Chapter 2

On the decay of a sawtooth profile in non-ideal magnetogasdynamics

2.1 Introduction

Discontinuity waves, such as shock waves, acceleration waves and weak waves are characterized by discontinuity in the normal derivative of the flow variable rather than the variable itself. Therefore, for nonlinear systems, the analysis of these waves has been the subject of great interest both from mathematical and physical point of view. For the physical phenomenon modelled by a system of quasi-linear hyperbolic partial differential equations, it is theoretically possible to find the progressive wave solution. Choquet Bruhat (1969) used the perturbation method to determine a shockless solution of a system of quasi linear hyperbolic partial differential equations that depend upon single phase function. Germain (1972), Fusco (1982), Fusco and Engelbrecht (1984), and Sharma *et al*. (1987) used the same technique to analyze the nonlinear wave propagation in various gasdynamic regimes. Hunter and Keller (1983) presented a method, known as ray method, to determine a small-amplitude high frequency wave solution of hyperbolic system. Singh *et al*. (2011) have studied the problem of propagation of acceleration waves along the characteristic path by using the characteristics of the governing system as the reference coordinate system.

In the present work, we deal with the study of propagation of weakly nonlinear waves in a non ideal gas permeated by a transverse magnetic field with infinite electrical

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conductivity. An evolution equation, characterizing the wave process in the high frequency domain, is derived. The growth equation for an acceleration wave is recovered as a special case. The propagation of a sawtooth profile that ends in a tail shock can be analyzed in similar manner.

2.2 Governing equations

The fundamental equations for one dimensional unsteady motion of a non-ideal gas in the presence of a transverse magnetic field may be written as (Whitham (1974), Korobeinikov (1976) and Wu (1996))

$$\rho_t + v\rho_x + \rho v_x + \rho m v x^{-1} = 0 \ , \tag{2.1}$$

$$v_t + v v_x + \rho^{-1}(p_x + h_x) = 0 \ , \tag{2.2}$$

$$p_t + v p_x + \rho d^2 (v_x + m v x^{-1}) = 0 \, , \tag{2.3}$$

$$h_t + v h_x + 2h(v_x + m v x^{-1}) = 0 \, , \tag{2.4}$$

where ρ is the density, v the fluid velocity, p the pressure, $d = \gamma p / \rho(1 - b\rho)$ is the speed of sound in non-ideal gas with γ as the adiabatic index, b is the van der Wall's constant, $h = \mu H^2 / 2$ the magnetic pressure with H as the magnetic field strength, μ is the magnetic permeability, t the time, and x the spatial coordinate. Here subscripts denote partial differentiation unless stated otherwise. The letter m takes values 0 for planar and 1 for cylindrically symmetric motion.

In matrix notation, Equation (2.1)-(2.4) can be written as

$$U_t + AU_x + B = 0 \, , \tag{2.5}$$

where,

$$U=\begin{bmatrix}\rho\\ v\\ p\\ h\end{bmatrix},\quad A=\begin{bmatrix}v & \rho & 0 & 0\\ 0 & v & \rho^{-1} & \rho^{-1}\\ 0 & \dfrac{\gamma p}{1-b\rho} & v & 0\\ 0 & 2h & 0 & v\end{bmatrix},\quad B=\begin{bmatrix}\rho m v x^{-1}\\ 0\\ \dfrac{\gamma p}{1-b\rho} m v x^{-1}\\ 2hmvx^{-1}\end{bmatrix}.$$

Equation (2.5) can be written as

$$U_t^i + A^{ij} + B^i = 0, \qquad i, j = 1, 2, 3, 4, \tag{2.6}$$

where U^i, A^{ij}, B^i are components of column vector U, matrix A and column vector B respectively.

The system (2.6) is hyperbolic and eigenvalues of the coefficient matrix A are $v-c,\ v,\ v$ and $v+c$. Here $c=(d^2+e^2)^{1/2}$ is the magneto-sonic speed with $d=(\gamma p/\rho(1-b\rho))^{1/2}$ as the speed of sound in non-ideal gas and $e=(2h/\rho)^{1/2}$ the Alfvén speed. The left and right eigenvectors of A corresponding to the eigenvalue $v+c$ are

$$l=(0,\ \rho c,\ 1,\ 1),\ r^T=(1,\ c/\rho,\ d^2,\ e^2), \tag{2.7}$$

where a superscript means transposition.

