CHAPTER-5

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5.1 Introduction

To find an algorithm for exact solutions of nonlinear partial differential equations, its method plays an important role. The nonlinear physical phenomena can be found in many branches of science and engineering, for example in fluid mechanics, plasma physics, atmospheric science, optical fiber communications etc. In the past few decades, there had been significant progress in the modifications where different methods are used viz. the inverse scattering method, Hirota's bilinear method, Backlund transformation method, Darboux transformation method, similarity transformation method, homogeneous balance method, the sine-cosine method, tanh function method, Jacobi elliptic function method, Painlev´e expansion method etc. Different phenomena in physics, like diffusion in a disordered or fractal medium, or in risk management, have been modelled using fractional partial differential equations. In general, there exists no method that gives exact solutions for these equations. However, in the past few decades, both mathematicians and physicists have dedicated their research to a considerable amount of study on explicit and numerical solutions of nonlinear differential equations of fractional order (Singh and Som (2012), Singh and Das (2013), Das et al. (2013), He (2011), Hristov (2015), Das (2009), Vishal et al. (2011)).

The contents of this chapter have been published in ***Nonlinear Sci. Lett. A.****, Vol.8, No.3, pp.276-288, 2017.*

The Homotopy Analysis Method (HAM) proposed by S.J. Liao (1992, 2003, 2004) is very effective and convenient for solving the linear and nonlinear ODEs and PDEs. The advantage of HAM is that it provides a simple way to adjust and control the convergence of the series solution by choosing proper values of auxiliary and homotopy parameters. A lot of researchers have used this method for solving fractional order differential equations.

The well-known (1+1) dimensional nonlinear fractional order Boussinesq-type equations are given by

$$u_t^{\alpha} + v_x + uu_x = 0,$$

$$v_t^{\beta} + (vu)_x + u_{xxx} = 0,\ 0 < \alpha, \beta < 1. \tag{5.1}$$

with initial conditions

$$u(x,0) = 2x,\ \ v(x,0) = x^2.$$

This equation appears in the modeling of nonlinear strings, which is a generalization of the classical Boussinesq equation (1872). Boussinesq shrinks into a nonlinear model with equations governing two-dimensional irrotational flows of an inviscid liquid into a uniform rectangular channel. Note that the equations in (5.1) are the perturbation of the classical linear wave equation which incorporates the basic idea of nonlinearity and dispersion. The fractional Boussinesq-type equation is the generalization of the case of the good means Boussinesq equation as compared to bad Boussinesq equation. This bad version appears in the research of water waves. Specifically, it is used to discuss a two-dimensional flow of a volume of water over a flat bottom with air above the water. It also appeared in a posterior study of Fermi-Pasta-Ulam (FPU) problem, which was performed to show that the finiteness of thermal conductivity of a harmonic lattice was coupled to nonlinear forces in the springs but it was not the case. This result is motivated by N. J. Zabusky and M. D. Kruskal (1965) to approach the FPU problem from the continuum point of view. The motivation of considering the fractional time derivative form of Boussinesq equations (5.1) is for their non-local behaviour, which provides a lot of flexibilities in the model.

In this chapter, The Homotopy analysis method is applied to solve the (1+1) dimensional nonlinear time-fractional Boussinesq-type equation, which is first of its kind. The salient feature of the problem is the graphical presentations and numerical discussion of the field variables $u\,(x,t)$ and $v\,(x,t)$ for various fractional Brownian motions and also for standard motion in different particular cases.

5.2 The basic idea of the Homotopy analysis method

In this section the method HAM which is applied to solve the (1+1) dimensional nonlinear fractional Boussinesq-type problem is discussed. In order to show the basic idea of HAM, consider the following differential equation

$$N[u(x,t)]=0, \tag{5.2}$$

where N is a non-linear operator, x and t are independent variables, $u(x,t)$ is the unknown function. By means of the HAM, let us first construct the so-called *zero*-th order deformation equation

$$(1-q)\,L[\phi(x,t;q)-u_0(x,t)]=\hbar\,H(x,t)N[\phi(x,t;q)], \tag{5.3}$$

where $q\in[0,1]$ is the embedding parameter, $\hbar\neq 0$, is a nonzero auxiliary parameter, $H(x,t)\neq 0$ is an auxiliary function, L is an auxiliary linear operator, $u_0\,(x,t)$ is the initial guess of $u(x,t)$, It is obvious that when the embedding parameter $q=0$ and $q=1$, equation (5.3) becomes $\phi(x,t;0)=u_0(x,t)$ and $\phi(x,t;1)=u(x,t)$ respectively. Thus, as q increases from 0 to 1, the solution $\phi(x,t;q)$ varies from the initial guess $u_0(x,t)$ to the exact solution $u(x,t)$. Expanding $\phi(x,t;q)$ in Taylor series with respect to q, one gets

$$\phi(x,t;q)=u_0(x,t)+\sum_{k=1}^{\infty}u_k(x,t)\,q^k \tag{5.4}$$

where, $$u_k(x,t)=\frac{1}{k!}\frac{\partial^k\phi}{\partial q^k}\bigg|_{q=0}. \tag{5.5}$$

