Vector Control Lyapunov Function Based Stabilization of Nonlinear Systems in Predefined Time

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***Abstract*—Predefined-time stability is the stability of dynamical systems whose solutions approach the equilibrium point within a predecided time duration. In this technical note, we develop general results of predefined-time stability of nonlinear systems using vector Lyapunov functions. A vector comparison system, which is predefined-time convergent, is constructed, and after that the stability of the original dynamical system is proved using differential inequalities and comparison principles. Moreover, we design predefined-time controllers for large-scale systems using vector control Lyapunov functions. Sliding-mode control is introduced in the design approach to mitigate matched bounded disturbances/uncertainties. Also, we aggregate comparison systems to reduce their dimensionality in order to effectively apply the derived results on practical systems. The theoretical results are implemented on a 2 DOF Helicopter model.**

***Index Terms*—Comparison principle, finite-time stability, Lyapunov function.**

I. Introduction

For nonlinear systems, various notions of stability, such as asymptotic and exponential stability, are used to describe the convergence of their trajectories to an equilibrium point in infinite time duration. However, as can be noted in the industrial and engineering sectors, several essential applications require a convergence of the trajectories to the equilibrium in finite/fixed-time or a prespecified time.

Researchers studied the finite-time stability of autonomous systems using continuous Hölder energy functions [1]. Finite-time stability was explored for several families of systems which include, in particular, homogeneous systems [2] and switched systems with uncertainty [3]. This stability was further investigated for higher order systems and using output feedback [4] as well. Finite-time estimation issues have been considered, e.g., in [5]. In most of these finite-time problems, the time of convergence primarily depends on initial conditions, which is indeed a crucial feature. The notion of fixed-time convergence to the equilibrium point has been introduced to overcome finite-time stability limitations. Fixed-time stability of systems has been investigated in particular in [6] in which uniformity relative to the initial conditions is required for computing the upper bound of the convergence time. Moreover, in the problems studied in [7], the fixed-time convergence depends on system parameters. Another notion of convergence that overcomes fixed-time problems' design constraints is prescribed-time convergence [8], which utilizes time scaling functions to obtain convergence precisely in the chosen time duration. However, in most of the problems, convergence time depends on initial conditions and system parameters. On the other hand, discontinuous controllers guaranteeing finite-time convergence were also developed in the literature [9]. Nevertheless, they result in chattering due to uncertainties or unmodeled dynamics in practical applications.

As a matter of fact, in various applications, it is advantageous to obtain the convergence of the trajectories in a predecided time, this is the case, for example, for differentiators and missile guidance. Hence, predefined-time convergent systems have been studied in [10], where scalar Lyapunov functions are the key tool of the proofs. The key feature of these systems is that the state and its derivative converge to zero as the time approaches the predefined time, independent of any initial condition. Vector Lyapunov functions (VLFs) were first introduced in [11] to relax certain strict conditions of scalar Lyapunov functions [12], [13], [14]. In particular, it is worth observing that the components of VLFs need not be all positive definite and that the derivative of the component of a VLF does not have to be necessarily negative or negative semidefinite to guarantee the stability of the studied systems. Hence, these functions enlarge the class of Lyapunov functions to analyze system stability. In this work, a general framework is developed to analyze the predefined-time stability of the equilibrium point of nonlinear systems using VLFs. Specifically, we formulate a vector comparison system in such a way that it is predefined-time stable and after that we relate these stability features with the stability features of the original system using differential inequalities and comparison principles. Besides, we design universal predefined-time convergent controllers for the large-scale systems and further discuss their robustness with respect to matched bounded disturbances. Moreover, in order to reduce the dimension of the comparison systems, we discuss the aggregation procedure of comparison systems, which provides a simple and efficient way to derive control for practical systems. At the end, the efficacy of the theoretic approach is verified using as example a 2 DOF Helicopter model. The control of this system is a very challenging problem as it represents a higher order, highly nonlinear, firmly coupled multi-input multi-output system.

The rest of this technical note is organized as follows. Section II is devoted to definitions and notations. Section III provides the main results of the predefined-time stability of nonlinear systems by the exploitation of VLFs. Universal predefined-time controllers are designed for large-scale systems and further their robustness with respect to

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matched bounded disturbances is discussed in Section IV. In addition, we discuss the aggregation procedure of comparison systems in order to apply the derived results effectively on practical systems. An illustrative example with the simulation results is given in Section V. Finally, Section VI concludes this article.

II. Mathematical Preliminaries

In this section, we provide the necessary notations and definitions. Let $\mathbb{R}$, $\mathbb{R}_{>q}$, and $\mathbb{R}_{\geq q}$ denote the sets of real numbers, real numbers greater than q, and real numbers greater than or equal to q. The set of the $n \times 1$ column vector is denoted by $\mathbb{R}^n$ and $[\cdot]^\top$ represents transpose. We denote $p \leq\leq q$, for $p = [p_1, p_2, \ldots, p_n]^\top$ and $q = [q_1, q_2, \ldots, q_n]^\top$, if $p_i \leq q_i$ for each $i = 1, 2, \ldots, n$. $\|\cdot\|_1$ and $\|\cdot\|$ denote the 1-norm and Euclidean norm in $\mathbb{R}^n$ or the induced matrix norms. Given $\zeta \in \mathbb{R}^n$, the Fréchet derivative of $V \in \mathbb{R}^p$ at ζ is denoted by $V'(\zeta)$. Define $d = [1, 1, \ldots, 1]^\top \in \mathbb{R}^p$. $C[E, F]$ denotes the set of the continuous functions from the nonempty set E to F where $E \subseteq \mathbb{R}^k$, and $F \subseteq \mathbb{R}^l$. For the set $U \in \mathbb{R}^n$, $\bar{U}$, and ∂U denote the closure and the boundary of this set, respectively. A square matrix M is known as a Metzler matrix if its off-diagonal entries are nonnegative. We denote the pseudoinverse of a nonsquare matrix $\mathcal{T}$ by $\mathcal{T}^+$. A function $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is known as a class $\mathcal{K}$ function, if it is continuous and strictly increasing with $\psi(0) = 0$. It is known as $\mathcal{K}_\infty$ function, if it is a $\mathcal{K}$ class function and $\psi(r) \to \infty$ as $r \to \infty$. Furthermore, to study finite-time, fixed-time, and predefined-time cases, we consider generalized functions [15]. A function $\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is known as a generalized $\mathcal{K}$ class function, if it is continuous with $\varphi(0) = 0$ and satisfies $\varphi(r_1) > \varphi(r_2)$, if $\varphi(r_1) > 0, r_1 > r_2$, and $\varphi(r_1) = \varphi(r_2)$, if $\varphi(r_1) = 0, r_1 > r_2$. A function $\mathit{\Lambda} : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be a generalized $\mathcal{KL}$ class function ($\mathcal{GKL}$), if for each fixed $t \geq t_0$, the function $\mathit{\Lambda}(r, t)$ with respect to r is a generalized class $\mathcal{K}$ function and the function $\mathit{\Lambda}(r, t)$ with respect to t is continuous and tends to zero as $t \to T, T < \infty$, for each fixed r. If T is some predefined time, then $\mathit{\Lambda}$ is called predefined $\mathcal{GKL}$ function ($\mathcal{PGKL}$ function).

