APPENDICES

Appendix-A1

Theorem (Boley (1962)):

Let the inverse of the Laplace transform of a function $f(t)$ be expressed as

$$f(t) = \frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty} \bar{F}(p)\, e^{g(p,t)} dp$$

If for large p, we have

$$F(p) = \frac{\widetilde{K}}{p^n}\left[1 - O(\frac{1}{p})\right], \qquad n > 0$$

where, $\widetilde{K}$ is free from p and if there exists a function $\xi(t)$ such that for large p

$$g(p,t) - p\xi(t) = O(\frac{1}{p})$$

then the discontinuity of $f(t)$ is obtained in the following way

$$[f(t)] = f(t+0) - f(t-0) = \begin{cases} 0 & \text{for large } \xi \neq 0 \\ = \left\{ \begin{array}{ll} 0 & x > 1 \\ \widetilde{K} & n = 1 \\ \infty & n < 1 \end{array} \right\} & \text{for } \quad \xi = 0 \end{cases}$$

Appendix-A2

Inversion of Laplace transforms (Bellman *et al.* (1966)):

The Laplace transform of a function $u(t)$ can be defined in the following form:

$$\bar{u}(s) = \int_0^\infty u(t)\, e^{-st} dt, \quad s > 0 \tag{B.1}$$

where, $u(t)$ is sufficiently smooth such that it can be approximated.

Now we assume $x = e^{-t}$ in Eq. (B.1), we obtain

$$\bar{u}(s) = \int_0^\infty x^{s-1} v(x)\, dx \tag{B.2}$$

where, $v(x) = u(-ln(x))$.

By using the Gaussian quadrature formula to Eq. (B.2), we obtain

$$\sum_{i=1}^{N} w_i x_i^{s-1} v(x_i) = \bar{u}(s) \tag{B.3}$$

where, x_i $(i = 1, 2, 3...N)$ are the roots of the shifted Legendre polynomial $P_N(x) = 0$ and w_i $(i = 1, 2, 3..., N)$ are the respective weights.

Eq. (B.3) can be written as

$$w_1 x_1^{s-1} v(x_1) + w_2 x_2^{s-1} v(x_2) + w_3 x_3^{s-1} v(x_3) + ... + w_N x_N^{s-1} v(x_N) = \bar{u}(s) \tag{B.4}$$

Now putting $s = 1, 2, 3, ..., N$ in Eq. (B.4) which can be written as a system of equations in the following form:

$$w_1 v(x_1) + w_2 v(x_2) + w_3 v(x_3) + ... + w_N v(x_N) = \bar{u}(1)$$

$$w_1x_1v(x_1)+w_2x_2v(x_2)+w_3x_3v(x_3)+...+w_Nx_Nv(x_N)=\bar{u}(2)$$

....

$$w_1x_1^{N-1}v(x_1)+w_2x_2^{N-1}v(x_2)+w_3x_3^{N-1}v(x_3)+...+w_Nx_N^{N-1}v(x_N)=\bar{u}(N)$$

Hence, from the system as expressed above, we obtain $v(x_i)$ in the terms as given below

$$\begin{bmatrix} v(x_1) \\ v(x_2) \\ ... \\ v(x_N) \end{bmatrix} = \begin{bmatrix} w_1 & w_2 & ... & w_N \\ w_1x_1 & w_2x_2 & ... & w_Nx_N \\ ... & ... & ... & ... \\ w_1x_1^{N-1} & w_2x_2^{N-1} & ... & w_Nx_N^{N-1} \end{bmatrix}^{-1} \begin{bmatrix} \bar{u}(1) \\ \bar{u}(2) \\ ... \\ \bar{u}(N) \end{bmatrix} \tag{B.5}$$

Using Eq. (B.5), we find the discrete values of $v(x_i)$, which is $u(t_i)$ and thereafter, by using interpolation method, we find the function $u(t_i)$.

