CHAPTER-3

Combination-Combination phase synchronization among non-identical fractional order complex chaotic systems via nonlinear control

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3.1 Introduction

Mathematical modeling of dynamical systems describes the evolution of a system with initial conditions. Several mathematical modeling problems can be recast as a global problem and can be solved with appropriate mathematical tools. Newtonian mechanics is the origin of dynamical systems, and later it was found in many disciplines like population dynamics in biology, chemical kinetics in chemistry, mechanics in physics, sociology, etc. Mathematical modeling of a dynamical system is expressed in the form of difference equation for discrete case and differential equation for continuous case.

Nonlinearity is quite common nature of dynamical systems. The nonlinear dynamical system has evolved as an important convergence between the engineering and mathematics disciplines. An interesting phenomenon associated with it is the possibility of chaos which means occurrence of irregular solution while equation of motion is deterministic.

The study of dynamic behaviour in nonlinear fractional order systems has become an interesting topic to the scientists and engineers. Fractional calculus is playing an important role in the analysis of nonlinear dynamical systems. Nowadays from factional order modeling has been an active field of research theoretical and applied perspectives.

Fractional calculus theory provides the generalization of the order of the derivative and integration from integer to any real number and complex number. Due to its memory property and nonlocal behaviour, fractional calculus are used in various physical areas of engineering and sciences such as viscoelasticity, biological model, material science,

The contents of this chapter have been communicated in ***International Journal of Dynamics and Control*** *2017* electromagnetic wave, dielectric polarization etc. (Bagley and Calico (1991), Magin (2010), Carpinteri et al. (2004), Heaviside (1971), Sun et al.(1984)).

The chaotic dynamical system is a kind of nonlinear dynamical systems which exhibits exponentially sensitive dependence on initial conditions with infinite unstable periodic nature and bounded unstable dynamic behaviour. The behaviour of sensitive dependence on initial conditions is popularly known as 'The Butterfly effect' (Alligood et al. (1997)). The research on chaotic dynamics of fractional order systems is growing rapidly in past few years ((Srivastava et al. (2014)), Erjaee and Taghvafard (2011), Li and Chen (2004), Luo and Wang (2013)).

Synchronization between two identical (or non-identical) chaotic systems named one as a drive (master) and another as response (slave) systems means, the trajectories of these two systems generated from different initial conditions will converge and will remain with each other and make the systems structurally stable. Synchronization of chaotic systems through a simple coupling was first introduced by Pecora and Carroll (1990). Synchronization of chaos can be useful in physical systems, secure communications, ecological systems, chemical systems, etc. (Lakshmanan and Murali (1996, 2003), Blasius and Huppert (1999), Haung (2005)). Chaotic synchronization of fractional order differential systems has become one of the most interesting subjects in chaos theory. Recently, various effective techniques have been presented and successfully applied to achieve chaos control and synchronization (Razminia et al. (2011), Abd-Elouahab et al. (2010), Hegazi et al. (2013), Wang et al. (2009)). Different types of synchronization have been proposed in recent years such as active control, adaptive control, linear and nonlinear and time delay feedback approach, sliding mode control, backstep design method, etc. and concept of synchronization has been extended and investigated to chaos control, complete synchronization, phase and, anti-phase synchronization, and also various types of synchronization viz., generalized, lag, projective, hybrid, dual, Compound, double compound, combination, combination-combination, finite-time combination-combination, finite-time stochastic combination, hybrid and reduced-order hybrid combination synchronizations (Ott et al. (1990), Mainieri and Rehacek (1999), Li (2007), Barajas-Ramirez et al. (2003), Liu and Davids (2000), Sun et al. (2013a, 2013b), Wu and Zhang (2014), Zhang and Deng (2014), Luo et al. (2011), Runzi and Yinglan

(2013), Wu and Fu (2013), Sun et al. (2013), Lin et al. (2013), Zhou et al. (2014), Sun et al.(2014), Ojo et al. (2014a, 2014b), Singh et al. (2017)). These have motivated the authors to study on the combination-combination phase synchronization of fractional order non-identical complex chaotic systems. In phase synchronization, the coupled chaotic systems keep their phase difference bounded by a constant while their amplitudes remain uncorrelated. The phase synchronization usually applied upon two waveforms of the same frequency with identical phase angles with each cycle. However, it can be applied if there is an integer relationship of frequency such that the cyclic signals share a repeating sequence of phase angles over consecutive cycles. There are few results about the phase synchronization for the fractional order chaotic systems (Eejaee and Taghvafard (2011), Yadav et al. (2015), Das et al. (2013a)).