Definition 1 (Quasi-monotone increasing function [16], [17]): Let $E \subseteq \mathbb{R}^n$ and let $e = [e_1, e_2, \ldots, e_n]^\top$ be an element of E. A function $Q = [Q_1, Q_2, \ldots, Q_n]^\top \in C[E, \mathbb{R}^n]$ is called quasi-monotone increasing on E if for every $i \in \{1, 2, \ldots, n\}$, Q_i is increasing in e_k for all $k = 1, 2, \ldots, i-1, i+1, \ldots, n$.

Let us consider the nonlinear system

$$\dot{\zeta} = F(\zeta, \tau), \qquad \zeta(t_0) = \zeta_0 \tag{1}$$

where state $\zeta \in D \subseteq \mathbb{R}^n$, $\tau \in \mathbb{R}^m$ is the control and $F : D \times \mathbb{R}^m \to \mathbb{R}^n$ is a continuous nonlinear vector field such that $F(0, \tau) = 0$, that is, origin $\zeta = 0$ is an equilibrium point of system (1) when control τ is applied. The following result is a fundamental comparison principle for nonlinear systems in the VLF framework.

Lemma 1 (see [18]): Let us consider system (1). Suppose that the continuously differentiable vector function $W : D \to l \subseteq \mathbb{R}^p_{\geq 0}$ is such that, for a specific τ, $W'(\zeta)F(\zeta, \tau) \leq\leq Q(W(\zeta)), \zeta \in D$, where $Q : l \to \mathbb{R}^p$ is a quasi-monotone increasing continuous function, such that $\dot{z}(t) = Q(z(t)), z(t_0) = z_0$, admits a unique solution $z(t)$ defined over $[t_0, \infty)$. If $W(\zeta_0) \leq\leq z_0$, $z_0 \in \mathbb{R}^p_{\geq 0}$, then $W(\zeta(t)) \leq\leq z(t)$ for all $t \geq t_0$, where $\zeta(t)$ is the solution of system (1) defined over $[t_0, \infty)$ when control τ is applied.

Now, we consider the time-varying differential system

$$\dot{\zeta} = -\phi(t, \zeta) := \begin{cases} \frac{-\gamma(e^{\zeta} - 1)}{e^{\zeta}(t_a - t)}, & \text{if} \quad t_0 \leq t < t_a \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

where $\zeta \in \mathbb{R}$, $\gamma \in \mathbb{R}_{>1}$, t_0 is the initial time and $t_a = T_A + t_0$, T_A is a predefined-time duration. It is easy to prove existence and uniqueness of the solutions of this system and to see that $\dot{\zeta}(t) = 0$ and $\zeta(t) = 0$ for all $t \geq t_a$ [10]. This system will be used when we establish the main results of the note.

Furthermore, consider the time-varying system

$$\dot{\zeta} = \mathcal{F}(t, \zeta, \tau, \sigma), \quad \zeta(t_0) = \zeta_0 \tag{3}$$

where state $\zeta \in D \subseteq \mathbb{R}^n$, $\sigma \in \mathbb{R}^p$ represent constant system parameters to be tuned, $\tau \in \mathbb{R}^m$ is the control, $\mathcal{F} : \mathbb{R}_{\geq 0} \times D \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^n$ is a continuous nonlinear vector field such that $\mathcal{F}(t, 0, 0, \sigma) = 0$ for all $t \geq 0$, that is, origin $\zeta(t) = 0$ is an equilibrium point of system (3). The following definition describes predefined-time stability.

Definition 2 (see [10]): System (3) is known as predefined-time stable at the origin for a control $\tau : \tau(t, \zeta, t_a)$ if

1) it is asymptotically stable and any solution $\zeta(t, t_0, \zeta_0)$ of (3) reaches the origin at some finite time, that is, $\zeta(t, t_0, \zeta_0) = 0$ for all $t \geq t_0 + T(t_0, \zeta_0)$, where $T : \mathbb{R}_{\geq 0} \times D \to \mathbb{R}_{\geq 0}$ denotes the convergence time;
2) it is possible to choose a predefined convergence time duration $T_A > 0$ $(t_a > t_0)$, which does not depend on initial conditions and can be chosen in advance;
3) the inequality $T_A \geq T_{tf}$ (weak predefined-time stable) can be established where T_{tf} denotes the true fixed time duration or actual time duration of convergence in which the system trajectories reach to the origin.

