Chapter 4

On the Cauchy problem for isentropic dusty gas

4.1 Introduction

The Cauchy problem for compressible flow is an initial value problem for the Euler equations supplemented by a particular initial data. The study of Cauchy problem is of great importance both from mathematical and physical point of view due to its wide applications in mathematical physics. For many systems, the singularity arises in the solution of Cauchy problem, despite the initial data being small and smooth, see Lax, (1964) and Lee (1994). Thus, to identify the conditions on the initial data, so that the Cauchy problem for hyperbolic systems has a unique global solution, turns out to be an important problem in both theory and applications. In recent years, many attempts have been made to study the existence of solution to the Cauchy problem in various gasdynamic regimes. Zheng (1987) studied the Cauchy problem for the system of gasdynamics equations with dissipation. Lions et al. (1996) proved the conditions for the existence and stability of entropy solutions for the hyperbolic system of conservation laws corresponding to isentropic gasdynamics. Yang and Zhu (2000) derived the conditions for the existence and non-existence of the Global solution to the Cauchy problem for p system with relaxation. Li and Liu (2009) investigated the critical threshold phenomena related to quasilinear hyperbolic relaxation system to obtain information associated to global time regularity and breakdown of solution. Cauchy problem for shallow water equations has been studied by Fu and Sharma

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(2013). Recently Fu and Sharma (2015) investigated the existence of the solution to the Cauchy problem for one dimensional isentropic magnetogasdynamics system. A systematic study about existence and non existence of global solutions can be found in Li (1994). Examples of dusty gas flow include a number of physical phenomena in astrophysics and gasdynamics such as underground explosion studied by Lamb (1992), interstellar mass flow see Laibe et al. (2014) and explosive volcanic eruptions see Pelanti (2006). Dusty gas is a mixture of gas and dust particles where dust particles occupy less than 5% of total volume. For basic understanding of gas particle flow the reader is referred to Pai (1977) and Rudinger (1980). Dusty gas is a subject of great interest in recent decades. Recently Gupta et al. (2016) studied the Riemann problem for dusty gas. In the present work, an attempt has been made to study the existence of smooth solution to Cauchy problem for dusty gas flow of Mie Grüneisen type, for planar and cylindrically symmetric flow. The effect of the parameters, characterizing the dusty gas, on the blow up phenomenon is observed and the restrictions on the initial data for the existence of smooth solutions are also obtained.

4.2 Governing equations

The basic equations governing the planar and cylindrically symmetric flow of an isentropic dusty gas obeying the equation of state of Mie Grüneisen type are given in Pai (1980) and Miura (1983)

$$\begin{aligned}
&\rho_t + v\rho_x + \rho v_x + \rho m v x^{-1} = 0, \\
&\rho(v_t + v v_x) + p_x = 0, \\
&\rho(0,x) = \rho_0(x),\ v(0,x) = v_0(x),
\end{aligned} \tag{4.1}$$

where p is the pressure, v is the particle velocity along x-axis, t is the time and ρ is the density. The constant $m = 0$ corresponds to planar flow and $m = 1$ to cylindrically symmetric flow.

The pressure p is defined as

$$p = k\left(\rho/(1-Z)\right)^{\Gamma},$$

where, k is a positive constant and

$$\Gamma = \gamma(1+\lambda\beta)/(1+\lambda\beta\gamma),\ \lambda = k_p/(1-k_p),\ \beta = c_{sp}/c_p,\ \gamma = c_p/c_v. \tag{4.2}$$

The entity $Z = V_{sp}/V_g$ is the volume fraction and $k_p = m_{sp}/m_g$ is the mass fraction of the solid particles in the mixture where m_{sp} and V_{sp} are the total mass and volumetric extension of the solid particles and V_g and m_g are the total volume and total mass of the mixture respectively. Here c_{sp} is the specific heat of the solid particles, c_p the specific heat of the gas at constant pressure, and c_v the specific heat of the gas at constant volume. The relation between the entities Z and k_p is given by $Z = \theta\rho$, $\theta = k_p/\rho_{sp}$, with ρ_{sp} as the species density of the solid particles.

