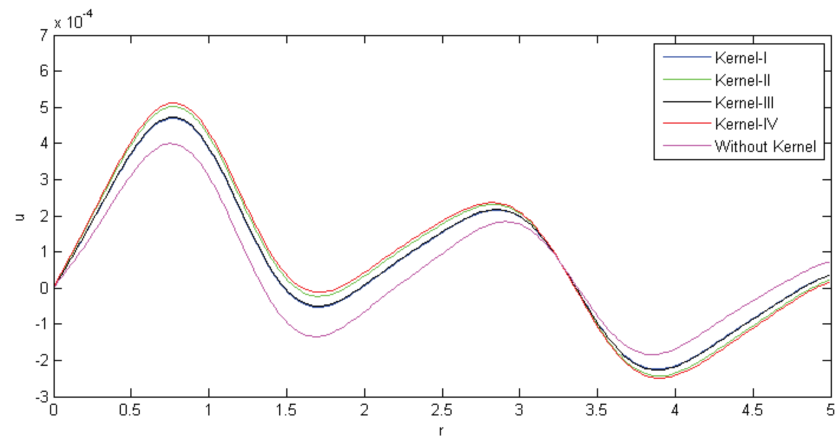


**Fig. 4.9** Displacement distributions with delay time 0.1 and non dimensional time 0.69

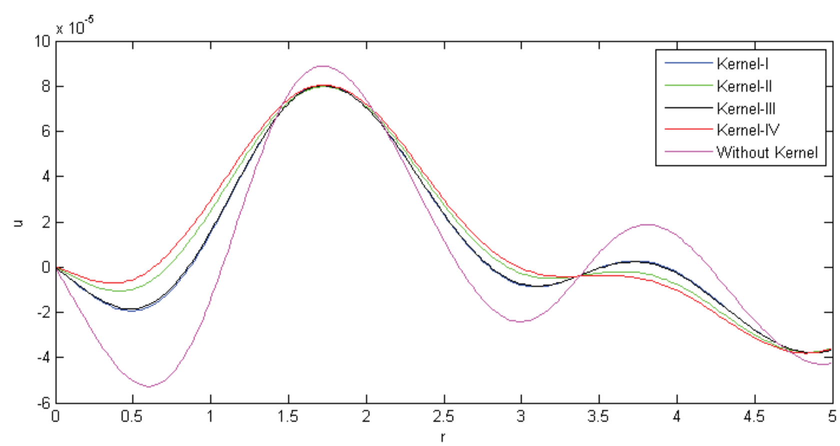


**Fig. 4.10** Displacement distributions with delay time 0.1 and non dimensional time 0.35

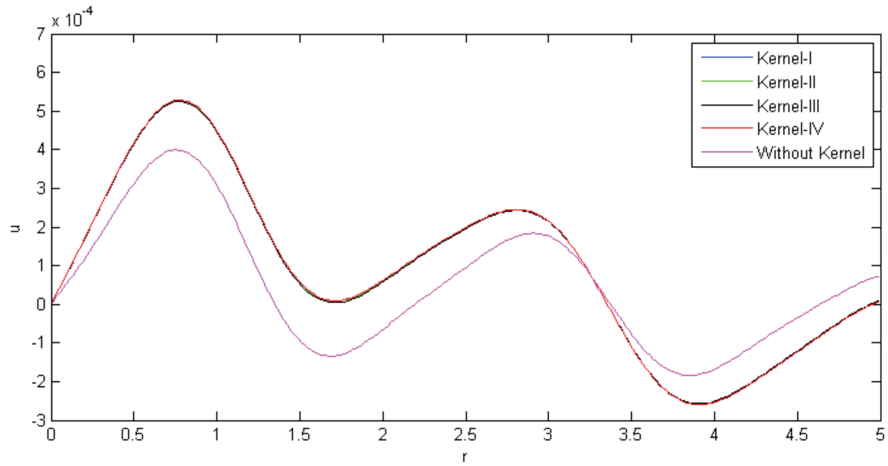


**Fig. 4.11** Displacement distributions with delay time 0.01 and non dimensional time 0.69

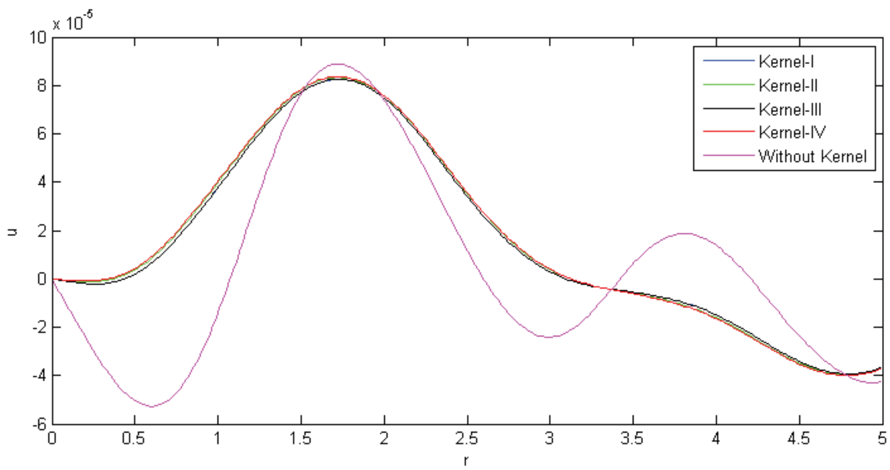


**Fig. 4.12** Displacement distributions with delay time 0.01 and non dimensional time 0.35

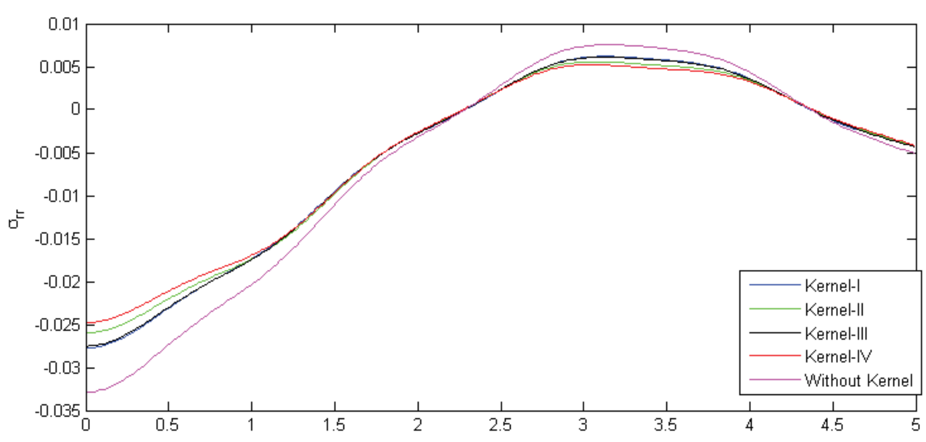


**Fig. 4.13** Radial stress distributions with delay time 0.2 and non dimensional time 0.69

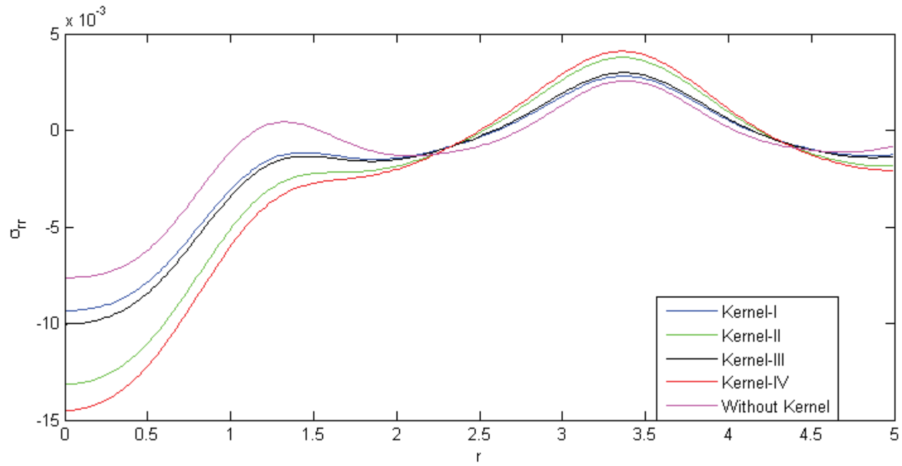


**Fig. 4.14** Radial stress distributions with delay time 0.1 and non dimensional time 0.35

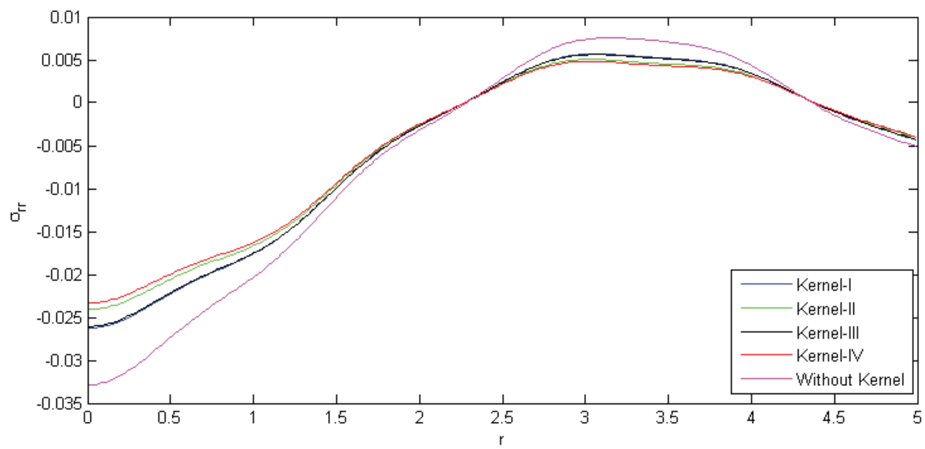