Remark 1: Note that T_A does not explicitly depends on any system parameter. In fact, T_A itself is an independent parameter, which is explicitly predefined in advance. Theoretically, one can choose any arbitrarily small value of T_A. However, we recall that due to inherent dynamics of practical systems (in particular, the actuator dynamics), these systems usually impose restrictions on assuming arbitrarily small values of T_A.

We provide following definitions to differentiate among finite-time, fixed-time, and predefined-time stability using the generalized $\mathcal{KL}$ class functions.

Definition 3: The origin of system (3) is called finite-time stable if there exists a $\mathcal{GKL}$ class function $\mathit{\Lambda}$ with $\mathit{\Lambda}(r, t) = 0$ when $t \geq T(r)$, where $T(r)$ is continuous with $T(0) = 0$ and $\|\zeta(t)\| \leq \mathit{\Lambda}(\|\zeta(t_0)\|, t)$.

Definition 4: The origin of system (3) is called fixed-time stable if it is finite-time stable and $\sup_{r \in \mathbb{R}_{\geq 0}} T(r) < \infty$.

Definition 5: The origin of system (3) is called predefined-time stable if there exist a $\mathcal{PGKL}$ class function $\mathit{\Lambda}$ with $\mathit{\Lambda}(r, t) = 0$ when $t \geq t_a$, where $t_a = T_A + t_0$, T_A is a predefined convergence time duration, which does not depend on initial conditions and can be chosen in advance, and α as a class $\mathcal{K}_\infty$ function, such that $\|\zeta(t)\| \leq \mathit{\Lambda}(\|\zeta(t_0)\|, t_a - t), \forall t \in [t_0, t_a)$ and $\zeta(t) \equiv 0 \forall t \geq t_a$ for all $\|\zeta(t_0)\| \leq \alpha(c)$.

III. Predefined-Time Stability Analyzed Via VLF

In this section, we derive results by using VLFs to analyze the predefined-time stability of nonlinear systems.

Theorem 1: Consider system (1). Suppose that there exist a continuously differentiable vector function $V = [V_1, V_2, \ldots, V_p]^\top : D \to S$, where $p \leq n$, $S \subset \mathbb{R}^p_{\geq 0}$ is an open and connected set, $0 \in S$ and a vector $r \in \mathbb{R}^p_{\geq 0}$ such that $r^\top V(\zeta)$ is a positive definite function, and there exists a control input $\tau(t, \zeta, t_a)$ such that

$$V'(\zeta)F(\zeta, \tau(t, \zeta, t_a)) \leq\leq M\Phi_{fr}(t, V(\zeta)), \quad \zeta \in D \tag{4}$$

where $\Phi_{fr}(t, V(\zeta)) := [\phi(t, V_1(\zeta)), \ldots, \phi(t, V_p(\zeta))]^\top$, ϕ is the function defined in (2), $M \in \mathbb{R}^{p \times p}$ is Metzler and Hurwitz, and such that $y^\top M \leq\leq -y^\top$ for all nonnegative vector $y \in \mathbb{R}^p_{\geq 0}$. Besides, suppose

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the following vector comparison system:

$$\dot{\eta}(t) = M\Phi_{fr}(t, \eta(t)), \quad \eta(t_0) = \eta_0, \quad \text{for all } t \geq t_0 \tag{5}$$

admits a unique solution $\eta(t) \in \mathbb{R}^p_{\geq 0}$ defined over $[t_0, \infty)$. Let $\zeta(t)$ be any solution of (1) with $\tau(t, \zeta, t_a)$ which satisfies (4), such that $V(\zeta_0) \leq\leq \eta_0$. Then, the solution $\zeta(t) = 0$ is predefined-time stable if $\gamma > p$.

Proof: Let us consider the comparison system (5). Observe that $M\Phi_{fr}(t, \eta)$ is a quasi-monotone increasing function of η uniformly in t_0. As a consequence, the solutions to (5) are nonnegative when $\eta_0 \in \mathbb{R}^p_{\geq 0}$ [23]. Now, let us consider the Lyapunov function $v = \eta^\top \eta, \eta \in \mathbb{R}^p_{\geq 0}$. Its time derivative along the trajectories of (5) is given by $\dot{v} = 2\eta^\top(t) M\Phi_{fr}(t, \eta(t))$. Then, from the definition of $\phi(\cdot)$ in (2), it follows that $\dot{v} = 0$ for all $t \geq t_a$. Now let us perform an analysis in the time interval $[t_0, t_a)$. Since $\eta^\top M \leq\leq -\eta^\top$, it follows that

$$\dot{v} \leq -2\eta^\top(t)\Phi_{fr}(t, \eta(t)), \quad \text{for all } t \in [t_0, t_a). \tag{6}$$

Let us introduce the function $\mathcal{M}ax$ defined by $\mathcal{M}ax(y) = \max_{i \in \{1,\ldots,p\}} y_i$. Observe that inequality (6) implies that

$$\dot{v} \leq -2\eta_1(t)\phi(t, \eta_1(t)), \ldots, \dot{v} \leq -2\eta_p(t)\phi(t, \eta_p(t)) \tag{7}$$

because $y\phi(t, y) \geq 0$ for all $y \in \mathbb{R}$ and $t \in [t_0, t_a)$. Suppose that at any particular instant $t \in [t_0, t_a)$, $\mathcal{M}ax(\eta(t)) = \eta_1(t)$. Then

$$||\eta(t)||^2 \leq p\eta_1(t)^2 \Rightarrow v(t) \leq p\eta_1(t)^2 \Rightarrow \sqrt{\frac{v(t)}{p}} \leq \eta_1(t). \tag{8}$$