2.3 Progressive wave solution

Let us consider the asymptotic solution of equation (2.6) which exhibits the feature of progressive waves. Consider the following asymptotic expansion

$$U^i(x,t) = U_0^i + \varepsilon U_1^i(x,t,\xi) + O(\varepsilon^2), \tag{2.8}$$

where U_0^i is a known constant solution of (2.6) such that $B^i(U_0)=0$. The remaining terms of Equation (2.8) are of progressive wave nature. The choice of ε depends upon the physical problem to be studied. Let τ_{ch} be the characteristic time scale for the medium and τ_a be the attenuation time; then we define a parameter $\varepsilon = \tau_{ch} / \tau_a << 1$. The variable ξ is a ''fast variable'' defined as $\xi = f(x,t)/\varepsilon$, where $f(x,t)$ is a phase function to be determined later. It may be noticed that the case $\varepsilon << 1$, which corresponds to the situation in which the characteristic frequency of the medium is very large than the attenuation frequency of the signal, characterizing a high frequency propagation (Seymour, 1970).

Introducing the Taylor's series expansion of A^{ij} and B^i in the neighborhood of the known constant solution U_0^i and using equation (2.8), we get

$$A^{ij} = A_0^{ij} + \varepsilon\left(\frac{\partial A^{ij}}{\partial U^k}\right)_0 U_1^k + O(\varepsilon^2), \tag{2.9}$$

$$B^i = B_0^i + \varepsilon\left(\frac{\partial B^i}{\partial U^k}\right)_0 U_1^k + O(\varepsilon^2) \ . \tag{2.10}$$

Substituting Equations (2.8)-(2.10) in (2.6) and cancelling the coefficient of ε^0 and ε^1, we get

$$(A_0^{ij} - \lambda\delta_j^i)\frac{\partial U_1^j}{\partial \xi} = 0, \tag{2.11}$$

$$(A_0^{ij} - \lambda\delta_j^i)\frac{\partial U_2^j}{\partial \xi} + \left(\frac{\partial U_1^i}{\partial t} + A_0^{ij}\frac{\partial U_1^j}{\partial x}\right) f_x^{-1} + U_1^k\left(\frac{\partial A^{ij}}{\partial U^k}\right)_0 \frac{\partial U_1^j}{\partial \xi} + f_x^{-1} U_1^k \left(\frac{\partial B^i}{\partial U^k}\right)_0 = 0, \tag{2.12}$$

where $\lambda = -f_t / f_x$, δ_j^i is the Krönecker delta and the subscript 0 means the quantity involved is evaluated at constant state U_0. Equation (2.11) yields the characteristic polynomial $\lambda^2(\lambda^2 - c^2) = 0$, providing non zero eigenvalues $\pm c_0$ of A_0. Considering the velocity $\lambda = c_0$ the corresponding left and right eigenvectors of A_0 are given by (2.7) with subscript 0. From equation (2.11) we see that $\partial u / \partial \xi$ is collinear to r_0 and therefore U_1 may be written as

$$U_1(x,t,\xi)=\alpha(x,t,\xi)r_0+W(x,t), \tag{2.13}$$

representing a solution of equation (2.11). Here $\alpha(x,t,\xi)$ is the amplitude factor to be determined and the W^i (the components of the column vector W) are integration constants which are not of progressive wave nature and therefore can be taken as zero. Now the phase function $f(x,t)$ is determined by

$$f_t+c_0 f_x=0 \,, \tag{2.14}$$

$f(x,0)=r-r_0$, then

$$f(x,t)=(x-x_0)-c_0t\,. \tag{2.15}$$

Multiplying equation (2.12) by l_0^i , and using (2.14) we obtain, the following evolution equation for α