Motivated by the above discussions, a combination of two drive systems and a combination of two response systems are synchronized in the present chapter, which is known as combination–combination synchronization. The complex systems have several important applications in the field of engineering and physics, for example secure communications and detuned lasers (Murali and Lakshmanan (2003), Mahmoud et al. (2009)) etc. The Combination-Combination synchronization has advantages over the other types of synchronization since it provides good security. Numerical simulations for C-C phase synchronization show the effectiveness and feasibility of the method. To the best of authors' knowledge the combination-combination phase synchronization for the non-identical complex chaotic systems for fractional order derivatives using nonlinear control method has not yet been solved.

3.2 Problem formulation and method

3.2.1 Problem Formulation

In this section our aim is to investigate the Combination-Combination phase synchronization between complex dynamical systems. Let us consider two mater systems as

$$\frac{d^q X}{dt^q} = AX + f(X),$$

where $X = X_1 + iX_2$ is complex state variable with

$$\left.\begin{aligned}\frac{d^q X_1}{dt^q} &= AX_1 + f_1(X_1)\\ \frac{d^q X_2}{dt^q} &= AX_2 + f_2(X_2)\end{aligned}\right\}. \tag{3.1}$$

and

$$\frac{d^q Y}{dt^q} = BY + g(Y),$$

where $Y = Y_1 + iY_2$ is complex state variable with

$$\left.\begin{aligned}\frac{d^q Y_1}{dt^q} &= AY_1 + g_1(Y_1)\\ \frac{d^q Y_2}{dt^q} &= AY_2 + g_2(Y_2)\end{aligned}\right\}, \tag{3.2}$$

where $X_1 = [X_{11}, X_{13}, ..., X_{1(2n-1)}]^T$, $X_2 = [X_{12}, X_{14}, ..., X_{1(2n)}]^T$ and $Y_1 = [Y_{11}, Y_{13}, ..., Y_{1(2n-1)}]^T$, $Y_2 = [Y_{12}, Y_{14}, ..., Y_{1(2n)}]^T$ are the state variables. $A, B \in R^{n \times n}$ are the constant matrix with proper dimensions, $f, g : C^n \to C^n$ are complex vector valued functions of the systems.

Next we consider two fractional order complex systems as response (slave) systems as

$$\frac{d^q Z}{dt^q} = CZ + h(Z) + U,$$

where $Z = Z_1 + iZ_2$ is complex state variable with

$$\left.\begin{aligned}\frac{d^q Z_1}{dt^q} &= CZ_1 + h_1(Z_1) + U_1\\ \frac{d^q Z_2}{dt^q} &= CZ_2 + h_2(Z_2) + U_2\end{aligned}\right\}. \tag{3.3}$$

and $\dfrac{d^q W}{dt^q} = DW + I(W) + V$,

where $W = W_1 + iW_2$ is complex state variable with

$$\left.\begin{aligned}\frac{d^q W_1}{dt^q} &= DW_1 + I_1(W_1) + V_1\\ \frac{d^q W_2}{dt^q} &= DW_2 + I_2(W_2) + V_2\end{aligned}\right\}, \tag{3.4}$$

where $Z_1 = [Z_{11}, Z_{13}, ..., Z_{1(2n-1)}]^T$, $Z_2 = [Z_{12}, Z_{14}, ..., Z_{1(2n)}]^T$ and $W_1 = [W_{11}, W_{13}, ..., W_{1(2n-1)}]^T$, $W_2 = [W_{12}, W_{14}, ..., W_{1(2n)}]^T$ are the state variables. $C, D \in R^{n\times n}$ are the constant matrix with proper dimensions, $h, I : C^n \to C^n$ are complex vector valued functions of the systems. $U = U_1 + iU_2$, $V = V_1 + iV_2$ are control functions, $U_1 = [U_{11}, U_{13}, ..., U_{1(2n-1)}]^T$, $U_2 = [U_{12}, U_{14}, ..., U_{1(2n)}]^T$ and $V_1 = [V_{11}, V_{13}, ..., V_{1(2n-1)}]^T$, $V_2 = [V_{12}, V_{14}, ..., V_{1(2n)}]^T$.

Now our goal is to obtain the combination-combination synchronization between master and slave systems. Defining the error function between the master and slave systems as

$e = W + Z - Y - X$, where $e = e_{11} + ie_{12}$. (3.5)

For combination-combination synchronization we use the nonlinear control method to design the control functions in such a way that the error dynamic becomes asymptotically stable i.e., $\lim_{t\to\infty}\| e \| = \lim_{t\to\infty}\|W + Z - Y - X\| = 0$.