Appendix-A3

The constants $a_i,\ i=1,2,3,4,5,6,7$ are given by

$a_1=\frac{1}{\sqrt{2a_0}}\left[(a_0+1+\varepsilon)+\sqrt{(a_0+1+\varepsilon)^2-4a_0}\right]^{\frac{1}{2}}$

$a_2=\frac{1}{\sqrt{2a_0}}\left[(a_0+1+\varepsilon)-\sqrt{(a_0+1+\varepsilon)^2-4a_0}\right]^{\frac{1}{2}}$

$a_3=\frac{1}{a_0}\left[\sqrt{(a_0+1+\varepsilon)^2-4a_0}\right]$

$a_4=\frac{1}{2a_0a_3}\left[(1+\varepsilon-a_0)+\sqrt{(a_0+1+\varepsilon)^2-4a_0}\right]$

$a_5=\frac{1}{2a_0a_3}\left[(1+\varepsilon-a_0)-\sqrt{(a_0+1+\varepsilon)^2-4a_0}\right]$

$a_6 = \frac{a_1}{a_3}$

$a_7 = \frac{a_2}{a_3}$

where,

$a_0 = \frac{K^*}{Kc^2\xi}$ and $\xi = \frac{\rho c_e}{K}$

Appendix-A4

The constants, a_{ij}, $i = 1, 2$; $j = 0, 1, 2, 3, 4$ are given by:

$a_{10} = (1 + \tau_2 + \varepsilon\tau_1 + a)/2$

$a_{11} = [(1+\varepsilon)a + b]/(2a)$

$a_{12} = \varepsilon c/a^3$

$a_{13} = -\varepsilon bc/a^5$

$a_{14} = -\varepsilon cd/(4a^7)$

$a_{20} = (1 + \tau_2 + \varepsilon\tau_1 - a)/2$

$a_{21} = [(1+\varepsilon)a - b]/(2a)$

$a_{22} = -a_{12}$

$a_{23} = -a_{13}$

$a_{24} = -a_{14}$

where,

$a = [1 + 2(\varepsilon\tau_1 - \tau_2) + (\varepsilon\tau_1 + \tau_2)^2]^{1/2}$

$b = \varepsilon - 1 + (\varepsilon + 1)(\varepsilon\tau_1 + \tau_2)$

$c = 1 + (\varepsilon + 1)(\tau_1 - \tau_2)$

$d = (\varepsilon + 1)^2 a^2 - 5b^2$

The constants, b_{ij}, $i = 1, 2$; $j = 0, 1, 2$ are given by:

$b_{i0} = (a_{i0})^{1/2}$

$b_{i1} = a_{i1}/(2b_{i0})$

$b_{i2} = (4a_{i0}a_{i2} - a_{i1}^2)/(8a_{i0}b_{i0})$

The constants, b_i, $i = 0, 1, 2, 3, 4, 5$ are given as

$b_0 = 1/a$

$b_1 = -b/a^3$

$b_2 = (b^2 - 2\varepsilon c)/a^5$

$b_3 = b(6\varepsilon c - b^2)/a^7$

$b_4 = \left[6(\varepsilon c - b^2)^{2-}5b^4\right]/a^9$

$b_5 = -b\left[240\varepsilon^2 c^2 - 20\varepsilon c b^2 + 43b^4\right]/(8a^{11})$

The constants, c_{ij}, $i = 1, 2$; $j = 1, 2, 3, 4$ are given by:

$c_{i0} = b_0(a_{i0} - 1)$

$c_{ij} = b_j(a_{i0} - 1) + \sum_{k=1}^{j} b_{j-k} a_{ik}$

The constants, M_i and T_i , $i = 1, 2$ are given as

$M_i = 2\sqrt{|\, b_{i2} \,|\, x k_i}$

$T_i = 2\sqrt{|\, b_{i2} \,|\, x(t - b_{i0})}$

where,

$k_i = \text{Max}(0, t - b_{i0} - l)$; $i = 1, 2$

Appendix-A5

The numerical method which is used to solve the Fredholm's integral equation of the first kind (see Delves and Mohammed (1985)):