$$\frac{\partial\alpha}{\partial\tau}+P_0\alpha\frac{\partial\alpha}{\partial\xi}+Q_0\alpha=0\,, \tag{2.16}$$

where $\frac{\partial}{\partial\tau}=\frac{\partial}{\partial t}+c_0\frac{\partial}{\partial x}$ is the ray derivative taken along the ray direction and

$$P_0=r_0^k\left(\frac{\partial(v+c)}{\partial U^k}\right)_0=\frac{(\gamma+1)d_0^2+3e_0^2(1-b\rho_0)}{2c_0\rho_0(1-b\rho_0)}>0 \quad ,$$

$$Q_0=\frac{l_0^j r_0^k}{l_0^i r_0^i}\left(\frac{\partial B^j}{\partial U^k}\right)_0=\frac{mc_0}{2x}\,.$$

Here Q_0^{-1} has the dimension of time and may be taken as having attenuation time τ_a characterizing the medium. Equation (2.16) is hyperbolic one and its characteristics curves can be obtained in the following form

$$\xi=\begin{cases}\xi_0+\tau P_0\phi(x_0,\xi_0), & \text{for } m=0,\\ \xi_0+2P_0\phi(x_0,\xi_0)\left(x_0/c_0\right)\left\{(1+c_0\tau/x_0)-1\right\}, & \text{for } m=1.\end{cases} \tag{2.17}$$

The existence of an envelope of the characteristics given by (2.17) gives evidence of the formation of a shock. It is evident that the shock is formed for $\tau > 0$ only by those characteristics for which $\partial\phi / \partial\xi_0 < 0$. The shock formation time for plane $(m = 0)$ and cylindrical $(m = 1)$ compressive waves turns out to be

$$\tau_{sh} = \begin{cases} \min\left(P_0 \left|\partial\phi/\partial\xi_0\right|\right)^{-1}, & \text{for plane waves.} \\ \min\left[\left(x_0/c_0\right)\left\{\left(1 + c_0/\left(2x_0 P_0 \left|\partial\phi/\partial\xi_0\right|\right)\right)^2 - 1\right\}\right], & \text{for cylindrical waves.} \end{cases} \tag{2.18}$$

where the minimum is evaluated over an appropriate range of the quantities x_0, ξ_0.

2.4 Acceleration waves

We can use the aforementioned analysis to study acceleration waves for the system of equations (2.1)-(2.4). Let us suppose that $f(x,t) = 0$; represents the acceleration front. Across such a front the velocity is continuous but its first and higher order derivatives undergo finite jump discontinuities. In the neighborhood of the front, the velocity v may be represented by an expansion

$$v = \varepsilon v_1(x,t,\xi) + O(\xi^2), \tag{2.19}$$

where $v_1 = 0$ for $\xi < 0$, and $v_1 = O(\xi)$ for $\xi > 0$. Now v_1 as an element of the column vector U_1 is given by (2.13), so we have (Germain, 1972)

$$\alpha(x,t,\xi) = \begin{cases} 0, & if\ \xi < 0, \\ \xi\beta(x,t) + O(\xi^2), & if\ \xi > 0, \end{cases} \tag{2.20}$$

with $\beta = (\rho_0 / c_0)\sigma$, where $\sigma = [\partial v / \partial x]$ denotes the jump in velocity gradient across the acceleration front.

Using (2.20) in (2.16), and evaluating the resulted equation at the front $f(x,t) = 0$; i.e., at the front $\xi = 0$, we obtain a Bernoulli type equation

$$\frac{d\sigma}{dt} + Q_0\sigma + \Pi_0\,\sigma^2 = 0\ , \tag{2.21}$$

where

$$\Pi_0 = \frac{1}{2}\{\gamma - 2 + (\gamma+1)b\rho_0 + 3\psi\}\psi^{-1}, \tag{2.22}$$

with $\psi = 1 + e_0^2 / d_0^2$, the Alfvén number, $Q_0 = mc_0 / 2x$, and the derivative d/dt of any quantity , which is supposed to be expressed on the front $f(x,t) = 0$, is the ordinary time derivative of the quantity. The solution of the equation (2.21), can be written as (Menon et al., 1981)