4.3 Solution of the problem

To carry out the characteristics analysis of system (4.1), we use the vector $V = \begin{pmatrix} \rho \\ v \end{pmatrix}$ of primitive variables. For smooth solutions system (4.1) is equivalent to

$$V_t + A(V)V_x + B(V) = 0, \tag{4.3}$$

where

$$A(V) = \begin{pmatrix} v & \rho \\ k\Gamma\rho^{\Gamma-2}/(1-\theta\rho)^{\Gamma+1} & v \end{pmatrix},\ B(V) = \begin{pmatrix} m\rho v/x \\ 0 \end{pmatrix}. \tag{4.4}$$

The eigenvalues of the matrix A can be calculated as $\lambda_1 = v - \sqrt{k\Gamma\rho^{\Gamma-1}/(1-\theta\rho)^{\Gamma+1}}$ and $\lambda_2 = v + \sqrt{k\Gamma\rho^{\Gamma-1}/(1-\theta\rho)^{\Gamma+1}}$.

Thus, the system is strictly hyperbolic. The Riemann invariants for the system (4.3)

Corresponding to the eigenvalues λ_1 and λ_2 are, respectively

$$R^{+} = v + \int_{\rho^*}^{\rho} \frac{\sqrt{k\Gamma y^{\Gamma-1} / (1-\theta y)^{\Gamma+1}}}{y} dy \quad \text{and} \quad R^{-} = v - \int_{\rho^*}^{\rho} \frac{\sqrt{k\Gamma y^{\Gamma-1} / (1-\theta y)^{\Gamma+1}}}{y} dy, \qquad (4.5)$$

where $\rho^* > 0$ is a fixed number.

From (4.5), it can be easily calculated that

$$\frac{\partial \lambda_1}{\partial R^-} = \frac{\partial \lambda_2}{\partial R^+} = \frac{1}{2} + \frac{\Gamma - 1 + 2\theta\rho}{4(1-\theta\rho)}, \frac{\partial \lambda_1}{\partial R^+} = \frac{\partial \lambda_2}{\partial R^-} = \frac{1}{2} - \frac{\Gamma - 1 + 2\theta\rho}{4(1-\theta\rho)}. \qquad (4.6)$$

For smooth solutions, system (4.3) is equivalent to

$$\begin{aligned} R_t^- + \lambda_1 R_x^- &= \sqrt{k\Gamma \rho^{\Gamma-1} / (1-\theta\rho)^{\Gamma+1}}\, m v x^{-1}, \\ R_t^+ + \lambda_2 R_x^+ &= -\sqrt{k\Gamma \rho^{\Gamma-1} / (1-\theta\rho)^{\Gamma+1}}\, m v x^{-1}, \ t > 0, \ x \in R, \end{aligned} \qquad (4.7)$$

subject to bounded and differentiable data

$$R^-(0,x) = v_0 - \int_{\rho^*}^{\rho_0} \frac{\sqrt{k\Gamma y^{\Gamma-1} / (1-\theta y)^{\Gamma+1}}}{y} dy, \ R^+(0,x) = v_0 + \int_{\rho^*}^{\rho_0} \frac{\sqrt{k\Gamma y^{\Gamma-1} / (1-\theta y)^{\Gamma+1}}}{y} dy. \qquad (4.8)$$

Through this reformulated system, there exists a uniform invariant region for the system (4.1), see Marcati et al. (2000). Thus there are constants $0 < \rho_{\min} < \rho_{\max}$ and $v_{\min} < v_{\max}$, depending only on the initial data (ρ_0, v_0), such that $(\rho(t,x), v(t,x)) \in D = [\rho_{\min}, \rho_{\max}] \times [v_{\min}, v_{\max}], \ \forall t \geq 0, x \in R.$

Therefore, the rest thing is to analyze the boundedness of (ρ_x, v_x), i.e., the boundedness of $R_x^{\pm}$.

4.4 Main result

To begin with, we need some conditions on the velocity v and the density ρ. When $m = 1$, the perturbation term $v\rho/x$ depends on the position x. We assume that the velocity v and the density ρ satisfy the following conditions as given in Stanyukovich (1960) and Sharma et al. (2010)

$$\begin{aligned} &v=(x/t)\,f_1(\xi)=t^{c_1-1}f_2(\xi)=x^{(c_1-1)/c_1}f_3(\xi),\; v(\infty,t)=0,\\ &\rho=t^{c_2}g_1(\xi)=x^{c_2/c_1}g_2(\xi). \end{aligned} \tag{4.9}$$

where $c_1, c_2 > 0,\; f_j\,(j=1,2,3)$ and $g_j\,(j=1,2)$ are bounded functions as the limit x or t goes to zero. Then, for sufficiently large x, v tends to zero and ρ tends to a constant.