Now using (7) and (8) and noting the fact that $-y_1\phi(t, y_1) \leq -y_2\phi(t, y_2)$ when $y_2 \leq y_1$, it is easy to obtain $\dot{v} \leq -2\sqrt{\frac{v}{p}}\phi\left(t, \sqrt{\frac{v}{p}}\right)$. Let us introduce the function: $w = \sqrt{\frac{v}{p}}$. Then, when $v(\eta_0) > 0$, the inequality $v(\eta(t)) > 0$, is satisfied for all $t \in [t_0, t_a)$. We deduce $\dot{w} = \frac{1}{2\sqrt{vp}}\dot{v} \leq \frac{-w\phi(t,w)}{\sqrt{vp}} \leq \frac{-\phi(t,w)}{p}$, for all$t \in [t_0, t_a)$. From the definition of $\phi(\cdot)$, it follows that $\dot{w} \leq \frac{-\gamma'(e^w - 1)}{e^w(t_a - t)}$, $\gamma' = \gamma/p$, for all $t \in [t_0, t_a)$. Using the fact that $\dot{v} = 0$ for all $t \geq t_a$, we deduce that $\dot{w} = 0$, for all $t \geq t_a$. Note that if $\gamma' > 1$ (i.e., $\gamma > p$), the dynamics of w is predefined-time stable [10]. Consequently, the dynamics of v is also predefined-time stable, which implies that the solution $\eta(t) = 0$ is predefined-time stable. Then, from the results of Lemma 1, we conclude that the solution $\zeta(t) = 0$ is predefined-time stable if $\gamma > p$. Note that a similar analysis can be carried out to show the predefined-time of convergence in the cases when $\mathcal{M}ax$ returns variables other than η_1. Let us observe that in the scalar case, i.e., $p = 1$, V reduces to V_1 and M is a constant m such that $m \leq -1$. Then, the condition in (4) reduces to $V_1'(\zeta)F(\zeta, \tau(t, \zeta, t_a)) \leq m\phi(t, V_1(\zeta)), \zeta \in D$, which directly ensures that if $\gamma > 1$, the dynamics is predefined-time stable. ■

Remark 2: It is important to discuss about the matrix M being a Metzler and Hurwitz matrix that satisfies $y^\top M \leq\leq -y^\top$ for $y \in \mathbb{R}^p_{\geq 0}$. Let us see some examples of M. The given condition leads to $y^\top(M + I) \leq\leq 0$, which can be alternatively written as $(M^\top + I)y \leq\leq 0$. One obtains, by selecting $M = \lambda I$, $(\lambda + 1)y \leq\leq 0$ which holds for all $\lambda \leq -1$. Although several other possibilities exist for M, above ones are the simplest.

Theorem 1 is generalized as follows.

Theorem 2: Consider system (1). Let us suppose that there exist a continuously differentiable vector function $V = [V_1, V_2, \ldots, V_p]^\top : D \to S$ where $p \leq n$, $S \subset \mathbb{R}^p_{\geq 0}$ is an open and connected set, $0 \in S$ and a vector $r \in \mathbb{R}^p_{\geq 0}$ such that $r^\top V(\zeta)$ is a positive definite function, and there exists $\tau(t, \zeta, t_a)$ such that

$$V'(\zeta)F(\zeta, \tau(t, \zeta, t_a)) \leq\leq Q(t, V(\zeta)), \quad \zeta \in D, t \geq t_0 \tag{9}$$

where $Q \in C[\mathbb{R}_{\geq 0} \times S, \mathbb{R}^p]$ is a quasi-monotone increasing function of V uniformly in t_0 with $Q(t, 0) = 0$ for all $t \geq t_0$. Besides, suppose the following vector comparison system:

$$\dot{\eta}(t) = Q(t, \eta(t)), \quad \eta(t_0) = \eta_0 \tag{10}$$

admits a unique solution $\eta(t) \in \mathcal{H} \subset \mathbb{R}^p_{\geq 0}$ defined over $[t_0, \infty)$ and is predefined-time stable. Let $\zeta(t)$ be any solution of (1) with $\tau(t, \zeta, t_a)$, which satisfies (9), such that $V(\zeta_0) \leq\leq \eta_0$. Then, $\zeta(t) = 0$ is predefined-time stable.

Proof: Let us assume that $U \subseteq \mathcal{H}$ is an open and bounded set such that $0 \in U$ and $\bar{U} \subset S$. Hence, ∂U is compact. We assume the same function $v(\cdot)$ as in Theorem 1 to be continuous, then, from the Weierstrass result, $v(\cdot)$ has a minimum on ∂U and $\alpha = \min_{\eta \in \partial U} v(\eta) > 0$. Suppose that $0 < \beta < \alpha$ and $D_\beta = \{\eta \in U : v(\eta) \leq \beta\}$. From the classical Lyapunov stability and positive definiteness of $v(\cdot)$, one can state that if $\epsilon > 0$, there exists $\delta > 0$ such that the ball B_δ satisfies, $B_\delta \subset D_\beta \subset \mathcal{H}$ and $\|\eta(t)\| \leq \epsilon, \forall\, t \geq t_0, \|\eta_0\| < \delta$. The abovementioned analysis establishes the boundedness of the solution $\eta(t)$. Now, we analyze the scalar case and the vector case one by one. First let us consider the scalar case, i.e., $p = 1$. In this case, we have $V = V_1$ and we replace $Q(t, V(\zeta))$ by $-\phi(t, V_1(\zeta))$ in (9) to obtain $V_1'(\zeta)F(\zeta, \tau(t, \zeta, t_a)) \leq -\phi(t, V_1(\zeta))$. Due to the continuity property of $V_1(\cdot)$, there exists $\delta_2 > 0$ such that $V_1(\zeta_0) < \delta, \forall\, \|\zeta_0\| < \delta_2$. Next, we replace $Q(t, \eta)$ by $-\phi(t, \eta)$ in (10) to obtain $\dot{\eta} = -\phi(t, \eta)$, whose solution is denoted by $\eta(t) = \eta(t, \eta_0)$. Let us choose the initial condition