$$\sigma = \begin{cases} \sigma_0/(1+\sigma_0\,\Pi_0\,t)\ ; & \text{for } m = 0, \\ \sigma_0\left(1+c_0t/x_0\right)^{-1/2} K^{-1}\ ; & \text{for } m = 1, \end{cases} \tag{2.23}$$

where

$$K = \left\{1 + 2\,\Pi_0/c_0\left(\left(1+c_0t/x_0\right)^{1/2} - 1\right)\sigma_0 x_0\right\}\ , \tag{2.24}$$

and σ_0 is the value of σ evaluated at $t = 0$.

2.5 Weak shock

The above analysis shows that a compression pulse always evolves in a shock in a finite time, however weak it may be in the beginning. The flow and field variables ahead and behind the shock designated respectively by the subscripts 0 and 1 and introducing the shock strength parameter $\delta = (\rho_1 - \rho_0)/\rho_0$, satisfy the following shock conditions (Korobeinikov, 1976)

$$\left.\begin{aligned} \rho_1 &= \rho_0(1+\delta), \\ v_1 &= \delta G/(1+\delta), \\ h_1 &= h_0(1+\delta)^2, \\ p_1 &= p_0 + \rho_0\delta G^2(1+\delta)^{-1} - h_0\delta(2+\delta), \end{aligned}\right\} \tag{2.25}$$

where the shock strength parameter δ and the shock velocity G are related by

$$G^2 = 2(1+\delta)\left\{d_0^2 + e_0^2\left((1-b\rho_0\delta)\left(1+\frac{\delta}{2}\right)-(\gamma-1)\frac{\delta}{2}(1+b\rho_0)\right)\right\}$$

$$\left\{2(1-b\rho_0\delta)-\delta(\gamma-1)(1+b\rho_0)\right\}^{-1}. \qquad (2.26)$$

For a weak shock $\delta << 1$ and therefore we have the first approximation to the equations in (2.25) and (2.26) as

$$\left.\begin{aligned} &\rho_1 = \rho_0(1+\delta), v_1 = c_0\delta, p_1 = p_0(1+\gamma\delta(1+b\rho_0)) \\ &h_1 = h_0(1+2\delta), G = c_0\left[1+\delta\prod_0/2\right] \end{aligned}\right\}. \qquad (2.27)$$

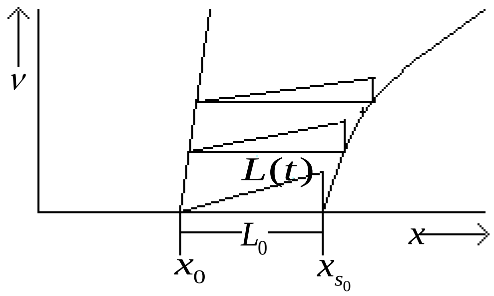

Figure 2.1 Sawtooth profile

2.6 Behaviour of sawtooth profile

The shock waves, after travelling a long distance from the source become weak enough so that we can apply the weak shock relations (2.27). Therefore we assume a shock, which is weak enough at the beginning and investigate the propagation of the fluid velocity disturbance given in the form of a sawtooth profile as shown in fig. 2.1, (Zirep, 1978).

The left portion of the profile which was situated initially at x_0 travels with the magnetosonic speed c_0 of the undisturbed fluid, whereas the shock at the right portion situated initially at x_{s_0} moves faster; suppose L_0 is the length of the sawtooth profile in the beginning. Suppressing the subscript 1 notation, let us denote by v and c the state at the rear side of the shock, which at time t is located at $x_s(t) = x_0 + c_0 t + L(t)$, where $L(t)$ is the length of the sawtooth profile at any time t. Then

$$G = \frac{dx_s}{dt} = c_0 + \frac{dL}{dt} \quad . \tag{2.28}$$

Also, from the second and fifth equation of (2.27), we obtain

$$G = c_0 + v\prod_0 /2 . \tag{2.29}$$

The fluid velocity v in the sawtooth course with constant $\partial v / \partial x$ can be described as

$$v = \sigma L(t) , \tag{2.30}$$

where $\sigma = (\partial v / \partial x)_{x - x_0 = c_0 t}$, the slope of the profile at any is time t, and is given by (2.23).