To state our main results, we introduce the following notations:

$$\eta=-(1/2)\ln\rho+(1/2)\ln\sqrt{k\Gamma\rho^{\Gamma-1}\Big/(1-\theta\rho)^{\Gamma+1}}, \tag{4.10}$$

$$r^{\pm}=e^{\eta}R_x^{\pm}=e^{\eta}\left(v_x\pm\left(\sqrt{k\Gamma\rho^{\Gamma-1}\Big/(1-\theta\rho)^{\Gamma+1}}\Big/\rho\right)\rho_x\right). \tag{4.11}$$

From (4.11), it is clear that the boundedness of (ρ_x, v_x) is equivalent to the boundedness of $r^{\pm}$.

Theorem:

Let $\rho_0(x)>0$ and $v_0(x)$ be smooth functions with bounded C^1 norm. Assume that (4.9) holds and x is sufficiently large when $v=1$. Then there exists constants $0<\rho_{\min}<\rho_{\max}$ and $v_{\min}<v_{\max}$, depending only on the initial data (ρ_0, v_0), such that $\forall x\in R, (\rho(t,x), v(t,x))\in D=[\rho_{\min},\rho_{\max}]\times[v_{\min},v_{\max}]$. Meanwhile, the first order derivatives of the C^1 solution to the Cauchy problem have the following two complementary conclusions:

i. If for at least one point $x\in R$, either $r^+(0,x)<0$ or $r^-(0,x)<0$ holds, then the solution of system (4.1) will experience a finite time blow up at

   $0<t_*\le t^*<\infty$,

   $\lim_{t\to t_*} r^+(t,x)=+\infty$, or $\lim_{t\to t_*} r^-(t,x)=-\infty$.

where $\int_0^{t^*} a(\tau)d\tau = -1/r_0^- \left(or\ -1/r_0^+\right)$.

Moreover,

$$\begin{cases} \partial t^*/\partial c_0 \ge 0, \ for\ R_{0,x}^- \ge 2c_0\rho_{0,x}/\rho_0 \\ \partial t^*/\partial c_0 < 0, \ for\ R_{0,x}^- < 2c_0\rho_{0,x}/\rho_0 \end{cases} \quad \text{or} \quad \begin{cases} \partial t^*/\partial c_0 \ge 0, \ for\ R_{0,x}^+ \ge -2c_0\rho_{0,x}/\rho_0 \\ \partial t^*/\partial c_0 < 0, \ for\ R_{0,x}^+ < -2c_0\rho_{0,x}/\rho_0 \end{cases},$$

with

$$c_0 = \sqrt{k\Gamma\rho_0^{\Gamma-1}/(1-\theta\rho_0)^{\Gamma+1}} \ .$$

(ii) The solution of the system (4.1) admits a unique global bounded solution satisfying $0 \le r^{\pm}(t,x) \le r^{\pm}(0,x)$, provided that for all $x \in R$, $r^{\pm}(0,x) \ge 0.$

Lemma 1:

The dynamical system for $r^{\pm}$ is given by the following

$$\begin{cases} r_t^- + \lambda_1 r_x^- + a(r^-)^2 + gr^- + hr^+ + k = 0, \ t>0, x\in R, \\ r_t^+ + \lambda_2 r_x^+ + a(r^+)^2 - gr^- - hr^+ - k = 0, \ t>0, x\in R. \end{cases} \tag{4.12}$$

where

$$\begin{cases} a = \dfrac{\partial\lambda_1}{\partial R^-} e^{-\eta}, \\ k = e^{\eta}\, mv\sqrt{k\Gamma\rho^{\Gamma-1}/(1-\theta\rho)^{\Gamma+1}}\Big/x^2, \\ g = -(m/2x)\left(v + c - v(\Gamma-1+2\theta\rho)/(1-\theta\rho)\right), \\ h = -(m/2x)\left(v - c - v(\Gamma-1+2\theta\rho)/(1-\theta\rho)\right). \end{cases} \tag{4.13}$$

Proof:

Set $s^{\pm} = R_x^{\pm}$. Then from (4.11) we have, $r^- = e^{\eta} s^-$.

Differentiating the point equation in (4.7) with respect to x, we have

$$s_t^- + \lambda_1 s_x^- + \left(\frac{\partial\lambda_1}{\partial R^-} s^- + \frac{\partial\lambda_1}{\partial R^+} s^+\right) s^- = \left(mv\sqrt{k\Gamma\rho^{\Gamma-1}/(1-\theta\rho)^{\Gamma+1}}\; x^{-1}\right)_x . \tag{4.14}$$

