Chapter 1

Introduction

1.1 Non linear waves

In many physical and real life problems such as supersonic flight of objects in a medium, detonation of explosives, flow in shock tube and dam break problem etc., the generated waves are in contrast with the linear theory of waves. In fact the motion of these waves is governed by the quasilinear partial differential equations. These waves are called nonlinear waves and the principles of superposition, reflection etc., do not apply to these waves. One of the distinguishing features of these waves is the occurrence of jump discontinuity. These discontinuities may be shock wave or at times contact wave. Across the shock wave a sudden change in the physical parameters of the flow such as velocity, pressure, density occurs. Across the contact wave the jump in flux occurs. A shock wave is an admissible discontinuity that satisfies the Rankine-Hugoniot conditions and the entropy criteria. The nonlinear nature of the flow may give rise to shock discontinuities even if, starting with a continuous motion. Other contrasting situation may occur, i.e. the initial discontinuity may be smoothed out (Courant and Friedrichs). The interaction of nonlinear waves is another interesting phenomenon. The decay behaviour of shock waves is most important in the case of weak shock waves.

When the thermodynamic properties are known, the instantaneous state of a moving gas can be characterized in terms of its physical parameters e.g. density, pressure, velocity etc, as a function of space and time. These functions are, in turn, related via partial differential equations that express the general laws of conservation of mass, momentum, and energy.

There is no general theory available for the solution of nonlinear PDEs. The determination of closed form solutions to quasilinear hyperbolic system of PDEs is of great interest but an uphill task. For most of the problems, closed form solution have not been possible until now. These problems are solved by approximate analytical and numerical methods to get insight of the physical process involved. The problems of high speed flows arising in the real life situations are highly nonlinear and complex. As the nonlinear theory is not fully developed, for most of the problems, exact or closed form solution is not possible. One has to resort to some approximate method. After development of personnel computers, numerical methods have got prominence due to their capability of prediction of the approximate solution. One takes recourse to numerical solution of a problem if the analytical solution is nearly impossible. Another situation is that the analytical solution of the problem can be found but it is time consuming and the error arising due to numerical approximation is tolerable.

Nonlinear partial differential equations related to the wave propagation occupy a position of substantial importance both from the point of view of the theory of partial differential equations and because of the various situations under which these equations are applicable. The notion of wave that we are concerned in this thesis is the propagation of disturbance that may or may not be of localized type. This form of wave is essentially associated with the motion involving the space $\mathbb{R}^n$ and dependence on time t. The partial differential equations representing the wave propagation can be hyperbolic or parabolic in nature; the present study is limited to the quasilinear hyperbolic systems only.

The concept of characteristic hypersurface and their geometry plays a central role in the mathematical theory of quasilinear hyperbolic partial differential equations. Across these hypersurfaces the solution may not exist, and when it does, the solution may have Lipschitz discontinuities in its first or higher order normal derivatives. The characteristic hypersurfaces behave like carrier of these discontinuities, when they exist, just as they also transport elements of a solution hypersurface when it is differentiable (smooth). In one dimensional time dependent equations these characteristic hypersurfaces reduce to the families of characteristics curves in the (x,t)- plane, along each of which may be transported a Lipschitz discontinuity in the first derivative of the solution normal to the characteristics. The solution hypersurface itself then reduces to an ordinary smooth surface on which a Lipschitz discontinuity in the first derivative of the solution normal to a characteristic curve manifests itself in the form of a crease on the surface. This crease in the solution surface, or its analogue in $\mathbb{R}^n \times t$, may be interpreted as representing a clearly defined propagating wavefront. The solution on the side of the wavefront towards which propagation takes place may then be regarded as being the ‘undisturbed solution’ ahead of the wavefront, whilst the solution on the other side may be regarded as a propagating disturbance wave’ which is entering a region occupied by the undisturbed solution.

1.2 Hyperbolic partial differential equations

A general quasilinear system of first order equations in $\mathbb{R}^n \times t$ may be written as

$$A_0(\underline{U}, \underline{x}, t)\underline{U}_t + \sum_{i=1}^{n} A_i(\underline{U}, \underline{x}, t)\underline{U}_{x_i} + B(\underline{U}, \underline{x}, t) = 0\,, \tag{1.1}$$

where the vector $\underline{U}(\underline{x}, t)$ is a column vector consisting of the elements $u_1(\underline{x}, t), ..., u_m(\underline{x}, t)$, $\underline{x} = (x_1, x_2, ..., x_n)$ is a vector in $\mathbb{R}^n$, $A_i(\underline{U}, \underline{x}, t)$ are $m \times m$ matrices with entries depending on $\underline{U}$, $\underline{x}$ and t, and $B(\underline{U}, \underline{x}, t)$ is a column vector having elements $b_1(\underline{U}, \underline{x}, t), b_2(\underline{U}, \underline{x}, t), ..., b_m(\underline{U}, \underline{x}, t)$. The suffixes t and x_i denote partial differentiation.

The fundamental idea for the hyperbolic type of a system is that the Cauchy problem should be well posed for it. With respect to the first order system (1.1) the Cauchy problem amounts to prescribing $\underline{U}$ at points on some initial manifold $\wp$ in $\mathbb{R}^{n-1} \times t$, so the system will be hyperbolic when this prescribed data is sufficient to determine a unique solution that depends continuously on the data prescribed at points of $\wp$.

With these concepts and keeping in mind the geometrical approach to wavefronts that has been considered so far, let us now seek to determine the possibility of grouping the terms of (1.1) in such a way that they express the derivative of $\underline{U}$ normal to $\wp$ in terms of derivative of $\underline{U}$ in $\wp$ and the remaining terms of (1.1).