Using (2.30) in (2.29) and comparing the resulting equation with (2.28), we obtain

$$\frac{dL}{dt} = \frac{\sigma L \prod_0}{2} . \tag{2.31}$$

Let σ_0, L_0 and G_0 be the value of σ, L and G, respectively at $t = 0$. Then equation (2.29) and (2.30) gives the following relation connecting σ_0, L_0 and G_0

$$\sigma_0 = 2(G_0 - c_0) / L_0 \prod_0 . \tag{2.32}$$

From Equation (2.31), we have the following relation for the length of the sawtooth profile

$$\frac{L}{L_0} = \begin{cases} (1 + \sigma_0 \prod_0 t)^{1/2} & ; \quad \text{for plane waves,} \\ K^{1/2} & ; \quad \text{for cylindrical waves,} \end{cases} \tag{2.33}$$

where K is given by (2.24). Using (2.23) and (2.33) in (2.30), we obtain the relation for the velocity of sawtooth profile as

$$\frac{v}{v_0}=\begin{cases}(1+\sigma_0\prod_0 t)^{-1/2}\ ; & \text{for plane waves,}\\ \left(1+c_0t/x_0\right)^{-1/2}K^{-1/2}; & \text{for cylindrical waves,}\end{cases} \tag{2.34}$$

where v_0 is the value of v evaluated at $t=0$.