The convergence of the series (5.4) depends upon the auxiliary parameter $\hbar$. If it is convergent at $q=1$, this leads to

$$\phi(x,t;q)=u_0(x,t)+\sum_{k=1}^{\infty}u_k(x,t),$$

which must be one of the solutions of the original nonlinear equation, as proven by Liao (2003). Now we define the vector as

$$\vec{u}_n(x,t)=\{u_0(x,t),u_1(x,t),u_2(x,t),........,u_n(x,t)\}, \tag{5.6}$$

So the *m*-th order deformation equations are

$$L[u_m(x,t)-\chi_m\, u_{m-1}(x,t)]=\hbar\, R_m(\vec{u}_{m-1}(x,t)), \tag{5.7}$$

with the initial conditions

$$u_m(x,0)=0\,, \tag{5.8}$$

where

$$R_m(\vec{u}_{m-1}(x,t))=\frac{1}{(m-1)!}\frac{\partial^{m-1}N[\phi(x,t;q)]}{\partial q^{m-1}}\bigg|_{q=0}$$

and $\chi_m=\begin{cases}0, & m\le 1,\\ 1, & m>1.\end{cases}$

Now, the solution of the *m*-th order deformation equation (5.7) for $m\ge 1$ becomes

$$u_m(x,t)=\chi_m u_{m-1}(x,t)+\hbar J_t^{\alpha}[R_m(\vec{u}_{m-1}(x,t))]+c\,, \tag{5.9}$$

where c is the integration constant determined by the initial condition (5.8) and $J_t^{\alpha}\left(f(t)\right)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\xi)^{\alpha-1}f(\xi)\,d\xi$. In this way, it is easy to obtain $u_m(x,t)$ for $m\ge 1$, and finally the solution is obtained as

$$u(x,t)=\sum_{m=0}^{N-1}u_m(x,t)\,. \tag{5.10}$$

5.3 Solution procedure by HAM

This section deals with the algorithm to find the solution of equations (5.1) by HAM under the initial conditions

$$u_0(x,\,t)=2x,\;\; v_0(x,t)=x^2. \tag{5.11}$$

Let us assume the linear operators as

$$L_u[\phi(x,t;q)]=\frac{\partial^{\alpha}\phi(x,t;q)}{\partial t^{\alpha}},\quad L_v[\psi(x,t;q)]=\frac{\partial^{\beta}\psi(x,t;q)}{\partial t^{\alpha}},\tag{5.12}$$

with the property

$$L_u[c_1]=0,\ L_v[c_2]=0,\tag{5.13}$$

where c_1 and c_2 are integral constants, coefficients ϕ and ψ are real functions. Furthermore, equation (5.1) suggests to define an equation of nonlinear operators as

$$N_u[\phi(x,t;q).\psi(x,t;q)]=\frac{\partial^{\alpha}\phi(x,t;q)}{\partial t^{\alpha}}+\frac{\partial\psi(x,t;q)}{\partial x}+\phi(x,t;q)\frac{\partial\phi(x,t;q)}{\partial x},$$

$$N_v[\phi(x,t;q).\psi(x,t;q)]=\frac{\partial^{\beta}\psi(x,t;q)}{\partial t^{\beta}}+\psi(x,t;q)\frac{\partial\phi(x,t;q)}{\partial x}$$

$$+\phi(x,t;q)\frac{\partial\psi(x,t;q)}{\partial x}+\frac{\partial^3\phi(x,t;q)}{\partial x^3},\tag{5.14}$$

where $q\in[0,1]$, $\phi(x,t;q)$ and $\psi(x,t;q)$ are real functions of (x,t). Let $\hbar_u$ and $\hbar_v$ denote the non-zero auxiliary parameters using the assumptions $H_u(x,t)=1$ and $H_v(x,t)=1$.

Now, the *zero*-th order deformation equations are constructed as follows:

$$(1-q)L_u[\phi(x,t;q)-u_0(x,t)]=q\hbar_u N_u[\phi(x,t;q).\psi(x,t;q)],$$

$$(1-q)L_v[\psi(x,t;q)-v_0(x,t)]=q\hbar_v N_v[\phi(x,t;q).\psi(x,t;q)].\tag{5.15}$$

Obviously, when $q=0$, and $q=1$, we have

$$\phi(x,\, t\,;\, 0) = u_0(x,t)\ ,\ \ \psi(x,t,0) = v_0(x,t), \tag{5.16}$$

$$\phi(x,\, t\,;\, 1) = u(x,t)\ ,\ \ \psi(x,t,1) = v(x,t).$$

The power series of q is given in following form as

$$\phi(x,t,q) = u_0(x,t) + \sum_{n=1}^{\infty} u_n(x,t) q^n, \tag{5.17}$$

$$\psi(x,t,q) = v_0(x,t) + \sum_{n=1}^{\infty} v_n(x,t) q^n .$$

According to Fundamental Theorem of HAM, the *n*-th order deformation equations are

$$L_u\left[u_n(x,t) - \chi_n u_{n-1}(x,t)\right] = \hbar_u R_n^u\left(\vec{u}_{n-1}, \vec{v}_{n-1}\right), \tag{5.18}$$