From equation (3.5), we will get the following error systems as

$e_{11} = W_1 + Z_1 - Y_1 - X_1$ and $e_{12} = W_2 + Z_2 - Y_2 - X_2$,

from which the error dynamics can be obtained as

$$\frac{d^q e_{11}}{dt^q} = De_{11} - (A-D)X_1 - (B-D)Y_1 + (C-D)Z_1 - f_1(X_1) - g_1(Y_1) + h_1(Z_1) + I_1(W_1) + U^*,$$

$$\frac{d^q e_{12}}{dt^q} = De_{12} - (A-D)X_2 - (B-D)Y_2 + (C-D)Z_2 - f_2(X_2) - g_2(Y_2) + h_2(Z_2) + I_2(W_2) + V^*, \tag{3.6}$$

where $U^* = U_1 + V_1$ and $V^* = U_2 + V_2$ are control functions.

3.2.2 Nonlinear control method

Theorem 3.1: If the nonlinear control functions are designed as

$$U^* = -k_1 e_{11} - De_{11} + (A-D)X_1 + (B-D)Y_1 - (C-D)Z_1 + f_1(X_1) + g_1(Y_1) - h_1(Z_1) - I_1(W_1),$$

$$V^* = -k_2 e_{12} - De_{12} + (A-D)X_2 + (B-D)Y_2 - (C-D)Z_2 + f_2(X_2) + g_2(Y_2) - h_2(Z_2) - I_2(W_2), \tag{3.7}$$

then the combination-combination synchronization among two master systems I, II and two response systems I, II are achieved.

Proof: Let us define the Lyapunov function to stabilize the error systems (3.6) as

$$V = \frac{1}{2}(e_{11}^2 + e_{12}^2) \tag{3.8}$$

taking the q-th order fractional derivative of equation (3.8) w. r. to t, we have

$$\frac{d^q V}{dt^q} \le e_{11}\frac{d^q e_{11}}{dt^q} + e_{12}\frac{d^q e_{12}}{dt^q}. \text{ (Using Lemma 1.4)} \tag{3.9}$$

After putting the values of $\frac{d^q e_{11}}{dt^q}$, $\frac{d^q e_{12}}{dt^q}$ and control functions from equation (3.6) and (3.7) in equation (3.9), we obtain

$$\frac{d^q V(e)}{dt^q} \le -(k_1 e_{11}^2 + k_2 e_{12}^2) < 0.$$

Thus we may conclude that for $k_1, k_2 > 0$, $\frac{d^q V(e)}{dt^q} < 0$ is negative definite and thus master systems I, II and response system I, II are combination-combination synchronized according to definition of error systems.

If the any eigen value of the error system is equal to zero, then another type of synchronization phenomenon called combination-combination phase synchronization occurs, in which the difference between various states of synchronized systems may not necessarily converge to zero, but is less than or equal to a constant.

3.3 Systems' descriptions

3.3.1 Fractional order Complex Lorenz system

The fractional order complex Lorenz system (Luo and Wang (2013)) is given by

$$\begin{aligned}
\frac{d^q x_1'}{dt^q} &= a_{11}(x_2' - x_1'), \\
\frac{d^q x_2'}{dt^q} &= a_{12}x_1' - x_2' - x_1'x_3', \\
\frac{d^q x_3'}{dt^q} &= \frac{1}{2}(\bar{x}_1'x_2' + x_1'\bar{x}_2') - a_{13}x_3',
\end{aligned} \tag{3.10}$$

where $x' = [x_1', x_2', x_3']^T$ is the state variable vector, $x_1' = x_{11} + ix_{12}$ and $x_2' = x_{13} + ix_{14}$ are complex variables while $x_3' = x_{15}$ is real variable and a_{11}, a_{12}, a_{13} are parameters'.