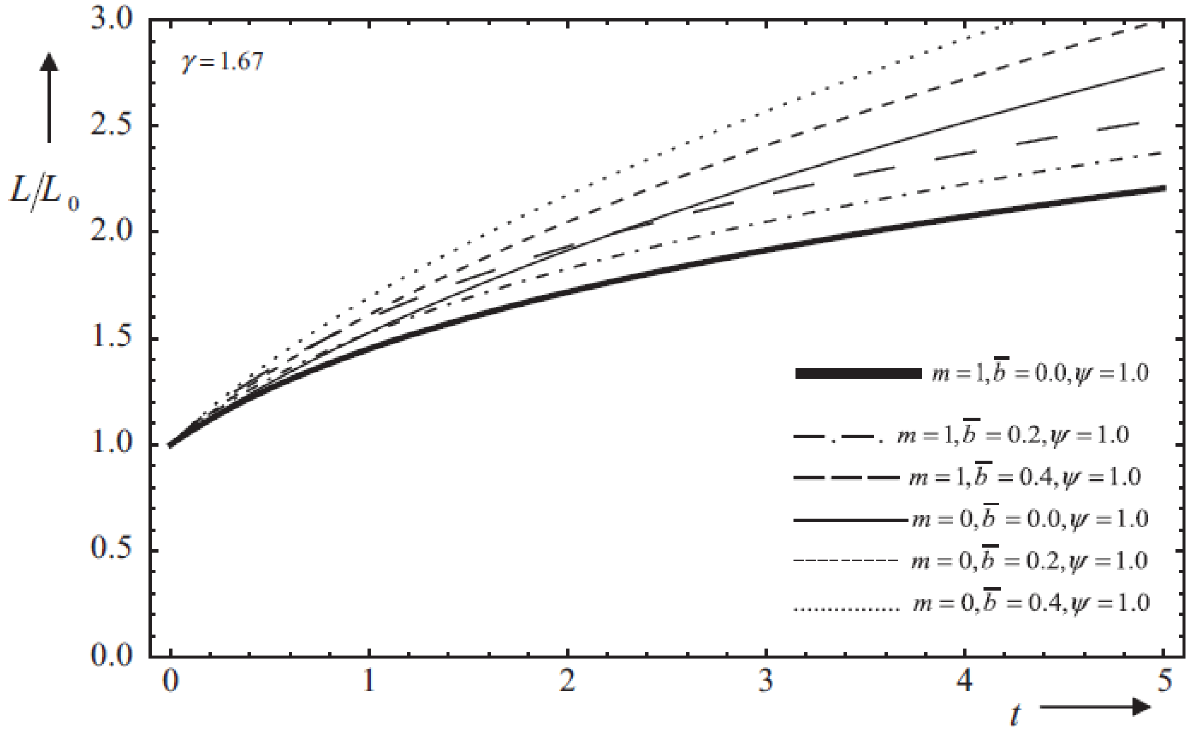

Figure 2.2 Length of sawtooth profile with time in non-magnetic case

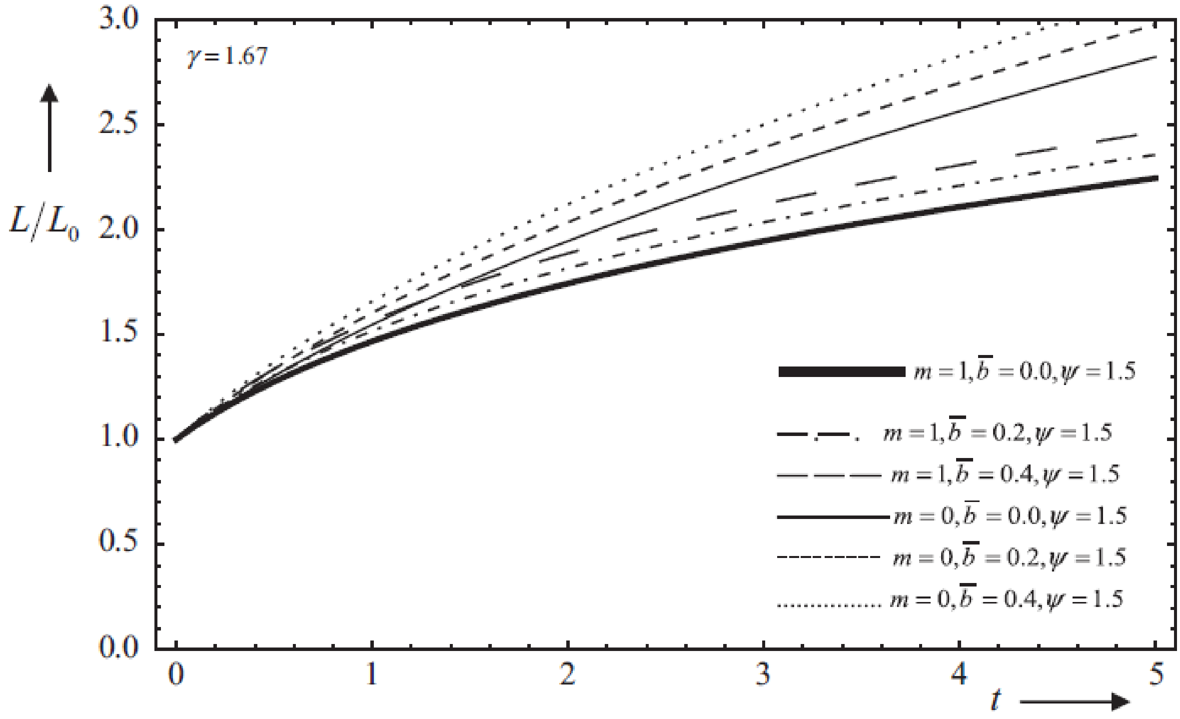

Figure 2.3 Length of sawtooth profile with time in magnetic case

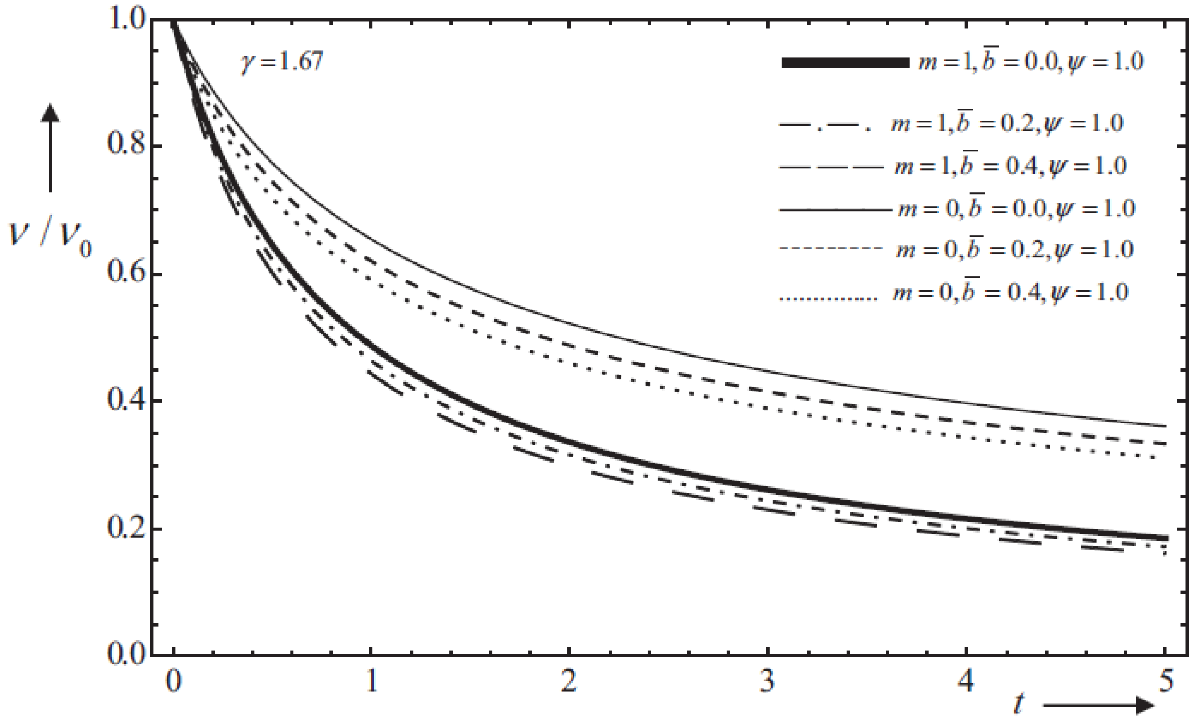

Figure 2.4 Velocity of sawtooth profile in non-magnetic case

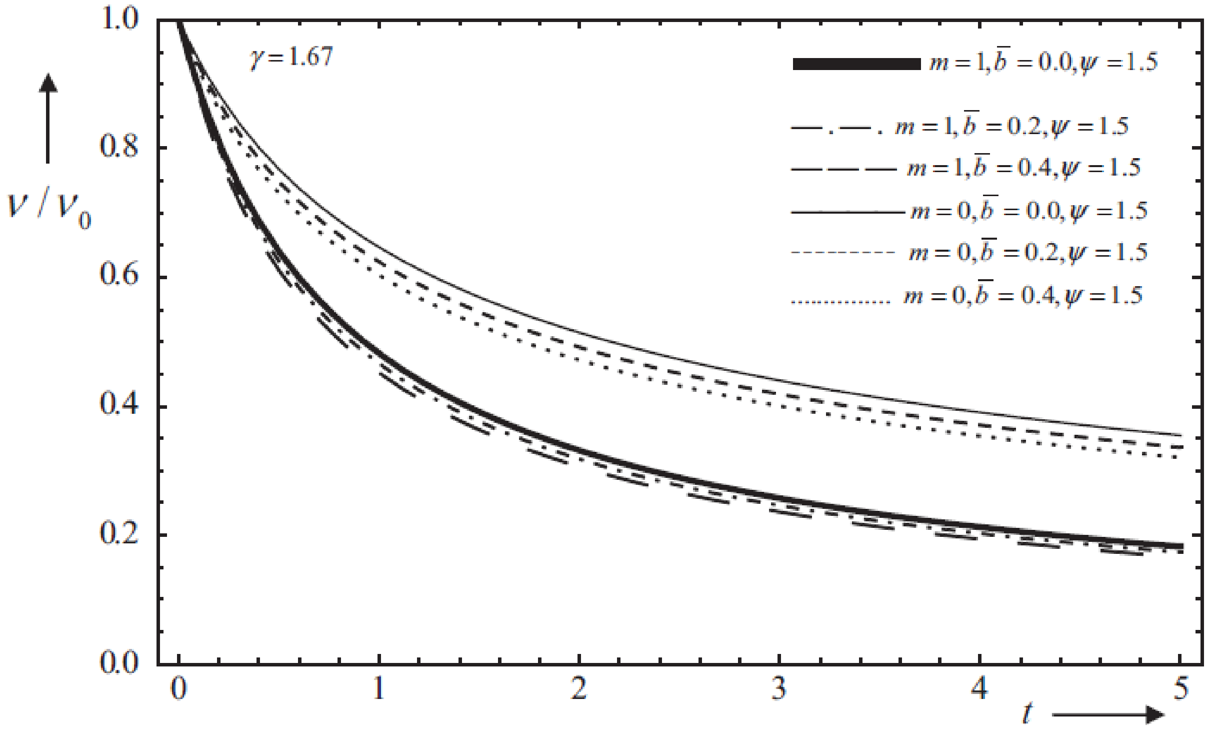

Figure 2.5 Velocity of sawtooth profile in magnetic case

2.7 Result and Discussion

Equations (2.33) and (2.34) govern the variation in the length and velocity of the sawtooth wave with time respectively. Figures (2.2) and (2.4) represent the graph of the length and velocity respectively, of sawtooth profile versus time in non-magnetic case. Figures (2.3) and (2.5) represent the graph of the length and velocity respectively, of sawtooth profile versus time when magnetic field is present. The length L/L_0 and velocity v/v_0 are computed using (2.33) and (2.34), after non-dimensionalising, for various values of parameter of non-idealness $\bar{b} = b\rho_0$ and magnetic field strength ψ for planar and cylindrically symmetric flows and presented in Figures (2.2), (2.3), (2.4) and (2.5) respectively. In case of an ideal magnetogasdynamics the results are in close agreement with Sharma et al. (1987).