$$L_v\left[v_n(x,t) - \chi_n v_{n-1}(x,t)\right] = \hbar_v R_n^v\left(\vec{u}_{n-1}, \vec{v}_{n-1}\right),$$

where

$$R_n^u\left(\vec{u}_{n-1}, \vec{v}_{n-1}\right) = \frac{\partial^\alpha u_{n-1}(x,t)}{\partial t^\alpha} + \frac{\partial v_{n-1}(x,t)}{\partial x} + \sum_{i=0}^{n-1} u_i(x,t) \frac{\partial u_{n-1-i}(x,t)}{\partial x}$$

$$R_n^u\left(\vec{u}_{n-1}, \vec{v}_{n-1}\right) = \frac{\partial^\alpha v_{n-1}(x,t)}{\partial t^\alpha} + \sum_{i=0}^{n-1} v_i(x,t) \frac{\partial u_{n-1-i}(x,t)}{\partial x} + \sum_{i=0}^{n-1} u_i(x,t) \frac{\partial v_{n-1-i}(x,t)}{\partial x} + \frac{\partial^3 u_{n-1}(x,t)}{\partial x^3}$$

and $\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases}$

Now the solutions of *n*-th order deformation equation for $n \ge 1$ become

$$u_n(x,t)=\chi_n u_{n-1}(x,t)+\hbar_u\left[L^{-1}(u_{n-1},v_{n-1})\right], \tag{5.19}$$

$$v_n(x,t)=\chi_n v_{n-1}(x,t)+\hbar_u\left[L^{-1}(u_{n-1},v_{n-1})\right].$$

For simplicity $\hbar_u=\hbar_v$ is taken.

Now proceeding as in the previous section 5.3, $u_i(x,t)$ and $v_i(x,t)$ $(i=1,2,3)$ are successively obtained as follows:

$$u_1(x,t)=\hbar\,6x\frac{t^{\alpha}}{\Gamma(\alpha+1)}, \tag{5.20}$$

$$u_2(x,t)=\hbar(\hbar+1)\,6x\frac{t^{\alpha}}{\Gamma(\alpha+1)}+24\hbar^2x\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+12\hbar^2x\frac{t^{\alpha+\beta}}{\Gamma(\beta+\alpha+1)}, \tag{5.21}$$

$$u_3(x,t)=\hbar(\hbar+1)^2\,6x\frac{t^{\alpha}}{\Gamma(\alpha+1)}+48\hbar^2(\hbar+1)x\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+24\hbar^2(\hbar+1)x\frac{t^{\alpha+\beta}}{\Gamma(\beta+\alpha+1)}$$

$$+108\hbar^3x\frac{t^{2\alpha+\beta}}{\Gamma(\beta+2\alpha+1)}+48\hbar^3\frac{t^{2\beta+\alpha}}{\Gamma(2\beta+\alpha+1)}+96\hbar^3x\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+36\hbar^3x\frac{t^{3\alpha}}{\left(\Gamma(\alpha+1)\right)^2}. \tag{5.22}$$

$$v_1(x,t)=\hbar\,6x^2\frac{t^{\beta}}{\Gamma(\beta+1)}, \tag{5.23}$$

$$v_2(x,t)=\hbar(\hbar+1)\,6x^2\frac{t^{\beta}}{\Gamma(\beta+1)}+30\hbar^2x^2\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+24\hbar^2x^2\frac{t^{2\beta}}{\Gamma(2\beta+1)}, \tag{5.24}$$

$$v_3(x,t)=\hbar(\hbar+1)^2\,6x^2\frac{t^{\beta}}{\Gamma(\beta+1)}+78\hbar^2(\hbar+1)x^2\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+60\hbar^2(\hbar+1)x^2\frac{t^{2\beta}}{\Gamma(2\beta+1)}$$

$$+72\hbar^3 x^2 \frac{t^{2\alpha+\beta}}{\Gamma(\beta+2\alpha+1)}+216\hbar^3 x^2 \frac{t^{2\beta+\alpha}}{\Gamma(2\beta+\alpha+1)}$$

$$+108\hbar^3 x^2 \frac{t^{2\beta+\alpha}\Gamma(\beta+\alpha+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(2\beta+\alpha+1)}+144\hbar^3 x^2 \frac{t^{3\beta}}{\Gamma(3\beta+1)}. \tag{5.25}$$

In this way the rest of the terms of u_n and v_n ; $n > 3$ can completely determined and the series solutions are thus entirely obtained.

Finally, the approximate analytical solutions of $u(x,t)$ and $v(x,t)$ are obtained by the truncated series as

$$u(x,t) = \lim_{N\to\infty} \Phi_N(x,t),\ v(x,t) = \lim_{N\to\infty} \Theta_N(x,t), \tag{5.26}$$

where $\Phi_N(x,t) = \sum_{n=0}^{N-1} u_n(x,t),\ \ \Theta_N(x,t) = \sum_{n=0}^{N-1} v_n(x,t),\ \ N \geq 1.$

The above series solutions generally converge very rapidly. The rapid convergence means that only few terms are required to get the accurate results.