$$\eta_0 = V_1(\zeta_0) \in B_\delta, \qquad \|\zeta_0\| < \delta_2. \tag{11}$$

Let us consider a scalar Lyapunov candidate function $v(\eta) = \eta^2$ whose time derivative along the trajectories of (10) is $\dot{v} = 2\eta\dot{\eta} = -2\eta\phi(t, \eta)$. This implies that $\dot{v} = 0$ for all $t \geq t_a$ and $\dot{v} \leq -2|\eta|\phi(t, |\eta|)$ for all $t \in [t_0, t_a)$. Noting the fact that $\sqrt{v(\eta)} = |\eta|$, we can write $\dot{v}(\eta) \leq -2\sqrt{v(\eta)}\phi(t, \sqrt{v(\eta)})$. Let us consider $w = \sqrt{v(\eta)}$, then, when $v(\eta_0) > 0$, the inequality $v(\eta(t)) > 0$ is satisfied for all $t \in [t_0, t_a)$. We deduce that $\dot{w} = \frac{1}{2\sqrt{v(\eta)}}\dot{v}(\eta) \leq -\phi(t, w)$ for all $t \in [t_0, t_a)$. We also see that $\dot{w} = 0$ for all $t \geq t_a$ leading to $w = 0$ for all $t \geq t_a$. Consequently, $v(\eta(t)) = 0$ for all $t \geq t_a$, from which it follows that

$$\eta(t) = 0, \text{for all} t \geq t_a, \quad \eta_0 \in B_\delta. \tag{12}$$

Note that the conclusion (12) can be reached directly by observing that $\dot{\eta} = -\phi(t, \eta)$ converges to the origin in predefined time t_a. Now, by using the comparison principle [24], for the considered initial condition (11) we have

$$V_1(\zeta(t)) \leq \eta(t), \quad \eta_0 \in B_\delta, \quad t \in [0, \infty). \tag{13}$$

From (12) to (13), it follows that $V_1(\zeta(t)) = 0$ for all $t \geq t_a$, $\|\zeta_0\| < \delta_2$. Consequently, $\zeta(t) = 0$ for all $t \geq t_a$. Thus, the solution $\zeta(t) = 0$ is predefined-time stable. Now, we consider the vector case, i.e., $p > 1$. Note that the vector comparison system (10) is assumed to be predefined-time stable, then it guarantees that the equality in (12) is also valid in the vector case of (10). Furthermore, we notice that $r^\top V(\zeta)$ is positive definite. Now, since $r^\top V(\zeta) \leq \max_{i=1,\ldots,p}\{r_i\} d^\top V(\zeta), \zeta \in D$, where d is a vector defined in Section II, we deduce that $d^\top V(\zeta)$ is also positive definite on $\zeta \in D$. Recalling the continuity property of $V(\cdot)$, there exists $\delta_2 > 0$ such that $\|V(\zeta_0)\| < \delta, \forall\, \|\zeta_0\| < \delta_2$. Let us choose $\eta_0 = V(\zeta_0) \in B_\delta$, for all $\|\zeta_0\| < \delta_2$. Then, from Lemma 1, it follows that $V(\zeta(t)) \leq\leq \eta(t)$. Utilizing (12), $d^\top V(\zeta(t)) \leq d^\top \eta(t) = 0, \forall\, t \geq t_a$ and since $d^\top V(\zeta(t))$ is nonnegative, it follows that $d^\top V(\zeta(t)) = 0, \forall\, t \geq t_a$. Since $d^\top V(\cdot)$ is positive definite, we conclude that $\zeta(t) = 0, \forall\, t \geq t_a, \forall\, \|\zeta_0\| < \delta_2$. Therefore, $\zeta(t) = 0$ is predefined-time stable. ■

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IV. Predefined-Time Stabilization of Large-Scale Nonlinear Systems

Let us consider the following nonlinear dynamical system consisting of p subsystems interconnected to each other:

$$\dot{\zeta}_i(t) = F_i(\zeta(t)) + H_i(\zeta(t))u_i(t) \tag{14}$$

where $F_i : \mathbb{R}^n \to \mathbb{R}^{n_i}$ with $F_i(0) = 0$ and $H_i : \mathbb{R}^n \to \mathbb{R}^{n_i \times m_i}$ with rank equal to $\min\{m_i, n_i\}$ for all ζ are the continuous functions, for $i = 1, \ldots, p$, $u_i \in \mathbb{R}^{m_i}$ is the control input, and $\zeta = [\zeta_1^\top, \zeta_2^\top, \ldots, \zeta_p^\top]^\top \in D \subseteq \mathbb{R}^n$ with $n = n_1 + n_2 + \cdots + n_p$, is the state. Furthermore, $u(t) \in \mathbb{R}^p$, where $p = m_1 + m_2 + \cdots + m_p$. It should be noted that the following control structure (16) is motivated by the Sontag's universal formula [19]. For brevity, we use $V_i(\zeta_i(t)) = V_i(\zeta_i)$.

Theorem 3: Consider system (14). Suppose that $V = [V_1, \ldots, V_p]^\top : D \to S$ with $V_i : \mathbb{R}^{n_i} \to \mathbb{R}$ is a continuously differentiable VLF, where $S \subset \mathbb{R}^p_{\geq 0}$ is an open and connected set, $0 \in S$ and $r \in \mathbb{R}^p_{\geq 0}$ is a vector such that $r^\top V(\zeta)$ is positive definite and

$$V_i'(\zeta_i)F_i(\zeta) \leq Q_i(t, V_i(\zeta_i)), \quad \zeta \in \mathcal{R}_i, i = 1, \ldots, p \tag{15}$$

where $\mathcal{R}_i = \{\zeta \in \mathbb{R}^n, \zeta \neq 0 : H_i^\top(\zeta)V_i'^\top(\zeta_i) = 0\}$. Let the proposed universal control be $u(t) = \tau(t, \zeta, t_a) = [\tau_1^\top(t, \zeta, t_a), \tau_2^\top(t, \zeta, t_a), \ldots, \tau_p^\top(t, \zeta, t_a)]^\top$