Let $\zeta_i = \zeta_i(\underline{x}, t)$ be differentiable function of their arguments, then we introduce the coordinate system $(\underline{\zeta}, t')$, where $\underline{\zeta} = (\zeta_1, \zeta_2, ..., \zeta_n)$ and $t = t'$. The manifold $\wp$ is taken to be associated with the coordinates ζ_k and to have the equation $\zeta_k(\underline{x}, t) = a\,$(const.) and, aside from this restriction, the other ζ_i will be chosen arbitrarily. The transformation thus becomes

$$t' = t,\ \zeta_i(\underline{x}, t) = \text{constant for } i = 1, 2, ..., n. \tag{1.2}$$

Also, it is supposed that initially the transformation is non-singular in the vicinity of $\wp$. Using the transformation (1.2), (1.1) reduces to the following form

$$A_0(\underline{U}, \underline{x}, t)\left(\frac{\partial u}{\partial t'} + \sum_{j=1}^{n} \frac{\partial \zeta_j}{\partial t}\frac{\partial u}{\partial \zeta_j}\right) + \sum_{i,j=1}^{n} A_i(\underline{U}, \underline{x}, t)\frac{\partial \zeta_j}{\partial x_i}\frac{\partial \underline{U}}{\partial \zeta_j} + B(\underline{U}, \underline{x}, t) = 0\ . \tag{1.3}$$

The manifold $\wp$ has been embedded in the family of coordinate manifolds $\zeta_k(\underline{x}, t) = \text{constant}$ and the derivative of $\underline{U}$ is to be obtained normal to $\wp$, and so the required derivative is $\partial\underline{U} / \partial\zeta_k$, which may be rewritten as

$$\Lambda\, \partial\underline{U} / \partial\zeta_k + R = 0\ , \tag{1.4}$$

where

$$\Lambda = \left[\frac{\partial \zeta_k}{\partial t} A_0(\underline{U}, \underline{x}, t) + \sum_{i=1}^{n} \frac{\partial \zeta_k}{\partial x_i} A_i(\underline{U}, \underline{x}, t)\right], \tag{1.5}$$

and R is a column vector consisting of m elements depending upon $\underline{U}$, $\underline{x}$, t and $\partial\underline{U} / \partial\zeta_i$ with $i \neq k$.

Hence, the derivative $\partial \underline{U} / \partial \zeta_k$ normal to $\wp$ may be obtained from (1.4), in case Λ^{-1} exists, which gives the condition

$$\det \Lambda \neq 0 . \tag{1.6}$$

Dividing $\det \Lambda$ by $|\nabla_x \xi_k| = [\sum_{i=1}^{n} (\partial \xi_k / \partial x_i)^2]^{1/2}$ and setting

$$-\lambda = \frac{\partial \xi_k / \partial t}{|\nabla_x \xi_k|}, \; v_i = \frac{\partial \xi_k / \partial x_i}{|\nabla_x \xi_k|} \text{ for } i = 1, 2, ..., n , \tag{1.7}$$

so that the normalized spatial gradient $\nabla_x \xi_k$ of ξ_k is given by the unit vector $\underline{v} = (v_1, v_2, ..., v_n)$. Using (1.7) the condition (1.6) can be written as

$$Q(P; \underline{v}, \lambda) \neq 0 , \tag{1.8}$$

where

$$Q(P; \underline{v}, \lambda) \equiv \left| \sum_{i=1}^{n} v_i A_i(P) - \lambda A_0(P) \right| . \tag{1.9}$$

Here, the notation $A_i(P)$ has been used to represent the value of $A_i(\underline{U}, \underline{x}, t)$ at point P of the manifold $\wp$. The expression $Q(P; \underline{v}, \lambda)$ which is a homogeneous polynomial of degree m in the quantities $\{-\lambda, v_1, v_2, ..., v_n\}$, is called the characteristic polynomial of the system (1.1) with respect to $\wp$.

It can also be noted that the normal derivative $\partial \underline{U} / \partial \xi_k$ will be indeterminate at any point P of a manifold $\wp$ for which

$$Q(P;\underline{v},\lambda)=0. \tag{1.10}$$

The manifolds $\wp$ on which (1.10) holds are called characteristic manifolds; and the manifolds for which (1.8) does not hold are called non-characteristic.

The system (1.1) is said to be strictly hyperbolic in t-direction at P if the zeroes $\lambda^{(1)},\lambda^{(2)},...,\lambda^{(m)}$ of the characteristic polynomial $Q(P;\underline{v},\lambda)$ are all real and distinct for all choices of the unit vector $\underline{v}$ and if the right eigenvectors $r^{(1)},r^{(2)},...,r^{(m)}$ satisfying

$$\sum_{i=1}^{m}[v_i A_i(P)-\lambda^{(j)}A_0(P)]r^{(j)}=0, \tag{1.11}$$

span the space E^m occupied by the m element eigenvectors. The system (1.1) will merely be said to be hyperbolic in the t-direction if the eigenvectors span the space E^m but the eigenvalues, although all real, are not all distinct.

1.3 Shock wave and Rankine-Hugoniot condition

Consider the system of equations introduced in (1.1) and rewrite it into following simpler form in $\mathbb{R}^3\times t$ with n dependent variables

$$F_t+\operatorname{div}G=H\ , \tag{1.12}$$

where $F=F(\underline{U}(\underline{x},t))$ and $H=H(\underline{U}(\underline{x},t))$ are n element column matrix vectors and $G=G(\underline{U}(\underline{x},t))$ an $n\times 3$ matrix. The matrix G in (1.12) is regarded as a tensor so that

$$\operatorname{div}G=\sum_{s=1}^{3}(\partial g^{(s)}/\partial x_s), \tag{1.13}$$

where $g^{(s)}$ is the s$^{\text{th}}$ column of G.