5.4 Results and discussion

In this section, the numerical values of $u(x,t)$ and $v(x,t)$ with the given initial conditions and the proper choices of $\hbar$, are obtained and the results are depicted through graphs. To demonstrate the efficiency of the method we compare the Homotopy Analysis Method solutions of fractional order (1+1) dimensional nonlinear Boussinesq equation, thus obtained for $\alpha=\beta=1$ with its exact solutions given in Taghizadeh (2013), obtained using HPM. The fact is to be noted that HAM series solutions of fractional order (1+1) dimensional nonlinear Boussinesq equation obtained by involving the auxiliary parameter $\hbar$ provide us with a simple way to adjust and control the convergence of the solution series. To obtain the appropriate value of $\hbar$, which ensures that the solutions series are convergent, as pointed out by Liao (2004) by finding out valid values of $\hbar$=-0.174578, and $\hbar$= -0.325283. It is also concluded for the HAM, the convergence of the values of $u(x,t)$ and $v(x,t)$ are found quite similar by controlling the values of auxiliary parameter $\hbar$ and homotopy parameter q . This clearly demonstrates the statement of Liao (1992) that the method provides great flexibility to choose initial approximation, homotopy parameter q , the auxiliary linear operator L , and the auxiliary parameter $\hbar$ to ensure the convergence of the series solutions.

Fig. 5.1 and Fig. 5.2 show the variations of $u(x,t)$ in $0 \le t \le 1$ at $x=1$ for $\hbar$=-0.174578 obtained by third order HAM solution given in equation (5.26) for $\alpha=0.25, 0.50, 0.75, 1$ at $\beta=1$ and for $\beta=0.25, 0.50, 0.75, 1$ at $\alpha=1$ respectively. In both cases $u(x,t)$ decreases as t increases.

Fig. 5.3 and Fig. 5.4 show the numerical solutions of $u(x,t)$ at $t=0.1$ in $-4 \le x \le 4$ for $\hbar$= -0.174578 at $\alpha=\beta=1$ and $\beta=\alpha=.25,.50,.75$. It is seen from the figures that $u(x,t)$ increases with increase in α and also $u(x,t)$ increases with decrease in β. The variations for $v(x,t)$ for similar cases are depicted through Figs. 5.5 – 5.8. Fig 5.9 and Fig. 5.10 present the comparison of $u(x,t)$ and $v(x,t)$ with the exact solutions Taghizadeh (2013) for standard order case i.e., for $\alpha=\beta=1$. The three dimensional variations of the field variables $u(x,t)$ and $v(x,t)$ w.r.t. x and t are displayed through Figs. 5.11 – 5.12 and Figs 5.13 – 5.14 respectively.

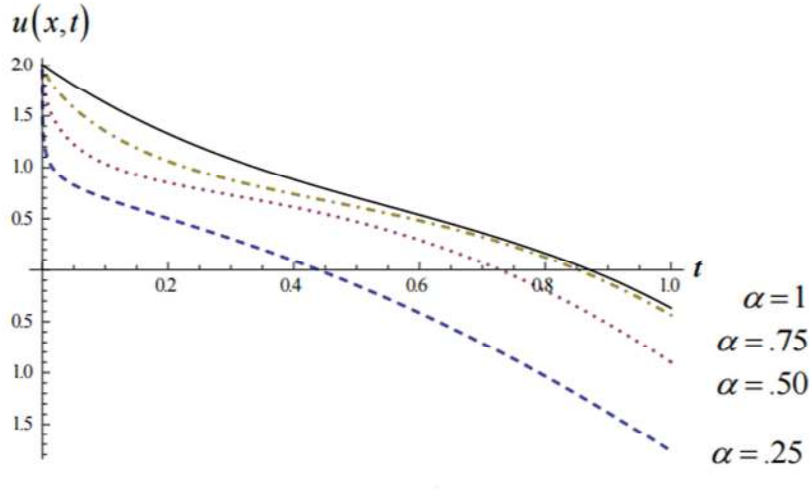

Fig. 5.1. Plot of $u(x,t)$ vs t at $\beta = 1$ and $x = 1$.

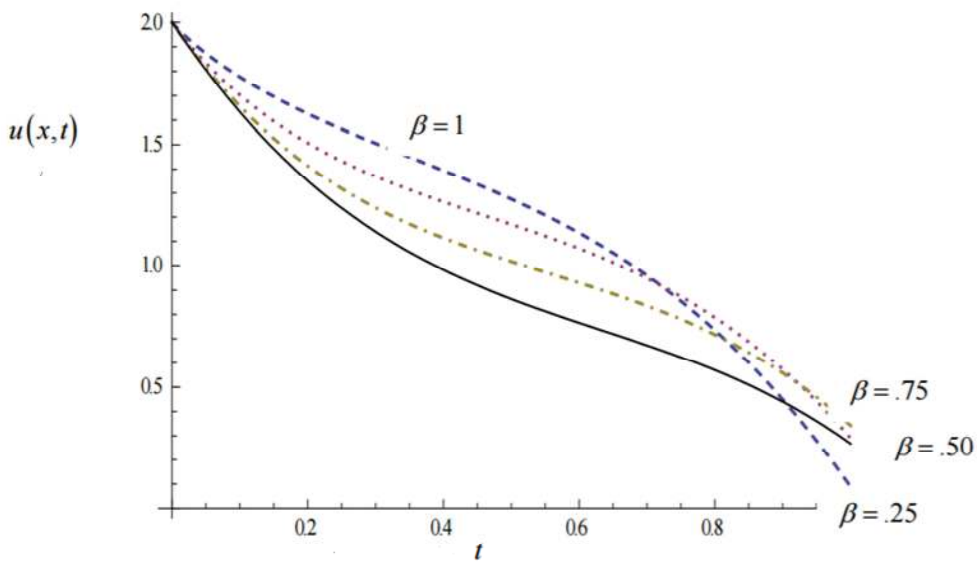

Fig. 5.2. Plot of $u(x,t)$ vs t at $\alpha = 1$ and $x = 1$.