**Fig. 4.15** Radial stress distributions with delay time 0.1 and non dimensional time 0.69

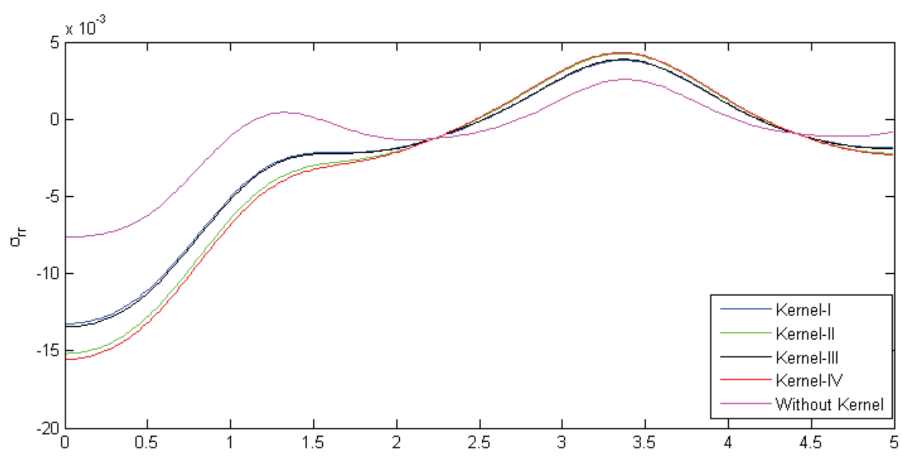


**Fig. 4.16** Radial stress distributions with delay time 0.1 and non dimensional time 0.35

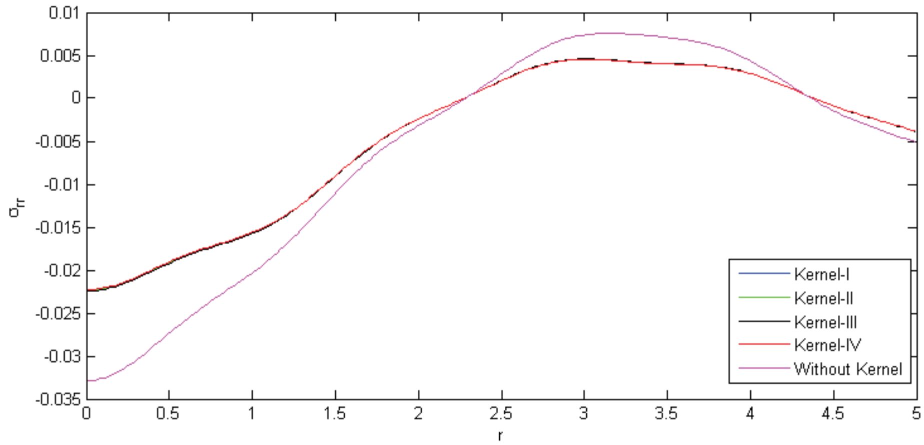


**Fig. 4.17** Radial stress distributions with delay time 0.01 and non dimensional time 0.69

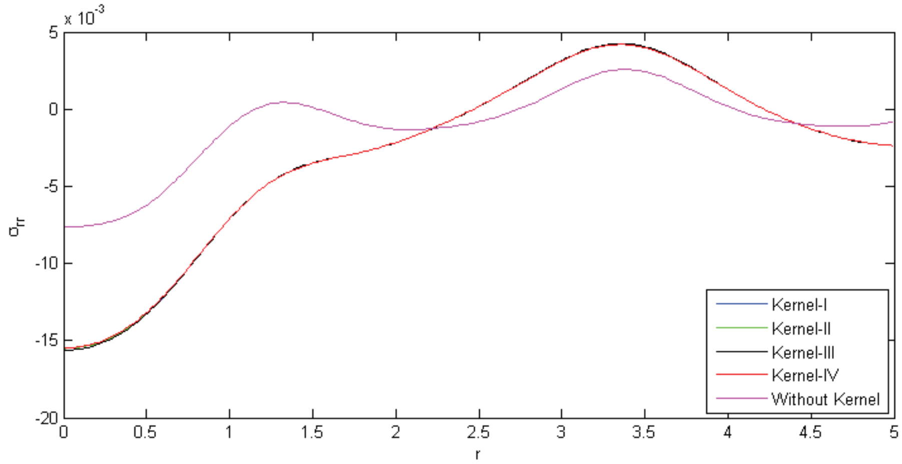


**Fig. 4.18** Radial stress distributions with delay time 0.01 and non dimensional time 0.35

## 4.6 Conclusion

In this chapter, we have considered the problem to investigate the effects of the memory dependent derivative in the generalized thermoelasticity with one relaxation parameter which is the new concept to use in the generalized thermoelasticity. To consider the memory effects we have taken four different kernel functions. The highlights of the present work is summarized in the following points:

- The present work is based on the generalized thermoelasticity with one relaxation time parameter using the memory dependent derivative with the help of four different kernels.

- The present results are compared with the generalized thermoelasticity with one relaxation parameter (i.e., under LS-model) as discussed in Chapter-3.

- The nature of all the physical field variables are similar in nature in the context of both the theories. However, there is prominent difference in predictions of values of different physical fields by the theory with memory dependent derivative as compared to the predictions by the generalized theory without memory dependent derivative.

- Each physical field variable corresponding to each kernel function has a prominent difference for larger values of the non-dimensional delay time parameter. However, this difference decreases with the decrease of the value of delay time.

- The difference is more prominent in all the physical field variables at initial time of thermoelastic interaction and the difference decreases with the increase of time.

- When the delay time parameter is considered to be very small, then each physical field variable shows almost the same value corresponding to all four different kernels. However, there is always a significant difference in the results between the cases when we consider the theory with memory dependent derivative and the theory without memory dependent derivative.