However, as $r_t^- = e^{\eta}(s_t^- + \eta_t s^-),\ r_x^- = e^{\eta}(s_x^- + \eta_x s^-)$, using (4.14) and (4.7), we have

$$r_t^- + \lambda_1 r_x^- = -\frac{\partial \lambda_1}{\partial R^-} e^{-\eta}\left(r^-\right)^2 + r^-[\eta_t + \lambda_1 \eta_x - \frac{\partial \lambda_1}{\partial R^+} s^+] + e^{\eta}\left(mv\sqrt{k\Gamma \rho^{\Gamma-1} \big/ (1-\theta\rho)^{\Gamma+1}}\ x^{-1}\right)_x . \tag{4.15}$$

It may be noticed that

$$\eta_t + \lambda_1 \eta_x - \frac{\partial \lambda_1}{\partial R^+} s^+ = \left(mv/2x\right)\left(1 - \left(\Gamma - 1 + 2\theta\rho\right)\big/\left(2\left(1-\theta\rho\right)\right)\right). \tag{4.16}$$

Next, we estimate $e^{\eta}\left(mv\sqrt{k\Gamma \rho^{\Gamma-1} \big/ (1-\theta\rho)^{\Gamma+1}} \big/ x\right)_x$. From (4.5), it is easy to check that

$$v_x = \left(s^+ + s^-\right)/2,\ c_x = \left(s^+ - s^-\right)\left(\Gamma - 1 + 2\theta\rho\right)\big/\left(4\left(1-\theta\rho\right)\right). \tag{4.17}$$

Noting that $r^- = e^{\eta} s^-$, we obtain

$$\begin{aligned} e^{\eta}\left(\frac{mv\sqrt{k\Gamma \rho^{\Gamma-1} \big/ (1-\theta\rho)^{\Gamma+1}}}{x}\right)_x &= \frac{m}{2x}\left(\sqrt{k\Gamma \rho^{\Gamma-1} \big/ (1-\theta\rho)^{\Gamma+1}} + \frac{\Gamma - 1 + 2\theta\rho}{2(1-\theta\rho)} v\right) r^+ \\ -\frac{mv\sqrt{k\Gamma \rho^{\Gamma-1} \big/ (1-\theta\rho)^{\Gamma+1}}}{x^2} e^{\eta} &+ \frac{m}{2x}\left(\sqrt{k\Gamma \rho^{\Gamma-1} \big/ (1-\theta\rho)^{\Gamma+1}} - \frac{\Gamma - 1 + 2\theta\rho}{2(1-\theta\rho)} v\right) r^- . \end{aligned} \tag{4.18}$$

Finally, combining (4.6), (4.15), (4.16) and (4.18), we end up with

$$r_t^- + \lambda_1 r_x^- + a(r^-)^2 + g r^- + h r^+ + k = 0 .$$

The second equation is derived in a similar way. Thus Lemma 2 is proved.

Proof of the Main result:

In this section, we shall prove our main result. First recall the following known result.

Lemma 2:

Assume that $A(t)$ satisfies the following ordinary differential equations:

$$dA/dt + q(t)[A - b_1(t)][A - b_2(t)] = 0,\ A(0) = A_0 , \tag{4.19}$$

with $\inf q > 0, b_1 \le b_2$ and that q, b_1, b_2 are uniformly bounded. We have:

(i) If $A_0 < \min b_1$, then solution to (4.19) will experience a finite time blow up at

$0 < t_* \le t^* < +\infty$,

$\lim_{t \to t_*} A(t) = -\infty,$

where t^* satisfies

$$\int_0^{t^*} q(\tau) d\tau = \left(1/(\min b_2 - \min b_1)\right) \ln\left[1 + \left(\min b_2 - \min b_1\right) / \left(\min b_0 - A_0\right)\right],$$

which equals $1/\left(\min b_2 - A_0\right)$ if $\min b_2 = \min b_1$.

(ii) If there exists a constant $\bar{b}$ such that

$b_1(t) \le \bar{b} \le b_2(t).$

Then, system (4.19) admit a unique global bounded solution satisfying

$\bar{b} \le A(t) \le \max\{A_0, \max b_2\}$, provided that, $A_0 \ge \bar{b}.$

The proof of Lemma 2 can be found in Li et al. (2009), Lemma 3.1. Based on Lemma 2, the proof of Theorem is given below.

Proof of Theorem:

Denoting $x = x(t, \beta)$ by the characteristic passing through any fixed point $(0, \beta)$ on the initial axis $t = 0$, a short calculation shows that $r^-(t, x)$ satisfies the following Cauchy problem for ordinary differential equation along $x = x(t, \beta)$

$$\frac{dr^-}{dt} + a(r^-)^2 + gr^- + hr^+ + k = 0,\ r^-(0, x) = r_0^- = e^{\eta} R_{0,x}^-(\beta), \tag{4.20}$$

where a, g, h and k are given by (4.13). It is well known that the smooth solution of the Cauchy problem (4.1) will develop singularities in the first derivative even for smooth initial data see Lax (1964), and the smooth solution will not exist for a large time t . In the sequel, we will discuss $m = 0$ and $m = 1$, respectively.