For the discontinuous solution of the system (1.13), we use the integral formulation.

If F be a $n \times 1$ column matrix whose components are continuous scalar functions of position and time in the volume $V(t)$, which is itself enclosed by a surface $S(t)$ moving with velocity $\underline{v}$, then the rate of change of the volume integral of F is given as

$$\frac{d}{dt}\int_{V(t)} F dV = \int_{V(t)} \frac{\partial F}{\partial t} dV + \int_{S(t)} F.\underline{V}d\underline{S} \ , \tag{1.14}$$

where $d\underline{S}$ is the vector element normal to the surface area. Let $\sigma(\underline{x},t) =$ constant be the surface of discontinuity for the vector $\underline{U}$ and hence for F, G and H, and let the volume $V(t)$ be enclosed by the surface $S(t)$ moving with velocity $\underline{v}$, such that an arbitrary part $S_0(t)$ of the discontinuity surface $\sigma(\underline{x},t) =$ constant divides it into two sub volumes V_1 and V_2. If S_1 and S_2 be the parts of the surface S which bound V_1 and V_2 respectively, excluding the surface $S_0(t)$, which is assumed to have the velocity $\underline{v}$, then the integration of (1.13) over $V(t) = V_1 \cup V_2$ yields

$$\int_{V_1 \cup V_2} (\partial F / \partial t) dV + \int_{V_1 \cup V_2} (div\, G) dV = \int_{V_1 \cup V_2} H dV \ . \tag{1.15}$$

Applying Gauss divergence theorem separately to V_1 and V_2, in which F and G are continuous and differentiable, we get

$$\int_{V_1 \cup V_2} (\partial F / \partial t) dV + \int_{S_1 \cup S_2} G\, d\underline{S} = \int_{V_1 \cup V_2} H dV \ , \tag{1.16}$$

where $G\, d\underline{S}$ denotes the scalar product of G, now considered as a tensor, and vector $d\underline{S}$. Now, from (1.14) and (1.16), and noting the fact that the dividing surface $S_0(t)$ is also moving with velocity $\underline{v}$, we get

$$\frac{d}{dt}\int_{V_1\cup V_2} F dV = \int_{S_1\cup S_2} (F.\underline{V} - G)d\underline{S} + \int_{V_1\cup V_2} H dV \ . \tag{1.17}$$

Subtracting from (1.17), the resulting expressions integrated over the separate volumes V_1 and V_2 yields

$$\int_{S_0(t)} (F\underline{V} - G)_1 d\underline{S}_1 + \int_{S_0(t)} (F\underline{V} - G)_2 d\underline{S}_2 = 0 \ , \tag{1.18}$$

where $d\underline{S}_1$ and $d\underline{S}_2$ are the outward directed surface elements associated with the volumes V_1 and V_2.

The surface elements $d\underline{S}_1$ and $d\underline{S}_2$ are both normal to discontinuity surface $\sigma(\underline{x}, t) = \text{constant}$, but are oppositely directed. Therefore, $\underline{n} = -\underline{n}_2$ and $d\underline{S}_1 = d\underline{S}_2 = \underline{n} dS_0$. The relation (1.18) may be rewritten as

$$\int_{S_0(t)} [(F\underline{V} - G)_1.\underline{n} - (F\underline{V} - G)_2.\underline{n}] dS_0 = 0 \ . \tag{1.19}$$

Since dS_0 is arbitrary, we have an algebraic jump condition across $\sigma(\underline{x}, t) = \text{constant}$ in the following form

$$(F\underline{V} - G)_1.\underline{n} - (F\underline{V} - G)_2.\underline{n} = 0 \ . \tag{1.20}$$

The above relation (1.20) can also be written as

$$\underline{\lambda}(F_1 - F_2) = (G_1 - G_2).\underline{n} \ , \tag{1.21}$$

where $\underline{\lambda} = \underline{v}_1.\underline{n} = \underline{v}_2.\underline{n}$ is the normal speed of propagation of the area elements $d\underline{S}_1$ and $d\underline{S}_2$ on opposite sides of, and moving with $\sigma(\underline{x}, t) = \text{constant}$ and also is continuous across $S_0(t)$.

Equation (1.21) can be written as

$$\underline{\lambda}[F]=[G].\underline{n} \ , \tag{1.22}$$

where $[Q]$ represents the jump in Q across discontinuity surface $S_0(t)$. The relation (1.22) is also called the generalized Rankine-Hugoniot relation for the system (1.12).

In the case of gasdynamics, a discontinuous solution for a system of equations written in conservation form satisfying the generalized Rankine-Hugoniot conditions and appropriate entropy conditions is called a shock.

1.4 Simple wave and Progressive wave

For one dimensional flow, consider the system of quasilinear hyperbolic partial differential equations

$$\underline{U}_t + A(\underline{U})\underline{U}_x = 0 \ , \tag{1.23}$$

where $\underline{U}$ is the vector in $\mathbb{R}^n$ and the matrix A is a function of $\underline{U}$.

The solution vector $\underline{U}$ is said to define a Simple wave if it can be written in terms of variables by means of a single function and the associated flows are called simple wave flow (Germain 1972, Whitham 1978).