It is observed that the length L/L_0 of sawtooth profile increases with time, whereas the velocity decreases with time which is expected. It is observed from figures (2.2) and (2.4) that the effect of increasing values of parameter of non-idealness $\bar{b}$ is to increase the length of sawtooth profile whereas the same effect produces a decreasing trend in the velocity of the sawtooth profile, see figures (2.4) and (2.5). This implies that the non-idealness of the gas causes an early decay of the sawtooth wave as compared to ideal case. From figures (2.2) and (2.3) it may also be noted here that the effect of non-idealness in the presence of magnetic field is to slow down the decay process as compared to non-ideal non-magnetic case. Also, the effect of non-idealness is more dominant in case of cylindrical symmetry as compared to plane case, as can be seen from figure (2.4) and (2.5). From (2.22) we also observe that for $\gamma = 2$ the magnetic field effects contribute in decay behaviour of sawtooth profile in non-ideal magnetogasdynamics which is in contrast to ideal magnetogasdynamics case given in Sharma (1987).

2.8 Conclusion

In the present study, a progressive wave analysis is used to determine the asymptotic solution of the system of nonlinear hyperbolic partial differential equations governing the non-ideal magneto-gasdynamic flow. The analysis leads to an evolution equation, which characterizes the wave process in the high frequency domain and points out the possibility of wave breaking at a finite time, is derived. The growth equation governing the behaviour of an acceleration wave is also recovered as a special case. Further, we consider a sufficiently weak shock at the outset and study the propagation of the disturbance given in the form of a sawtooth profile. It is observed that the non-idealness of the gas causes an early decay of the sawtooth wave as compared to ideal case however the presence of magnetic field causes to slow down the decay process as compared to non-ideal non-magnetic case. The effect of non-idealness, in the presence of magnetic field, on the formation of shock is more dominant in case of cylindrical symmetry as compared to planar case. Also, as an important case, for $\gamma = 2$, the magnetic field effect contributes in decay process of the sawtooth profile in non-ideal magnetogasdynamics which is in contrast to ideal magnetogasdynamics case.