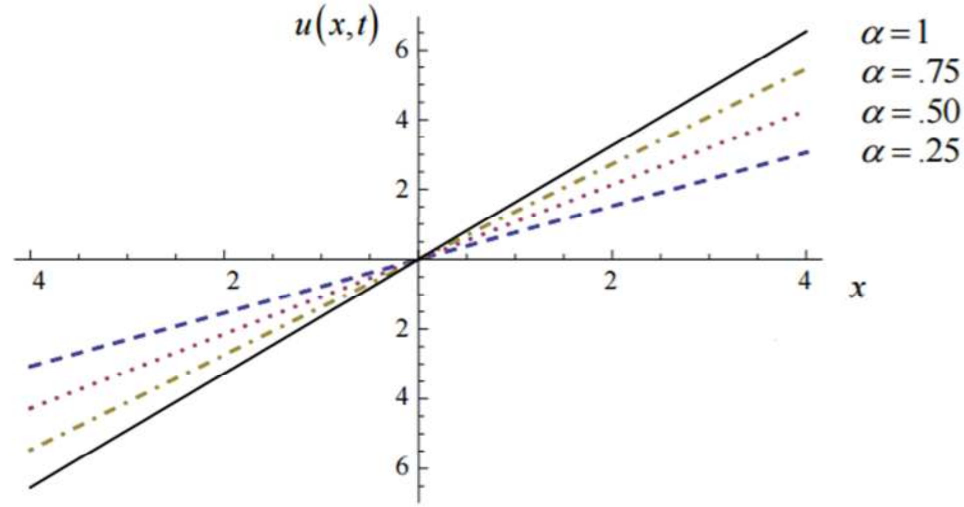

Fig. 5.3. Plot of $u(x,t)$ vs x at $\beta = 1$ and $t = 0.1$.

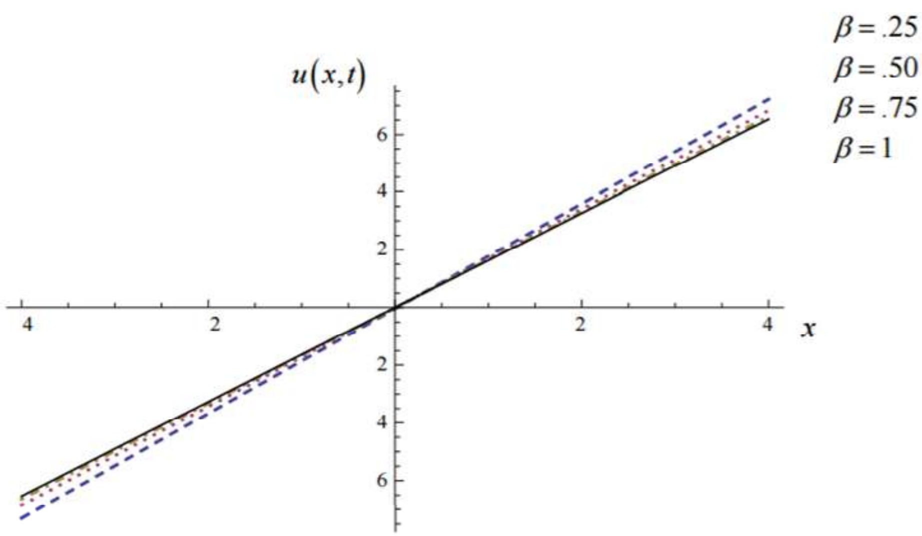

Fig. 5.4. Plot of $u(x,t)$ vs x at $\alpha = 1$ and $t = 0.1$.

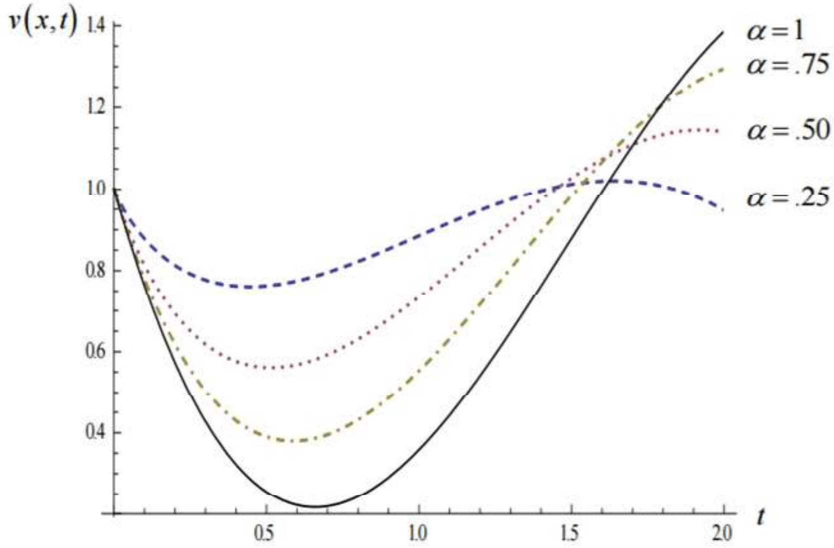

Fig. 5.5. Plot of $v(x,t)$ vs t at $\beta = 1$ and $x = 1$.