As the system (1.23) is hyperbolic, the matrix A has n real eigenvalues $\lambda_i,\ i=1,...,n$ with corresponding eigenvectors r^i , i.e,

$$A r^i = \lambda_i r^i. \tag{1.24}$$

The simple wave solutions of the system (1.23) are solutions of the form

$$\underline{U}(x,t) = \underline{U}(\underline{X}), \tag{1.26}$$

where

$$\underline{X} = \underline{X}(x,t). \tag{1.27}$$

Thus, equation (1.23) gives

$$A\left(\frac{d\underline{U}}{d\underline{X}}\right) = -\left[\left(\partial\underline{X}/\partial t\right)/\left(\partial\underline{X}/\partial x\right)\right]\left(\frac{d\underline{U}}{d\underline{X}}\right), \tag{1.28}$$

which suggests that $d\underline{U}/d\underline{X}$ is an eigenvector of the matrix $A(\underline{U})$ corresponding to the eigenvalues

$$\lambda_i = -\left(\partial\underline{X}/\partial t\right)/\left(\partial\underline{X}/\partial x\right). \tag{1.29}$$

The function $\underline{U}(\underline{X})$ can be determined by the solution of the equation

$$\frac{d\underline{U}}{d\underline{X}} = r^i, \tag{1.30}$$

i.e., of the system

$$\frac{dU_1}{r_1^i} = \frac{dU_2}{r_2^i} = \ldots = \frac{dU_n}{r_n^i} = d\underline{X}. \tag{1.31}$$

Once we know the function $\underline{U}(\underline{X})$, the value of $\underline{X}(x,t)$ can be obtained by integration of (1.29). Thus, we have

$$x = r^i\, t + f(\underline{X}), \tag{1.32}$$

where, the function $f(\underline{X})$ is arbitrary. The curves along which the function $\underline{X}$ is constant are called Simple waves, i.e., Simple waves are the straight lines given by the eq. (1.32). The surface $\underline{X} = \text{constant}$ are often called wavelets. Therefore, $\underline{U}$ remains constant if and only if one stays on a wavelet.

The solution $\underline{U}(x,t)$ is said to describe a progressive wave if there exists a family of propagating wavelets $\underline{X} = \text{constant}$, with

$$F(x,t) = \underline{X}, \tag{1.33}$$

such that the magnitude of the rate of change of $\underline{U}$ - or eventually its derivatives, when x is moving with such a wavelet is small in comparison to the magnitude of rate of change of $\underline{U}$ when x is kept fixed (Germain, 1972).

To obtain the progressive wave solution the function $\underline{X}$ is introduced either by substituting one of the independent variables like t or to add an additional one which characterizes the wavelet. In light of this definition we can write

$$\underline{X} = \psi X = F(x,t), \tag{1.34}$$

where ψ is a small parameter. Now $\underline{U}$ may be written as

$$\underline{U}(x,t) = \underline{U}(x,t,X) \ . \tag{1.35}$$

From the right hand side of equation (1.35), we can see that $\underline{U}$ is a function of three independent scalar variables. Using (1.34) in (1.35), the rates of changes are given by

$$\frac{\partial \underline{U}}{\partial t} = \left(\frac{1}{\psi}\right)\frac{\partial \underline{U}}{\partial X} F_t + \frac{\partial \underline{U}}{\partial t}, \qquad F_t = \frac{\partial F}{\partial t}, \tag{1.36}$$

$$\frac{\partial \underline{U}}{\partial x} = \left(\frac{1}{\psi}\right)\frac{\partial \underline{U}}{\partial X} F_X + \frac{\partial \underline{U}}{\partial x}, \qquad F_X = \frac{\partial F}{\partial X} \ . \tag{1.37}$$

Since ψ is a small parameter, we may suppose that the first partial derivative of $\underline{U}$ and F are bounded, say $O(1)$. For any given curve $(x(\eta), t(\eta))$, we can write

$$\frac{\partial \underline{U}}{\partial \eta} = \left(\frac{1}{\psi}\right)\left(\frac{\partial \underline{U}}{\partial x}\right)\left(\frac{\partial F}{\partial \eta}\right) + \left(\frac{\partial \underline{U}}{\partial t}\right)\left(\frac{dt}{d\eta}\right) + \left(\frac{\partial \underline{U}}{\partial x}\right)\left(\frac{dx}{d\eta}\right). \tag{1.38}$$

In the above equation the left hand side is of the order of $O(\psi^{-1})$; but if $F(\eta)$ is constant it is only $O(1)$. The general discussion for linear and nonlinear systems can be found in Ludwig (1960), Lewis (1965), Vaillant (1968) and Courant & Hilbert (1962).

1.5 Non-Ideal gas

The equation of state of an ideal gas is written under two assumptions: the gas molecules are so small that they have no volume and that the molecules are non-interacting. It is given as

$$PV = nRT,$$

where n is the number of molecules of the gas, R is the gas constant, T is the absolute temperature, P is the pressure and V is the volume of the gas. It is a good description of most gases in the low density regime, where on average molecules are far apart. But it is not true for real gases.

However, if the temperature of the gas is very high and density is too low then the hypothesis that the gas is ideal is no longer valid. Then there is no choice but to relax the assumptions of ideal gas. The most preferred equation of state is the Van der Waals equation of sate.

Dutch Physicist Van der Waals derived an equation of state, known as the Van der Waals equation of state without the assumptions of ideal gas, which is written as,

$$p = \frac{RT}{V-b} - \frac{a}{V^2} ,$$

where a and b are two small constants and V is the molal volume.

In the first approximation, for the molal volume V there must be substituted the covolume $V-b$, where the constant b is proportional to the sum of the volumes of all the molecules in one mole of the gas. The equation of state therefore becomes

$$p = RT/(V-b) ,$$

as the covolume equation of state. Now, if the molecules of the gas do interact at a distance, say, attract each other, and then the internal pressure due to this attraction must be taken into account. When the density of the gas in a given vessel is changed by adding more gas or subtracting it, all the internal forces change in the ratio $1/V^2$. Since the pressure is defined as the force per unit area, this applies also to the internal pressure and we obtain for it the expression a/V^2 which is added to p in Van der Waals' equation.