In this appendix, we are going to describe the method which is used to solve the Fredholm's integral equation of first kind. We assume an integral equation in the following form (see Delves and Mohammed (1985), Sherief and El-Maghraby (2005)):

$$\int_{r_1}^{r_2} \bar{L}_1(z, x, s)\, \varphi(z, s) dz = \bar{L}_2(x, s) \qquad \text{(A.1)}$$

We can write the above equation briefly in the following form:

$$\bar{L}_1 \varphi = \bar{L}_2 \qquad \text{(A.2)}$$

Here, $\bar{L}_2(x, s)$ will have some finite accuracy k. Hence, we try to obtain $||\bar{L}_1 \varphi - \bar{L}_2|| \leq k$. The all functions φ which satisfy this relation, we use only the smoothest function in the sense that for some linear operator L, $||L\varphi||$ has the minimum value which produce the constrained minimization problem as given below:

$$minimize_{\varphi}||L\varphi|| \tag{A.3}$$

subjected to $||\bar{L}_1\varphi - \bar{L}_2|| \leq k$.

The above minimizing problem can be solved in any given norm which is comparatively difficult to do so (see Delves and Mohammed (1985)). Therefore, the Eqs. (A.1)-(A.2) can not be solved analytically in the easy way, but can be easily solved numerically in (see Delves and Mohammed (1985), Sherief and El-Maghraby (2005)).

We know that the value of objective function $||L\varphi||$ decreases with the increase of the value of k which means, as the constraint weaken. Therefore at the minimum of (A.1)-(A.2), the constraint will be binding that is $||\bar{L}_1\varphi - \bar{L}_2|| = k$. Now, if we solve the unconstrained problem for any fixed α as given below

$$minimize_{\varphi}||\bar{L}_1\varphi - \bar{L}_2||^2 + \alpha\,||L\varphi||^2 \tag{A.4}$$

Now we will have to obtain some minimum value β of $||\bar{L}_1\varphi - \bar{L}_2||$. If we take $\alpha \to 0$, and $\beta \to 0$ provided that the solution of (A.1)-(A.2) exists, and for some value of α, $\beta = k$. The solution of the problem (A.4) is identical to the solution of the original problem (A.1)-(A.2) (see Delves and Mohammed (1985)). We know that (A.4) is an unconstrained problem, it is easier to solve than (A.1)-(A.2). This problem is referred as the regularization problem and the method which is based on a numerical solution of (A.4) is known as the regularization method (see Delves and Mohammed (1985)).

The most common possibility for the operator L, can be taken as $L = 1$; $\frac{d}{dx}$; $\frac{d^2}{dx^2}$. $L = 1$ and the natural norm L^2 have been taken for the computational work.

Now, the Eq. (A.4) can be written as given below

$$minimize_{\varphi}\ Z(\varphi) = < \bar{L}_1\varphi - \bar{L}_2, \bar{L}_1\varphi - \bar{L}_2 > + \alpha < \varphi, \varphi > \tag{A.5}$$

where, $< . >$ is the scalar product in L^2 norm.

After simplifying the above Eq. (A.5), we obtain the following

$$Z(\varphi) =< \varphi, \left\{\bar{L}_1^+ \bar{L}_1 + \alpha I\right\} \varphi > - < \varphi, \bar{L}_1^+ \bar{L}_2 > - < \bar{L}_1^+ \bar{L}_2, \varphi > + < \bar{L}_2, \bar{L}_2 >$$

where, $\bar{L}_1^+$ is the Hermitian conjugate of $\bar{L}_1$.