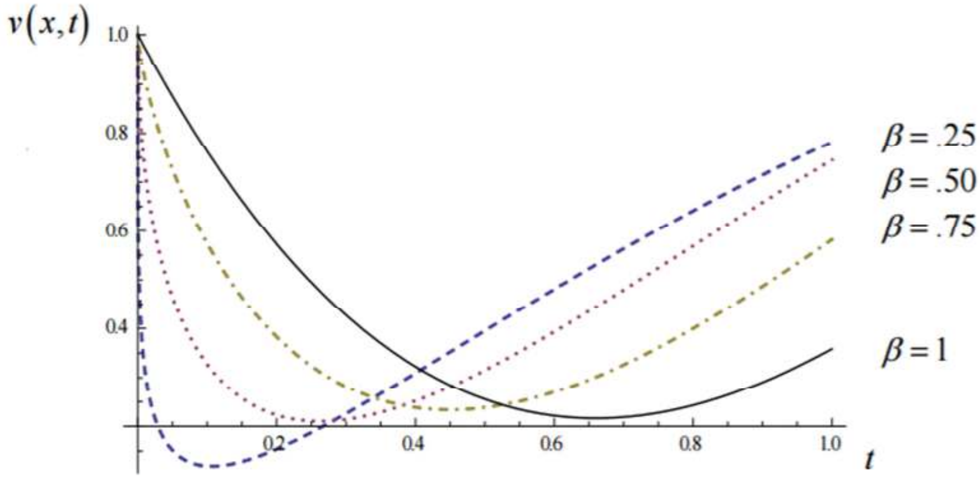

Fig. 5.6. Plot of $v(x,t)$ vs t at $\alpha = 1$ and $x = 1$.

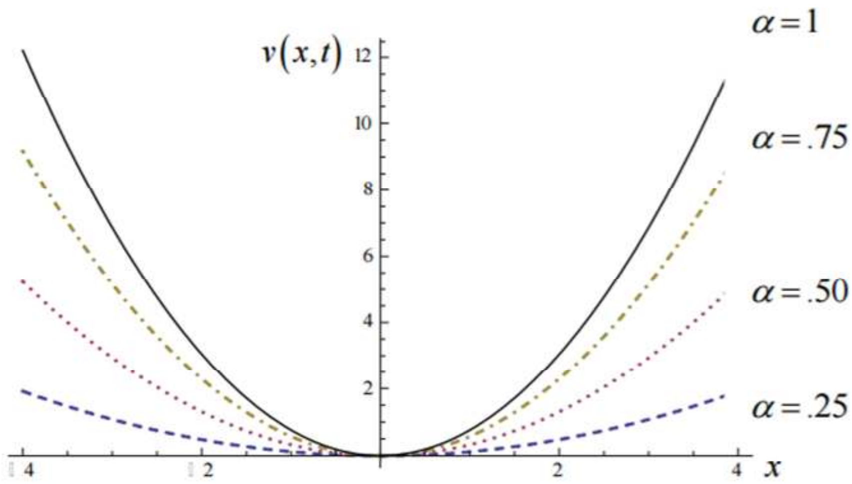

Fig. 5.7. Plot of $v(x,t)$ vs x at $\beta = 1$ and $t = 0.1$.

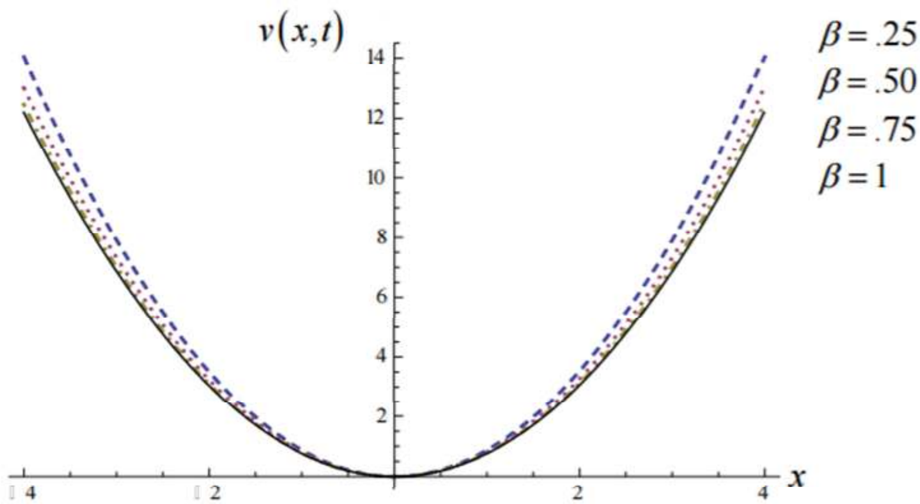

Fig. 5.8. Plot of $v(x,t)$ vs x at $\alpha = 1$ and $t = 0.1$.