Roberts and Wu (1996) determined conditions for the stability of strong spherical implosions for both ideal and Van der Waals gases. When the Van der Waals excluded volume is sufficiently large and the attractive constant is neglected, they have shown that a new type of solution is found and the shock may be linearly stable. Wu and Roberts (1996) studied the properties of strong spherical shock waves whereas Somogyi and Roberts (2007) investigated the numerical stability of an imploding spherical shock wave in Van der Waals gas. Zhao *et al.*(2011) investigated the problem of admissibility of shock waves and shock-induced phase transition in a Van der Waals fluids.

1.6 Magnetogasdynamics

The theory dealing with the study of interaction between moving electrically conducting gases and electromagnetic field is known as magnetogasdynamics. The governing equations for the study of interaction between gasdynamics and magnetic field comprises of the equations from gasdynamics and electromagnetic theory. Electromagnetic interaction takes place in many natural and man-made flows. Examples of magnetogasdynamic fluids are plasma, liquid metals and electrolysis. These are typically used in industry to heat, pump, stir and levitate liquid metals. Magnetic field influences our everyday life.

There is terrestrial magnetic field maintained by the fluid motion in earth core and supposed to be main force behind the earth's rotation, the magnetic field of sun which is responsible for sunspots and solar flares, and the galactic magnetic field which contribute significantly to total pressure which balances the interstellar medium against gravity and affect gas flow in interstellar medium and formation of stars. The interaction between the gasdynamic phenomena and the magnetic field is investigated by combining the magnetic field equations with those of gasdynamics.

The interaction between the magnetic field and gasdynamics is studied by associating the field equations with the equations of gasdynamics. In most of the problems concerning electromagnetism, the Maxwell's displacement currents are generally ignored (Pai (1972), Kantrowitz and Petschek (1966), Anile and Greco (1978) and Spitzer (1967)). In magnetogasdynamics, the magnetic permeability of the media under consideration differs slightly from unity and therefore in the application, it is taken as unity. Thus the field equations are

$$\nabla \times \underline{E} = -\frac{1}{c}\underline{H}_t, \tag{1.39}$$

$$\nabla \times \underline{H} = \frac{4\pi}{c}\underline{J} = \frac{4\pi}{c}\sigma\left(\underline{E} + \frac{\underline{u} \times \underline{H}}{c}\right), \tag{1.40}$$

$$\nabla \bullet \underline{H} = 0, \tag{1.41}$$

where $\underline{E}$ is the electrical intensity, c is the speed of light, $\underline{H}$ is the magnetic field, $\underline{J}$ is the current density, $\underline{u}$ is the velocity of fluid and σ is electrical conductivity.

Suppose that σ is uniform in the medium. Using equation (1.41) in the equation (1.40) we have

$$\underline{H}_t - \nabla \times (\underline{u} \times \underline{H}) = \frac{c^2\nabla^2\underline{H}}{4\pi\sigma} \; . \tag{1.42}$$

If the electrical conductivity is infinite, the equation (1.42) reduces to the following form

$$\underline{H}_t + (\underline{u}.\nabla)\underline{H} + \underline{H}(\nabla.\underline{u}) - (\underline{H}.\nabla)\underline{u} = 0 \tag{1.43}$$

Using equation (1.41), the equation (1.43) may be written as

$$\underline{H}_t + (\underline{u}.\nabla)\underline{H} + \underline{H}(\nabla.\underline{u}) = 0. \tag{1.44}$$

The equation (1.44) is employed in concurrence with gas dynamic flow equations to study the consequence of magnetic field interaction.

Due to complexity in the solution of system of non-linear partial differential equations, initial efforts in this area were devoted to the propagation of gas dynamic shocks and electromagnetic waves. To study the hydrodynamic shocks, the electrical conductivity of the medium is supposed to be infinite. This supposition suggests that the self-induction will stop changes to magnetic field of the medium at rest (de Hoffman and Teller (1950) and Kulikovski and Liubimov (1961)). Also the basic equations degenerate into non-convex hyperbolic system, for which the corresponding characteristic surface may have unpredicted singularities, making the wave structure much more intricate than aerodynamic case (Courant and Hilbert (1962) and Jeffrey and Taniuti (1964)). Non-ideal magneto-hydrodynamics bids for striking potential applications, but also give rise to many subtle questions (Kantrowitz and Petschk (1966)).

1.7 Cauchy problem:

Let S be an open subset of $\mathbb{R}^n$, and let $f_j, 1 \le j \le m$, be m smooth functions from S into $\mathbb{R}^n$. Then for the general system of conservation laws in several space variables

$$\frac{\partial \underline{u}}{\partial t} + \sum_{j=1}^{m} \frac{\partial}{\partial x^j} f_j(\underline{u}) = 0,\ x = (x_1, ..., x_m) \in \mathbb{R}^m,\ t > 0\ , \tag{1.45}$$

where $\underline{u} = (u_1, ..., u_n)^T$ is a vector valued function from $\mathbb{R}^m \times [0, +\infty[$ into S, the set S is called the set of states and the functions $f_j = (f_{1j}, ..., f_{nj})^T$ are called flux functions. Then the Cauchy problem for (1.45) is to find a function $\underline{u} : (x,t) \in \mathbb{R}^m \times [0, \infty[\to \underline{u}(x,t) \in S$, that is a solution of (1.45) satisfying the initial condition

$$\underline{u}(\underline{x}, 0) = \underline{u}_0(\underline{x}),\ \ \underline{x} \in \mathbb{R}^m, \tag{1.46}$$

where $\underline{u}_0 : \mathbb{R}^m \to S$ is a given function.