Separating into real and imaginary parts, we get the system (3.10) as

$$\frac{d^q x_{11}}{dt^q} = a_{11}(x_{13} - x_{11}),$$

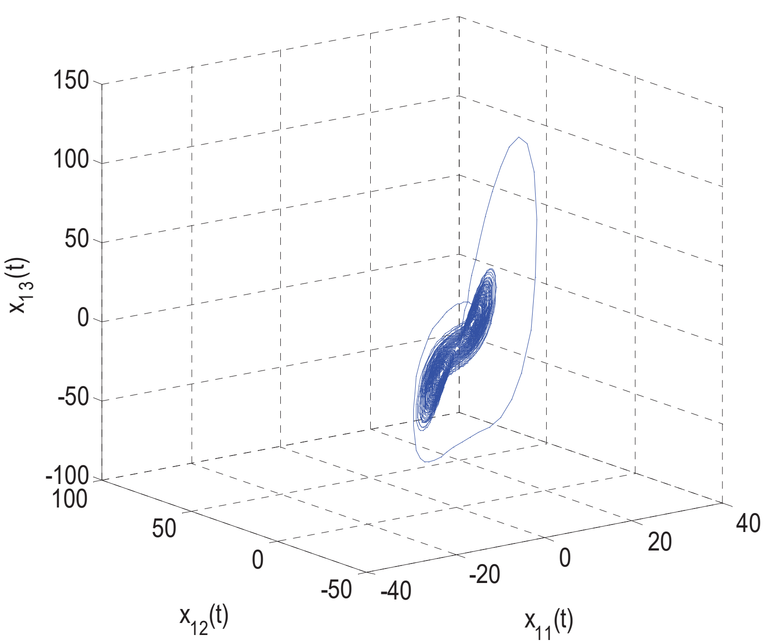
$$\frac{d^q x_{12}}{dt^q} = a_{11}(x_{14} - x_{12}),$$

$$\frac{d^q x_{13}}{dt^q} = a_{12}x_{11} - x_{13} - x_{11}x_{15}, \tag{3.11}$$

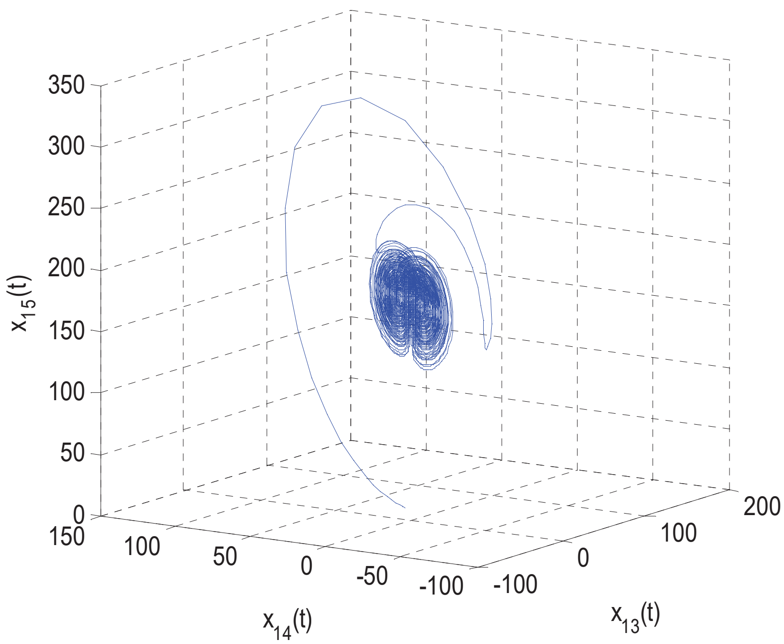
$$\frac{d^q x_{14}}{dt^q} = a_{12}x_{12} - x_{14} - x_{12}x_{15},$$

$$\frac{d^q x_{15}}{dt^q} = x_{11}x_{13} + x_{12}x_{14} - a_{13}x_{15}.$$

Taking the values of the parameters' as $a_{11} = 10,\ a_{12} = 180,\ a_{13} = 1$ and initial conditions $x(0) = [2,\ 3,\ 5,\ 6,\ 9]^T$ at the fractional derivative $q = 0.95$, the system (3.11) possesses the chaotic attractor given in Fig. 3.1.



(a)



(b)

Fig. 3.1. Phase portraits of fractional order complex Lorenz system at $q = 0.95$: (a) in $x_{11} - x_{12} - x_{13}$ space, (b) in $x_{13} - x_{14} - x_{15}$ space.

3.3.2 The fractional order complex T-system

The fractional order complex T-system (Liu et al. (2014), Yadav et al. (2016)) is

$$\frac{d^q y_1'}{dt^q} = b_{11}(y_2' - y_1'),$$
$$\frac{d^q y_2'}{dt^q} = (b_{12} - b_{11})y_1' - b_1 y_1' y_3', \tag{3.12}$$
$$\frac{d^q y_3'}{dt^q} = \frac{1}{2}(\bar{y}_1' y_2' + y_1' \bar{y}_2') - b_{13} y_3',$$

where $y' = [y_1', y_2', y_3']^T$ is the state variable vector of the system, $y_1' = y_{11} + iy_{12}$ and $y_2' = y_{13} + iy_{14}$ are complex variables, $y_3' = y_{15}$ is real variable and b_{11}, b_{12}, b_{13} are parameters. Separating complex variables into real and imaginary parts, the system (3.12) is reduced to