Now we know that the following condition will be satisfied for the minimum value of Z at the point φ for any function g

$$\frac{\partial Z(\varphi + kg)}{\partial k} |_{k=0} = 0$$

using the above equation, we find that φ is a minimum point if the following is satisfied

$$\left\{\bar{L}_1^+ \bar{L}_1 + \alpha I\right\} \varphi = \bar{L}_1^+ \bar{L}_2 \tag{A.7}$$

Now, simplify the above equation, we find

$$\int_{r_1}^{r_2} \hat{\bar{L}}_1(z, x, s)\, \varphi(z, s) dz + \alpha \varphi(x, s) = \hat{\bar{L}}_2(x, s) \tag{A.8}$$

where,

$$\hat{\bar{L}}_1(z, x, s) = \int_{r_1}^{r_2} \bar{L}_1^*(z, r, s)\, \bar{L}_1(z, r, s)\, dr$$

$$\hat{\bar{L}}_2(x, s) = \int_{r_1}^{r_2} \bar{L}_1^*(z, x, s)\, \bar{L}_2(z, s)\, dz$$

where, the asterisk notation represents the complex conjugation and Eq. (A.8) is a Fredholm's integral equation of the second kind with iterated kernels and the parameter α can be chosen to be equal to 10^{-5} (see Sherief and El-Maghraby (2005)).

We are now going to introduce the method which is used to solve the Fredholm's integral equation of the second kind. For this, we take an integral equation given below (see Delves and Mohammed (1985), Sherief and El-Maghraby (2005)):

$$w(u, s) + \int_{r_1}^{r_2} \bar{L}_1(u, v, s)\, w(v, s)\, dv = \bar{L}_2(u, s) \tag{A.9}$$

The above equation can be approximated in the following form:

$$w(u,\, s) + \sum_{i=0}^{n} d_i \bar{L}_1(u,\, v_i,\, s)\, w(v_i,\, s) \approx \bar{L}_2(u,\, s) \tag{A.10}$$

where, the points $v_i\,(i = 0,\, 1,\, 2,\, 3,\, ...,n)$ are the equally spaced points in the interval $[r_1,\, r_2]$ and d_i's are the corresponding weights. We know that two sides of the any equation must be equal at all the points of the domain. Therefore, from Eq. (A.10), we have the system of $n+1$ linear equations for, $j = 0,\, 1,\, 2,\, 3,\, ..,\, n$ which can be written as

$$w(v_j,\, s) + \sum_{i=0}^{n} d_i \bar{L}(v_j,\, v_i,\, s)\, w(v_i,\, s) \approx \bar{L}_2(v_j,\, s) \tag{A.11}$$

The above equation is having the $n+1$ unknowns, $w(v_0,\, s), ..., w(v_n,\, s)$, that denote the approximate values of the unknown function $w(v,\, s)$ at the $n+1$ chosen points. We now introduce the following notations for simplification:

$$w_j = w(v_j,\, s),\ \bar{L_2}_j = \bar{L}_1(v_j,\, s),\ \bar{L}_{1ji} = \bar{L}(v_j,\, v_i,\, s); \quad j,\, i = 0,\, 1,\, 2, ..., n$$

Therefore, Eq. (A.1) can be written in the following form:

$$w_j + \sum_{i=0}^{n} d_i \bar{L}_{1ji}\, w_i \approx \bar{L}_{2j}, \quad j = 0,\, 1,\, 2,\, 3, ..., n \tag{A.12}$$

Now if we assume the variables, w_j and $\bar{L}_{2j}$ are the components of the w and $\bar{L}_2$, respectively and define the matrix $\bar{L}_1 = \left[\bar{L}_{1jk}\right]$, then the system of Eq. (A.12) can be written as

$$w + \bar{L_1}\, d\, w = \bar{L}_2$$

where, $d = \left[d_j \delta_{ji}\right]$ is a diagonal matrix containing successive weighting coefficients. Therefore, the above set of equations can be written as

$$(I + \bar{L}_1\, d)\, w = \bar{L}_2 \tag{A.13}$$

where, I is the identity matrix of order $n+1$.

Now we have used the Simpson's rule of integration, for computing the numerical values of all the integration, and we take the following weights:

$$d_0 = d_n = \frac{w}{3};\ d_{2j-1} = \frac{4w}{3},\ j = 1 2, 3, ..., \frac{n}{2};\ d_{2j} = \frac{2w}{3},\ j = 1, 2, 3, ..., n-1.$$