$$\tau_i = \begin{cases} -\left(\frac{A + \sqrt{A^2 + (b_i^\top(\zeta)b_i(\zeta))^2}}{b_i^\top(\zeta)b_i(\zeta)}\right) b_i(\zeta), & b_i(\zeta) \neq 0 \\ 0, & b_i(\zeta) = 0 \end{cases} \tag{16}$$

where $A = a_i(\zeta) - Q_i(t, V_i(\zeta_i)) + \beta V_i(\zeta_i)$, $a_i(\zeta) = V_i'(\zeta_i)F_i(\zeta)$, $b_i(\zeta) = H_i^\top(\zeta)V_i'^\top(\zeta_i)$, $i = 1, \ldots, p$, $\beta > 0$, and $Q \in C[\mathbb{R}_{\geq 0} \times S, \mathbb{R}^p]$ is a quasi-monotone increasing function of V uniformly in t_0 with $Q_i(t, 0) = 0$ for all $t \geq t_0$. Besides, suppose the following vector comparison system:

$$\dot{\eta}(t) = Q(t, \eta(t)), \quad \eta(t_0) = \eta_0 \tag{17}$$

admits a unique solution $\eta(t) \in \mathbb{R}^p_{\geq 0}$ defined over $[t_0, \infty)$ and is predefined-time stable. Let $\zeta(t)$ be any solution of (14) with $\tau(t, \zeta, t_a)$, which satisfies (15), such that $V(\zeta_0) \leq\leq \eta_0$. Then, the solution $\zeta(t) = 0$ is predefined-time stable.

Proof: Simple calculations give, for $i = 1$ to p

$$\dot{V}_i(\zeta_i) = a_i(\zeta) + b_i^\top(\zeta)u_i(t). \tag{18}$$

First case: $b_i(\zeta) \neq 0$. Using the proposed universal control (16), (18) becomes, $\dot{V}_i(\zeta_i) \leq Q_i(t, V_i(\zeta_i))$. *Second case*: $b_i(\zeta) = 0$. Control $\tau_i(t, \zeta, t_a) = 0$. The chosen VLF V_i satisfies $V_i'(\zeta_i)F_i(\zeta) \leq Q_i(t, V_i(\zeta_i))$.

Thus, the derivative of VLF along the solutions of system (14) with the control $u(t)$ satisfies $\dot{V}_i(\zeta_i) \leq Q_i(t, V_i(\zeta_i))$. Since, it is assumed that the comparison system (17) is predefined-time stable. Then, from Theorem 2, the solution $\zeta(t) = 0$ of system (14) is predefined-time stable when the control (16) is selected if $V(\zeta_0) \leq\leq \eta_0$. ■

A. Robust Predefined-Time Stabilization

It can be noted that the control structure (16) is a nominal control that stabilizes system (14) within the predefined time. The presence of matched disturbances can be handled with the incorporation of sliding-mode control. This results in discontinuous right-hand side differential equations whose solutions are realized in the Filippov's sense [25]. We consider system (14) with bounded nonvanishing disturbances

$$\dot{\zeta}_i(t) = F_i(\zeta(t)) + H_i(\zeta(t))(u_i(t) + \mathcal{D}_i(t)) \tag{19}$$

where $\mathcal{D}_i \in \mathbb{R}^{m_i}$ with $\|\mathcal{D}_i(t)\|_1 \leq \mathcal{D}_{0i}$ is the matched disturbance, which persists even when ζ has converged to zero for all $t \geq 0$ and $H_i(\zeta(t)) \neq 0$ for all ζ. In this case, we design the control $u(t)$ to make the solutions of system (19) converge to the origin in predefined time despite of the disturbances.

First case: $b_i(\zeta) \neq 0$. We propose the robust control $u(t) = \tau(t, \zeta, t_a) = [\tau_1^\top(t, \zeta, t_a), \tau_2^\top(t, \zeta, t_a), \ldots, \tau_p^\top(t, \zeta, t_a)]^\top$

$$\tau_i = -\left(\frac{A}{b_i^\top(\zeta)b_i(\zeta)}\right) b_i(\zeta) - K_i H_i^\top \mathrm{Sign}(V_i'(\zeta_i)) \tag{20}$$

where $K_i \geq \frac{\|H_i\|_1}{\|H_i H_i^\top\|_1}\mathcal{D}_{0i}$ for all ζ, $i = 1, 2, \ldots, p$, is a constant gain and $\mathrm{Sign}(V_i'(\zeta_i)) = [\mathrm{sign}(V_i'(\zeta_{i1})), \mathrm{sign}(V_i'(\zeta_{i2})), \ldots, \mathrm{sign}(V_i'(\zeta_{in_i}))]^\top$.

Second case: $b_i(\zeta) = 0$. We select $\tau_i(t, \zeta, t_a) = 0$. This choice ensures that $V_i'(\zeta_i)F_i(\zeta) \leq Q_i(t, V_i(\zeta_i))$.

Thus, the derivative of VLF along the solutions of system (19) with the constructed control $u(t)$ satisfies $\dot{V}_i(\zeta_i) \leq Q_i(t, V_i(\zeta_i))$, when $K_i \geq \frac{\|H_i\|_1}{\|H_i H_i^\top\|_1}\mathcal{D}_{0i}$ for all ζ. Hence, we consider the comparison system: $\dot{\eta}(t) = Q(t, \eta(t)), \eta(t_0) = \eta_0$ where $\eta \in \mathbb{R}^p_{\geq 0}$, and assume that it is predefined-time stable. If $V(\zeta_0) \leq\leq \eta_0$, then from Theorem 2, the solution $\zeta(t) = 0$ of system (19) is predefined-time stable despite of the bounded disturbances when the control (20) is selected. It can be observed that the aforementioned control structure (20) is discontinuous due to the inclusion of the signum multivalued function to mitigate matched bounded disturbances. Note that such control scheme cannot be constructed using Sontag's universal formula [19] for these cases.