For $m = 0$, from (4.13), we know that $g = h = k = 0.$

Thus the equation (4.20) can be rewritten as

$$\frac{dr^-}{dt}+a(r^-)^2=0, r^-(0,x)=r_0^-=e^{\eta(\rho_0)}R_{0,x}^-(\beta)\,. \tag{4.21}$$

It can be easily calculated that

$$2c'(\rho)\rho\le c(\rho)\,. \tag{4.22}$$

Next, for $m=1$, from (4.9), (4.13) and (4.22), for large x, we know that $|g|,|h|,|k|\approx 0$.

Therefore, for sufficiently large x, $0<|g|,|h|,|k|<<1$. Then r^- is a perturbation of the ordinary differential equation (4.21).

On the other hand, from (4.6), we have

$$a=\rho^{1/2}\left(k\Gamma\rho^{\Gamma-1}\big/(1-\theta\rho)^{\Gamma+1}\right)^{-1/4}\left(1/2+(\Gamma-1+2\theta\rho)\big/\left(2(1-\theta\rho)\right)\right),$$
$$r_0^-=\rho_0^{-1/2}\left(k\Gamma\rho_0^{\Gamma-1}\big/(1-\theta\rho_0)^{\Gamma+1}\right)^{1/4}R_{0,x}^-(\beta). \tag{4.23}$$

From (4.23), we know that $a>0$ is uniformly bounded for all time. Thus considering the initial data r_0^-, by Lemma 2, we have the following two cases:

(i) If $r_0^-<0$, i.e., $R_{0,x}^-(\beta)<0$, there exists a finite time t_*, such that

$$\lim_{t\to t_*}r^-(t)=-\infty, 0<t_*<t^*<\infty\ ,$$

where t^* satisfying

$$\int_0^{t^*}a(\tau)d\tau=-1\big/r_0^-=-\rho_0^{1/2}c_0^{-1/2}\big/R_{0,x}^-\ ,$$

with $c_0=c(\rho_0)$. The above equation implies that t^* depends on c_0, indeed differentiating the above equation with respect to c_0, we have

$$a(t^*)\,\partial t^*\big/\partial c_0=\rho_0^{1/2}c_0^{-3/2}\big/2(R_{0,x}^-)^2\left[R_{0,x}^- - 2\,c_0\rho_{0,x}\big/\rho_0\right]\ .$$

Thus, for $2c_0\rho_{0,x}/\rho_0 \le R^-_{0,x}$, we have $\partial t^*/\partial c_0 \ge 0$, , and for $R^-_{0,x} < 2c_0\rho_{0,x}/\rho_0$, we have $\partial t^*/\partial c_0 < 0$.

(ii) If $r_0^- \ge 0$, i.e., $R^-_{0,x}(\beta) \ge 0$, then system (4.21) admits a unique global solution satisfying $0 \le r^-(t) \le r_0^-$.

The analysis of $r^+(t)$, in system (4.12) is performed in a similar way. It should be pointed out that the blow up time t^*, satisfies

$$\int_0^{t^*} a(\tau)d\tau = -1/r_0^+ = -\rho_0^{1/2}c_0^{-1/2}/R^+_{0,x} \ .$$

Differentiating above equation with respect to c_0 , we have

$$a(t^*)\,\partial t^*/\partial c_0 = \left(\rho_0^{1/2}c_0^{-3/2}/2(R^+_{0,x})^2\right)\left(R^+_{0,x} + 2c_0\rho_{0,x}/\rho_0\right) \ .$$

Thus, for $-2c_0\rho_{0,x}/\rho_0 \le R^+_{0,x} < 0$, we have $\partial t^*/\partial c_0 \ge 0$, and for $R^+_{0,x} < -2c_0\rho_{0,x}/\rho_0$, we have $\partial t^*/\partial c_0 < 0$.

Hence, the proof.

4.5 Conclusion:

This chapter deals with the issues of global solution in time regularity and finite singularity formation for a dusty gas system. By using Riemann invariants, we have shown that the system has global smooth solutions under certain reasonable assumptions. When the assumptions on the initial data do not hold, we have observed the blow up phenomena of the C^1 solution to the system. It can be observed that the blow up phenomena of the solution also depend on the mass fraction, volume fraction and specific heat of dust particles.