1.8 Riemann problem

The study of Riemann problem started with the work " theory of waves of finite amplitude" by great mathematician G. F. B. Riemann (1859), which was not limited to a single progressive wave and suited to calculate the propagation of planar waves of finite amplitude proceeding in both directions. Mathematically, the Riemann problem for one dimensional flow, consists of a one dimensional conservation law together with piecewise constant initial data. It is a particular type of Cauchy problem.

For one dimensional time dependent Euler equations, Riemann problem is the initial value problem for the conservation laws

$$U_t + F(U)_x = 0, \tag{1.47}$$

with

$$U = \begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix}, \; F = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ u(E+p) \end{bmatrix},$$

and the initial conditions

$$U(x,0) = U^{(0)}(x) = \begin{cases} U_L \text{ if } x < 0, \\ U_R \text{ if } x > 0, \end{cases}$$

where the domain of interest in the $x-t$ plane are points (x,t) with $-\infty < x < \infty$ and $t > 0$.

Riemann problem has many real world applications. In gasdynamics one dimensional Riemann problem for Euler equations is the generalization of the shock tube problem, which consists of two stationary gases in a tube separated by a diaphragm. When the diaphragm is broken suddenly, it produces a nearly centered wave system. The middle wave is always a contact discontinuity and the other two being shock or rarefaction waves equitably. In the case of shallow water equations the example of Riemann problem is the dam break problem. In dam break problem the water is obstructed by a dam. When the dam is broken or the spillway is open, the mathematical explanation of flow of water is given by using the solution of Riemann problem. Other applications of Riemann problem can be found in the study of traffic flow, haemodynamics, glacial flow, sediment transport etc.

Recently Shekhar and Sharma (2010) presented the solution for one dimensional Riemann problem and elementary wave interactions. Further, Shekhar and Sharma (2012) and Singh and Singh (2015) presented the solution of Riemann problem for magnetogasdynamic flow. Alcrudo et. al. (2001) presented the exact solution to the Riemann problem for shallow water equations with a bottom step. The exact solution for Riemann problem for shallow water equations with variable bottom is given by Bernetti et. al. (2009).

1.9 Dusty gas and its equation of state

Dusty gas is a mixture of gas and small solid particles where solid particles occupy less than 5% of total volume. We consider the thermodynamic equilibrium condition such as $T_p = T_g = T$.

The density of the mixture is given as

$$\rho = z\rho_{sp} + (1-z)\rho_g = \bar{\rho}_p + \bar{\rho}_g \, . \tag{1.48}$$

The mass concentration of the pseudo fluid of the solid particles is defined as

$$k_p = \bar{\rho}_p / \rho_m = Z\rho_{sp} / \rho_m \, . \tag{1.49}$$

The pressure of the mixture is

$$p = p_p + p_g \, . \tag{1.50}$$

The total pressure of the mixture is given as

$$p = R\rho_g T_g \; . \tag{1.51}$$

From (1.48), (1.50) and (1.51), the relation between the pressure and density of the mixture as a whole is given as

$$p_m = p = R\rho_g T_g = R\left(\frac{\rho_m - Z\rho_{sp}}{1-Z}\right)T_g \; ,$$

$$= R\rho_m \frac{(1 - Z\rho_{sp} / \rho_m)}{(1-Z)} T_g \, ,$$

$$= R\rho_m \frac{1-k_p}{1-Z} T.$$

Therefore,

$$p_m = \frac{\rho_m R_m T}{1 - Z}, \tag{1.52}$$

where

$$R_m = (1 - k_p)R. \tag{1.53}$$

Here, R_m may be considered as an effective gas constant of the mixture and subscript m refers to the value of the gas constant in the mixture as a whole.

The internal energy of the mixture per unit mass e_m is related to the internal energies of the two species by the following relation

$$\rho_m e_m = Z\rho_{sp} C_{sp} T_p + (1 - Z)\rho_g C_v T,$$

$$= k_p C_{sp} T_p + \left(\frac{1}{\rho_m} - \frac{Z}{\rho_m} \right) \rho_g C_v T,$$

$$= k_p C_{sp} T_p + \left(\frac{1}{\rho_m} - \frac{k_p}{\rho_{sp}} \right) \rho_g C_v T.$$

On simplification, we have

$$e_m = k_p C_{sp} T_p + (1 - k_p) C_v T \ , \tag{1.54}$$

where $C_{sp} = C_s + C_{vp}$ and we assume that C_{sp} and C_v are constant for simplicity. For thermodynamic equilibrium condition, we have the specific heat of the mixture at constant volume C_{vm} as follows

$$C_{vm} = k_p C_{sp} + (1 - k_p) C_v \ , \tag{1.55}$$

where C_v is the specific heat of the gas at constant volume.

For thermodynamic equilibrium condition the specific heat of the mixture at constant pressure is

$$C_{pm} = k_p C_{sp} + (1 - k_p) C_p \ .$$

The specific heats of the mixture are independent of the volume fraction Z but depend on the mass fraction k_p of solid particles. The ratio of the specific heat of the mixture is

$$\Gamma = \frac{C_{pm}}{C_{vm}} = \frac{(1 - k_p) C_p + k_p C_{sp}}{(1 - k_p) C_v + k_p C_{sp}} \ ,$$

or

$$\Gamma = \gamma \frac{1 + \lambda \beta}{1 + \lambda \gamma \beta} \ , \tag{1.56}$$

where, $\gamma = \dfrac{C_p}{C_v}$, $\beta = \dfrac{C_{sp}}{C_p}$ and $\lambda = \dfrac{k_p}{1 - k_p}$.

The ratio is always smaller than γ of the gas if k_p is different from zero. Also, if $k_p = 0$, then $\Gamma = \gamma$.