B. Aggregation of Comparison Systems

In order to make the results derived above simpler and more elegant, we aggregate comparison systems to reduce their dimension. To that end, consider the following aggregation procedure for the linear systems: $\dot{\zeta} = \mathcal{A}\zeta + \mathcal{B}\tau$ where $\zeta \in \mathbb{R}^n$ is the state vector, $\tau \in \mathbb{R}^p$ is the control input and $\mathcal{A}, \mathcal{B}$ are constant matrices with appropriate dimensions. We use the transformation as $z = \mathcal{T}\zeta$ to convert this linear system into the aggregated model: $\dot{z} = \mathcal{P}z + \mathcal{G}\tau$, where $z \in \mathbb{R}^m$ is the state vector, $\mathcal{T} = [\cdot]_{m \times n}$ is a nonsquare matrix with $m < n$ and the matrices $\mathcal{P}$ and $\mathcal{G}$ are $\mathcal{P} = \mathcal{T}\mathcal{A}\mathcal{T}^+(\mathcal{T}\mathcal{T}^\top)^+and\mathcal{G} = \mathcal{T}\mathcal{B}$ [20] under the assumption that $\mathcal{T}$ is a full rank matrix which possesses a pseudoinverse [21]. It is also assumed that $\zeta \in N(\mathcal{T})$ if and only if $\zeta = 0$, where the nullspace $N(\mathcal{T})$ is defined as $N(\mathcal{T}) = \{\zeta : \mathcal{T}\zeta = 0\}$. In a similar way, we can aggregate the nonlinear system of the form

$$\dot{\zeta} = F(\zeta, \tau) \tag{21}$$

where $\zeta \in \mathbb{R}^n$ represents the state, $\tau \in \mathbb{R}^p$ is the control, and F is a nonlinear vector field. Let us apply the transformation $z = \mathcal{T}\zeta$, where $\mathcal{T}$ is a full-rank matrix that possesses a pseudoinverse to convert system (21) into $\dot{z} = f(z, \tau)$, where $z \in \mathbb{R}^m$ is the state vector with $m < n$ and $f(z, \tau) = \mathcal{T}F(\mathcal{T}^+z, \tau)$.

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V. Simulation Example

Consider the nonlinear dynamical equation of a 2 DOF Helicopter system

$$\mathcal{M}(x)\ddot{x} + \mathcal{C}(x,\dot{x})\dot{x} + g(x) = U + \mathcal{D} \tag{22}$$

where $x = [x_1, x_3]^\top$ denotes the pitch and yaw angles, $\dot{x} = [x_2, x_4]^\top$ denotes the pitch and yaw velocities, $U = [U_1, U_2]^\top = [K_{pp}V_{mp} + K_{py}V_{my}, K_{yp}V_{mp} + K_{yy}V_{my}]^\top$ is the control input vector, where $K_{pp}, K_{py}, K_{yp}, K_{yy}$ are the constant gains and V_{mp}, V_{my} are the input voltages to the pitch and yaw motors, respectively. The disturbance $\mathcal{D} = [\mathcal{D}_1, \mathcal{D}_2]^\top$ is supposed to be bounded: $|\mathcal{D}_i(t)| \le \mathcal{D}_{0i}, i = 1,2$ for all $t \ge 0$

$$\mathcal{M} = \begin{bmatrix} m_{11} & 0 \\ 0 & m_{22} \end{bmatrix}, \mathcal{C} = \begin{bmatrix} c_{11} & c_{12} \\ -2c_{12} & c_{22} \end{bmatrix}, g = \begin{bmatrix} m_h gl\cos(x_1) \\ 0 \end{bmatrix}$$

where $m_{11} = J_p + m_h l^2,\ m_{22} = J_y + m_h l^2\cos^2(x_1),\ c_{11} = B_p,$ $c_{12} = m_h l^2 x_4 \sin(x_1)\cos(x_1),\ c_{22} = B_y$. The parameters description and their values are specified in [22]. Now, consider a regulation problem to design a feedback control law U so that x tracks a constant reference x_r in predefined time despite of the bounded disturbances. Let $e = x - x_r$. Then, e satisfies the differential equation: $\mathcal{M}(x)\ddot{e} + \mathcal{C}(x,\dot{x})\dot{e} + g(x) = U + \mathcal{D}$. Our aim is to stabilize this system at $(e = 0, \dot{e} = 0)$, but this point is not an equilibrium point when $U = \mathcal{D} = 0$. Let $U = g(x) - k_p e + \tau$, where k_p is a positive definite diagonal matrix with entries k_{p1} and k_{p2}, and $\tau = [\tau_1, \tau_2]^\top$ is a control to be chosen appropriately. Now substituting U, we get, $\mathcal{M}(x)\ddot{e} + \mathcal{C}(x,\dot{x})\dot{e} + k_p e = \tau + \mathcal{D}$. This can be written as the set of state space equations:

$$\begin{aligned} \dot{e}_1 &= e_2 \\ \dot{e}_2 &= -9.27e_2 - 0.55e_4^2\sin(e_1)\cos(e_1) - 11.59k_{p1}e_1 \\ &\quad + 11.59(\tau_1 + \mathcal{D}_1) \\ \dot{e}_3 &= e_4 \\ \dot{e}_4 &= \frac{-2c_{12}}{m_{22}}e_2 - \frac{0.318}{m_{22}}e_4 - \frac{k_{p2}}{m_{22}}e_3 + \frac{(\tau_2 + \mathcal{D}_2)}{m_{22}}. \end{aligned} \tag{23}$$