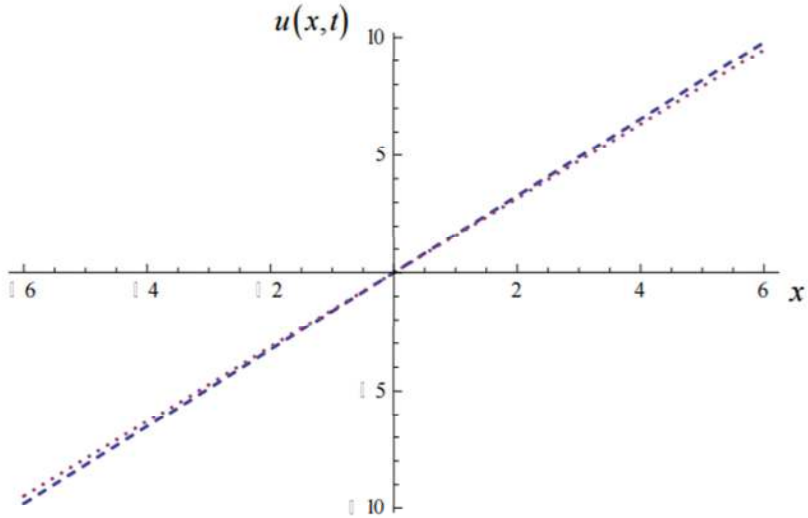

Fig. 5.9. Comparision of u(x, t) with exact solution at $\alpha = 1$ and $\beta = 1$ w.r.t x.

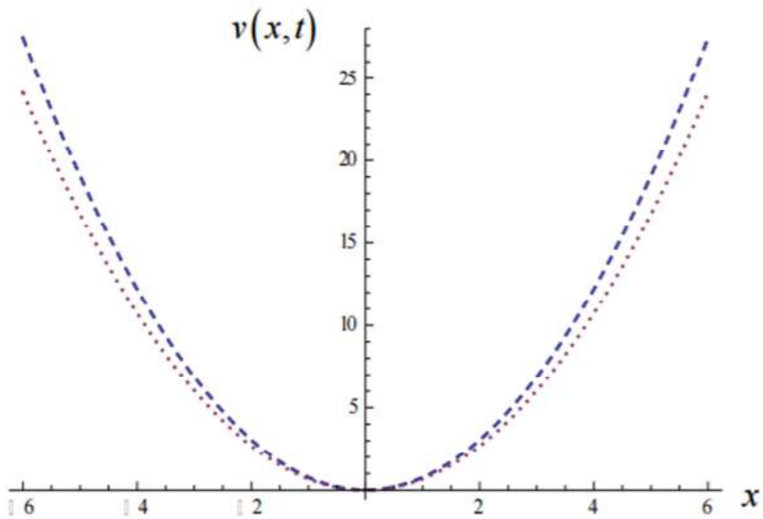

Fig. 5.10. Comparision of $v(x, t)$ with exact solution at $\alpha = 1$ and $\beta = 1$ w.r.t x.

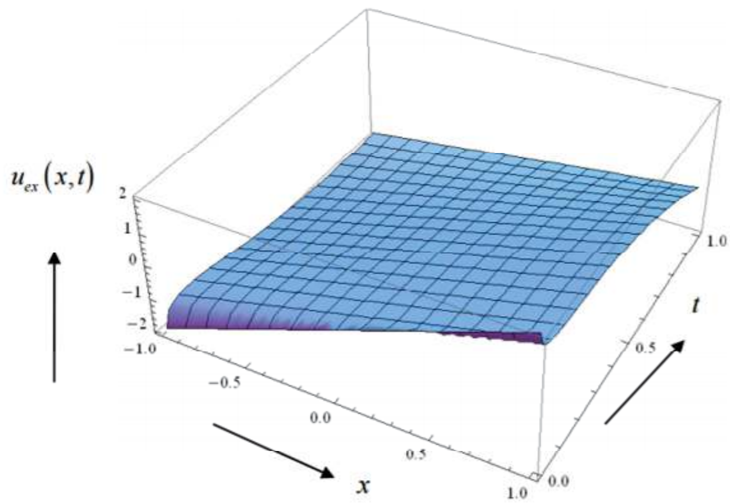

Fig. 5.11. Plot of $u_{ex}(x,t)$ w.r.t x and t.