If we consider the mixture as a homogeneous medium, the first law of thermodynamics for the mixture gives

$$dQ = de_m - \frac{1}{\rho_m^2} pd\rho_m , \tag{1.57}$$

where dQ is the heat addition to the mixture. The above equation is the energy equation of the mixture as a whole.

For isentropic change of state of the gas - particle mixture $dQ = 0$, so we have

$$de_m = \frac{1}{\rho_m^2} pd\rho_m = \frac{p}{\rho_m}\frac{d\rho_m}{\rho_m} . \tag{1.58}$$

Also, from equation

$$e_m = k_p C_{sp} T_p + (1 - k_p) C_v T , \tag{1.59}$$

and thus,

$$de_m = (k_p C_{sp} + (1 - k_p) C_v) dT \; .$$

Therefore,

$$(k_p C_{sp} + (1 - k_p) C_v) dT = \frac{p}{\rho_m}\frac{d\rho_m}{\rho_m} ,$$

or

$$C_{vm} dT = \frac{p}{\rho_m}\frac{d\rho_m}{\rho_m} . \tag{1.60}$$

Using equation (1.52) in (1.60), we have

$$C_{vm} dT = \frac{R_m T}{1 - Z}\frac{d\rho_m}{\rho_m} .$$

After simplification, we have

$$\frac{1}{\Gamma-1}\frac{dT}{T}=\frac{1}{1-Z}\frac{d\rho_m}{\rho_m}. \tag{1.61}$$

If $Z<<1$, the isentropic change of state of the mixture has a similar relation as that for a pure gas with an effective ratio of specific heats Γ. In general, the volume fraction Z has some influence on the isentropic change of the mixture.

Similarly, from equation (1.52), for a given k_p and $T_p=T$, we have

$$p_m=\frac{\rho_m R_m T}{1-Z} . \tag{1.62}$$

So,

$$dp_m=\frac{R_m d(\rho_m T)}{1-Z}+\frac{\rho_m R_m T dZ}{(1-Z)^2},$$

or

$$\frac{dp_m}{p_m}=\frac{dT}{T}+\frac{d\rho_m}{\rho_m}+\frac{dZ}{1-Z}.$$

Using equation (1.61) and simplifying, we get

$$\frac{dp_m}{p_m}=\frac{\Gamma}{(1-Z)}\frac{d\rho_m}{\rho_m} .$$

On integration, we have

$$p_m\left(\frac{\rho_m}{1-Z}\right)^{-\Gamma}=\text{constant}. \tag{1.63}$$

Again, if $Z << 1$, the equation (1.63) is identical in form for the corresponding relation of an ideal gas but with an effective ratio of specific heats.

Using the equation (1.63), the equilibrium speed of sound of the mixture a_m as

$$a_m^2 = \frac{\partial p}{\partial \rho_m} = \frac{\Gamma R_m T}{(1-Z)^2} \ . \tag{1.64}$$

Early works for the study of mixtures of gas and solid particles are those of Marble (1963), Murray (1965), Soo (1967) and Vasiliev (1969). High speed flows of a mixture of a gas and small solid particles are encountered in several branches of science and engineering. Some of them can be found in Marble (1970), Boothroyd (1971) and Rudinger (1980).

1.10 Review of literature

The study of waves goes back to the ancient times when philosophers, such as Pythagoras, studied the relation between pitch and length of string in musical instruments. Brook Taylor (1685-1731) was the first to give analytical solution for a vibrating string. Afterwards, the study of waves was moved forward by Daniel Bernoulli (1700-82), Leonard Euler (1707-83) and Jean d'Alembert (1717-83) who found the first solution to the linear wave equation. It was not until the second part of nineteenth century that the study of nonlinear waves started with the pioneering work of Stokes (1847) and Riemann (1858). Afterwards it has progressed, with significant advancement made in recent years. When a body is in a relative motion with respect to the fluid (inside the fluid), the disturbance (if sufficiently small) produced by the body moves through the fluid with the speed of sound. These disturbances can be rarefaction waves or compression waves. The compressions of finite amplitude usually give rise to a discontinuous growth of pressure leading to a shock wave in the flow field. There is a likewise increase in temperature, density, entropy and other fluid properties. If initially, the fluid is at rest and the shock wave is moving then, after the passage of the shock, the fluid will move in the direction of the shock. Gas compressions, which have finite amplitude, travel faster than the speed of sound, as in the case of strong explosions. The instant of formation of shock waves was largely investigated by many authors. Basics of gasdynamics can be found in Zierep (1978). The general theory of propagation of shock waves was presented by Boillat (1965). Colemann and Gurtin (1967) and Chen and Gurtin (1970) studied the growth and decay of shock waves with internal state variables. Shifrin (1970) studied the formation of a shock wave for planar flow of a perfect gas. Ardavan-Rhad (1970) studied the propagation of plane shock wave into a non-viscous, non-isentropic and non-heat conducting medium. Saldatov (1970) determined the instant of formation of a shock wave in a symmetric two-way traffic flow, by using the Riemann method. Macpherson (1971) used the molecular-dynamic approach to study the formation of a shock wave in dense Argon. Flack and Wittig (1971) presented the general solution for the normal shock wave moving in a medium where all flow properties vary arbitrarily. Chen (1971) studied the propagation of shock waves in elastic non-conductors. The effect of thermodynamic properties on the propagation of shock waves has been studied by Chen (1973). Bowen and Chen (1974) studied the same problem in the ideal mixture with several temperature layers. Sod (1977) numerically studied the propagation of one-dimensional shock wave with cylindrical and spherical symmetry. Ruggeri and Boillat (1979) investigated the problem of reflection and transmission of discontinuity waves through a shock wave.

Sharma *et al.* (1989) studied a traffic flow problem in which shock wave appears. In the past decades various authors have studied the problem of growth and decay behaviour of shock waves propagation in several material media.