We apply the aggregation procedure as discussed in Section IV. Let us apply the transformation as $z = \mathcal{T}\zeta$ with $\mathcal{T} = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 0 & 0 & 3 & 2 \end{bmatrix}, \zeta = [e_1, e_2, e_3, e_4]^\top, z = [z_1, z_2]^\top$ to transform system (23) into

$$\begin{aligned} \dot{z}_1 =& -2.3901z_1 - 0.0262z_2^2\sin(0.2308z_1)\cos(0.2308z_1) \\ &- 5.3499k_{p1}z_1 + 23.18(\tau_1 + \mathcal{D}_1) \\ \dot{z}_2 =& \ 0.461z_2 - \frac{0.308C_1z_1}{M_1} - \frac{(0.462k_{p2} + 0.098)z_2}{M_1} \\ &+ \frac{2(\tau_2 + \mathcal{D}_2)}{M_1} \end{aligned} \tag{24}$$

where $C_1 = -0.0147z_2\sin(0.2308z_1)\cos(0.2308z_1)$ and $M_1 = 0.0432 + 0.0478\cos^2(0.2308z_1)$. We design controls τ_1 and τ_2 as discussed in Section IV to make the solutions of system (24) converge to the origin in predefined time under the effect of disturbances $\mathcal{D}_1$ and $\mathcal{D}_2$. Let us consider the VLF, $V = [V_1, V_2]^\top$ with $V_1 = \frac{(z_1 - z_2)^2}{2}$ and $V_2 = \frac{(z_1 + z_2)^2}{2}$. It is easy to check that $r^\top V$ is positive definite, where $r = [1, 1]^\top$. In this case, $a_1 = (z_1 - z_2)E_1, a_2 = (z_1 + z_2)E_2, b_1 = H_1(z_1 - z_2),\quad b_2 = H_2(z_1 + z_2), A_1 = (a_1 - \phi_1(t, V_1) + \beta_1 V_1),$ and $A_2 = (a_2 - \phi_2(t, V_2) + \beta_2 V_2)$, where $E_1 = -2.3901z_1 - 0.0262z_2^2\sin(0.2308z_1)\cos(0.2308z_1) - 5.3499k_{p1}z_1,\quad E_2 = 0.461z_2 - \frac{0.308C_1z_1}{M_1} - \frac{(0.462k_{p2} + 0.098)z_2}{M_1},\ H_1 = 23.18,\ H_2 = \frac{2}{M_1},$ $\beta_i > 0$, and $\phi_i(t, \cdot), i = 1,2$ is the function defined in (2) with $\gamma = \gamma_i$. Now, the derivative of V_1 along the trajectories of (24) for all $t \in [t_0, t_a)$ after substituting controls designed according to (20), where $i = 1,2$, becomes, when $K_1 \ge \frac{1}{|H_1|}\mathcal{D}_{01}$: $\dot{V}_1 \le \frac{-\gamma_1(e^{V_1} - 1)}{e^{V_1}(t_a - t)}$. In a similar way, when $K_2 \ge \frac{1}{|H_2|}\mathcal{D}_{02}$, $\dot{V}_2 \le \frac{-\gamma_2(e^{V_2} - 1)}{e^{V_2}(t_a - t)}$. Also, note that the designed controls τ_1 and τ_2 will maintain $z(t) = 0$ for all $t \ge t_a$, hence, $\dot{V}_i = 0$ for all $t \ge t_a$. Thus, the comparison system constructed over $t \in [t_0, t_a)$ is $\dot{w}_i = \frac{-\gamma_i(e^{w_i} - 1)}{e^{w_i}(t_a - t)}$. For all $t \ge t_a, \dot{w}_i = 0$, for $i = 1,2$. The comparison system is quasi-monotone increasing and predefined-time stable in time t_a with $\gamma_i > 2$ as $p = 2$. Hence, it follows from Theorem 1 that the pitch and yaw angles regulate at the desired position in the set predefined time t_a despite of $\mathcal{D}_1, \mathcal{D}_2$ with the designed input voltages V_{mp} and V_{my}, respectively. The simulation results are shown in Fig. 1 with $K_i = 0.002, \gamma_i = 30.5, k_{p1} = k_{p2} = 1,$ $\beta_i = 5i = 1,2,\quad \mathcal{D}_1(t) = 0.01\sin 10t,$ and $\mathcal{D}_2(t) = 0.005\sin 10t$ with predefined time $t_a = 5$ s, and $t_a = 10$ s, respectively.

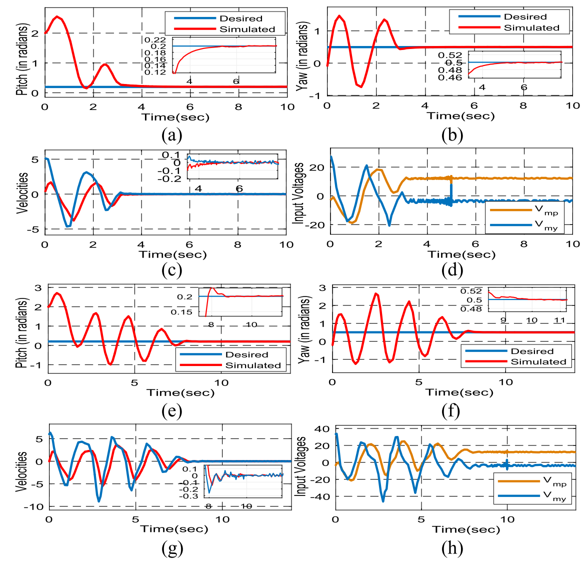

Fig. 1. States and input voltages of system (22) in predefined time, 1(a)–1(d) $t_a = 5$ s and 1(e)–1(h) 10 s.

VI. Conclusion

In this article, we presented the generalized control design approach to stabilize nonlinear systems in predefined time. We have shown that it is robust to matched bounded disturbances by using the framework of VLFs and comparison systems. We designed control so that the comparison system is predefined-time stable. After that, we relate these stability conditions with that of the original system by employing comparison principles. Furthermore, we aggregated the comparison system to reduce its dimension in order to make the proposed approach efficient and straightforward. Finally, we assessed through an example accompanied by simulations the efficacy of the mathematical results. In the future, the proposed work can be implemented on experimental setups.

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