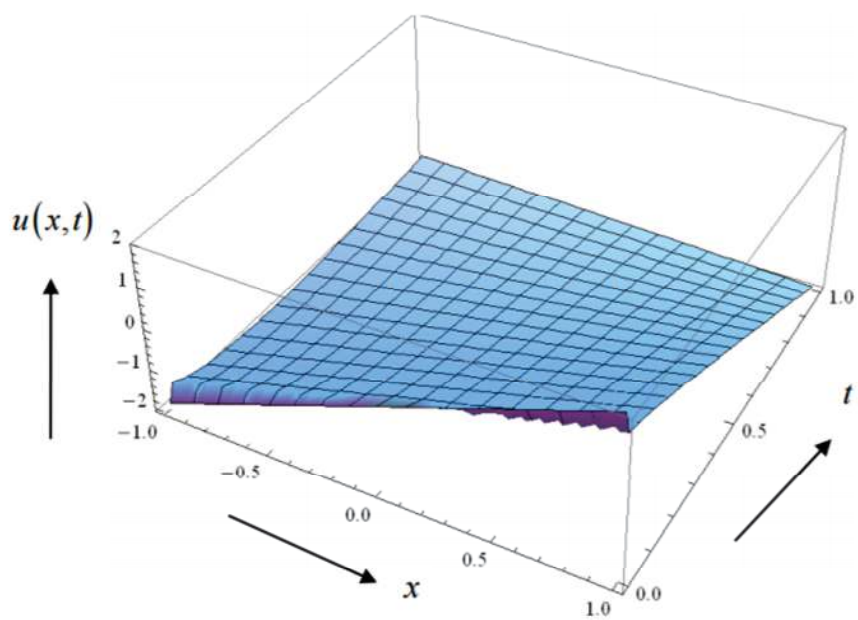

Fig. 5.12. Plot of $u(x,t)$ w.r.t x and t for $\alpha = 0.5$ and $\beta = 0.5$.

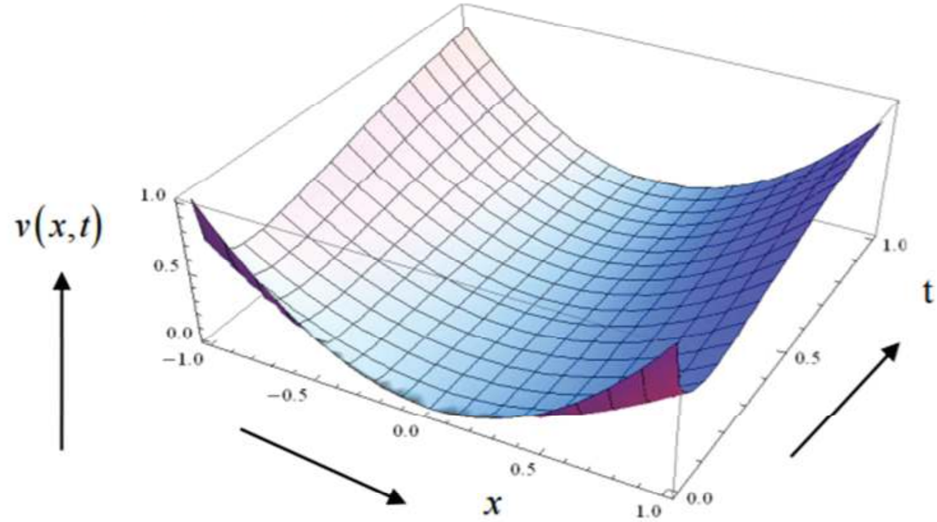

Fig. 5.13. Plot of $v_{ex}(x,t)$ w.r.t x and t.

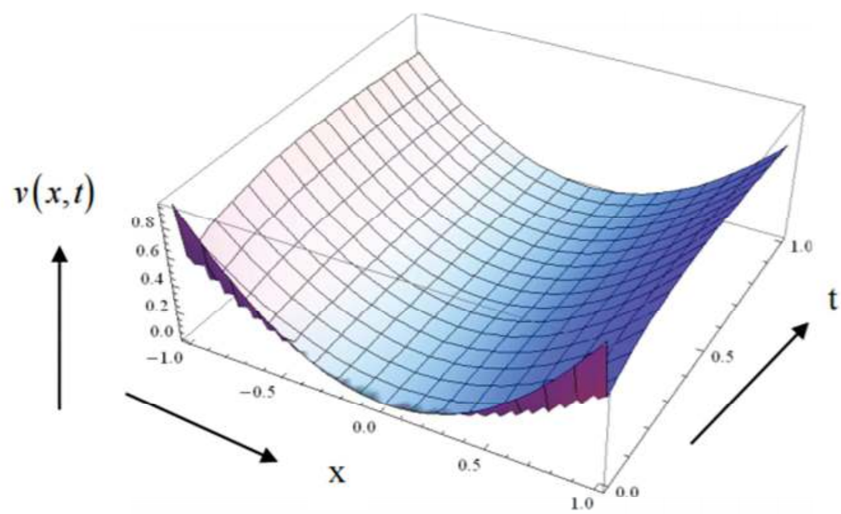

Fig. 5.14. Plot of $v(x,t)$ w.r.t x and t for $\alpha = 0.5$ and $\beta = 0.5$.

5.5 Conclusion

In this chapter the Homotopy analysis method has been successfully applied to obtain approximate analytical solution of fractional order (1+1) dimensional nonlinear Boussinesq equation. It has been explained that HAM solution of the problem in fractional order converges very rapidly to the exact one by choosing an approximate auxiliary parameter $\hbar$. In present study, it has proven that the HAM is a powerful and efficient technique in finding the approximate analytical solution of fractional order (1+1) dimensional nonlinear Boussinesq equation and can be used for many other fractional evolution equations arising in various fields.

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