One of the interesting features of the shock waves is the problem of determining the differential effects of shock fronts on the rear flow field. To study this problem Thomas (1947) developed a tensorial approach which was further extended by Kanwal (1958) for three dimensional shocks in stationary, pseudo-stationary and unsteady flows of non-conducting gases. The problem of vorticity generation by a shock has also been investigated by various authors like Trusdell (1952), Hayes (1957), Kanwal (1960) and Ram (1978). Boillat and Ruggeri (1979) analyzed the evolution of weak discontinuities for quasilinear hyperbolic system. The study of shock structure also got prominence in the recent decades. Shock structure was studied by Goldman and Sirovich (1969), Kuznetsov (1979), Boillatt and Ruggeri (1998). Wave fronts which are concave in the direction of propagation exhibit different kinds of behaviour depending on the strength of the wave-front. Generally, wave front propagates normal to it and therefore has a tendency to converge. The shocks of weak strength are called weak shocks. Focusing of weak shock wave is an important problem. Wanner *et al*. (1972) studied the problem of focusing of weak shock waves. Atomic explosions give the evidence of the shock waves of strong strength, called blast wave. Ram (1981) has obtained a closed form self similar solution to a MHD flow disturbed by propagating blast waves. In case of blast waves, the shock becomes very strong and the pressure ahead is generally neglected in comparison to the pressure behind the shock wave. This leads to similarity formulation of the problem. In this problem, the ratio of distance to a particular power of time is known as similarity exponent, which is not known a priori. Various authors developed the numerical and analytical techniques for the determination of similarity exponent of the problem e.g. Guderley (1942), Taylor (1950), Butler (1954), Sedov (1959), Stanyukovich (1960), Welsh (1967), Zel'dovich & Raiser (1967) and Lazarus (1981), Chisnell (1998). Zen'kevich and Stepanov (2007) obtained analytical solution of self similar equations in Lagrangian mass coordinate, expressing the dynamics of the explosion and the propagation of a strong shock wave. Branover (1978) investigated the magnetohydrodynamic flow in ducts. Taylor and Cargill (2001) investigated the problem of self-similar expansion waves in magnetogasdynamic flows. Lock and Mestel (2008) studied the possibility of self-similar, imploding, finite annular z-pinch solutions to the equations of magnetogasdynamics for a perfect gas at infinite magnetic Reynolds number.

In 1859, the great mathematician G. F. B. Riemann presented his famous Theorie der Wellen endlicher Schwingungsweite ("Theory of Waves of Finite Amplitude"). After the pioneering work of Riemann, the study of Riemann problem got significance but due to nonlinear nature of the equations involved, most of the problems do not have closed form solution. For the simplest case, a polytropic ideal gas, it was shown by Courant and Friedrichs (1948) and Landau and Lifshitz (1959), that the structure of the elementary waves and the solutions of Riemann problems can be determined analytically. Weyl (1949) had extended this analysis to include more general equation of state. The conditions on the equation of state that are necessary and sufficient for uniqueness of solutions are determined by Smith (1979). In the recent decades, many attempts have been made to develop Riemann solvers. The credit for the development of first exact Riemann solver for the Euler equations goes to Godunov (1959). Later, Godunov (1976) presented a second exact Riemann solver which was more convenient from the point of view of computation. Chorin (1976) presented improvements to the Godunov's first Riemann solver. Another improvement to the Godunov's first Riemann solver was presented by Leer (1979). Smoller (1994) suggetsted a different approach and Dutt (1986) implemented the scheme. Gottlieb and Groth (1988) presented a Riemann solver for ideal gas, which was more efficient than preceding one's. Other Riemann solvers are presented by Toro (1989), Schleicher (1993) and Pike (1993). Colella and Glaz (1985) presented the Riemann solver with general equation of state. Menikoff and Plohr (1989) studied the Riemann problem for fluid flow of real material with arbitrary equation of state. He also discussed the shock stability and nonuniqueness of the Riemann problem. Saurel, Larini and Loraud (1994) presented the exact and approximate Riemann solvers for real gases.

For the computation of Godunov flox, Harten, Lax and van Leer (1983) presented an approach for the approximate solution of Riemann problem. The Riemann solver they developed is known as HLL Riemann solver. HLL Riemann solver became a practical method after using the wave speed estimates of Davis (1988) and Einfeldt (1988). The HLL solver with the wave speed estimate of Einfeldt is known as HLLE solver. HLL solver was extended by Toro, Spruce and Speares (1992) to include the middle wave. The resulting solver is known as HLLC Riemann solver.

Euler equations with source term occur in many real world problems. Examples of Euler equations with source term are flows under gravity, cylindrical and spherically symmetric flows and flows in discontinuous duct. Many attempts have been made to develop the numerical methods for the solution of Euler equations with source term. Lefloch et al. (2003) solved the Riemann problem for fluid flows in a nozzle with discontinuous cross section. Desveaux et al. (2004) developed a well balanced scheme for the numerical solution of Euler equation with gravitation. Fuchs (2010) developed a well balanced finite volume scheme for the simulation of wave propagation in stratified magnetic atmosphere. Kroner et al. (2005) obtained the numerical solution for compressible flow in a nozzle with variable cross section. Leveque (1999) developed the wave propagation methods for the conservation laws with source terms. Xing et al. (2013) developed the high order WENO scheme for Euler equations under gravitational fields. Tauma et al. (2016) developed the central finite volume methods for Ripa system. Kappeli et al. (2014) developed a well balanced scheme for the Euler equation with gravitation. Fjordholm et al. (2011) developed a well balanced and energy stable scheme for the shallow water equations with discontinuous bottom topography.