$$\frac{d^q y_{11}}{dt^q} = b_{11}(y_{13} - y_{11}),$$

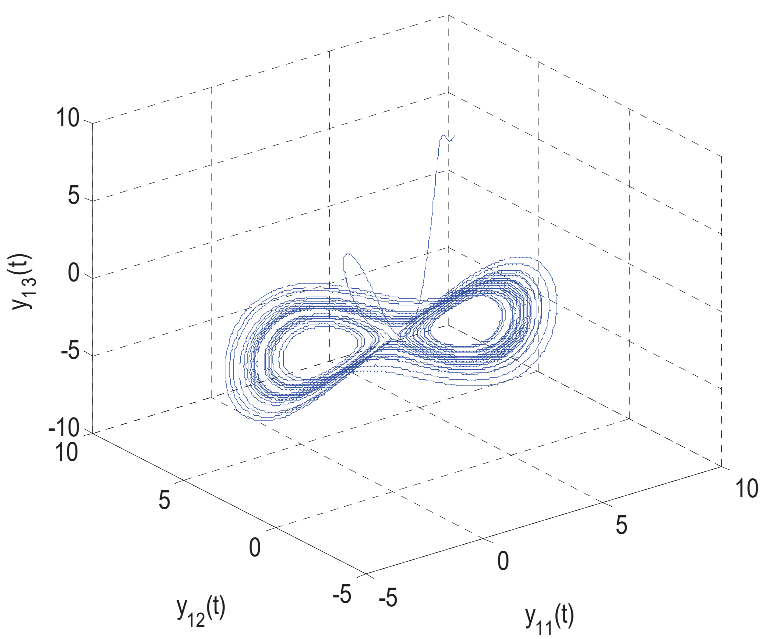
$$\frac{d^q y_{12}}{dt^q} = b_{11}(y_{14} - y_{12}),$$

$$\frac{d^q y_{13}}{dt^q} = (b_{12} - b_{11})y_{11} - b_{11}y_{11}y_{15}, \tag{3.13}$$

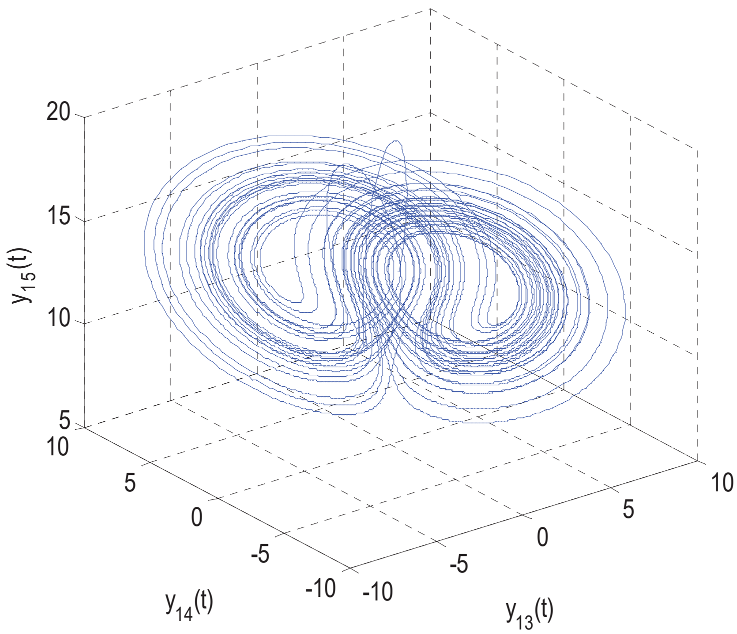
$$\frac{d^q y_{14}}{dt^q} = (b_{12} - b_{11})y_{12} - b_{11}y_{12}y_{15},$$

$$\frac{d^q y_{15}}{dt^q} = y_{11}y_{13} + y_{12}y_{14} - b_{13}y_{15}.$$

The system (3.13) possesses a chaotic attractors which are shown through Fig. 3.2 for the values of the parameters' $b_{11} = 2.1,\ b_{12} = 30,\ b_{13} = 0.6$, initial condition $y(0) = [8,\ 7,\ 6,\ 8,\ 7]^T$ at $q = 0.95$.



(a)



(b)

Fig. 3.2. Phase portraits of fractional order complex T-system at $q = 0.95$: (a) in $y_{11} - y_{12} - y_{13}$ space, (b) in $y_{13} - y_{14} - y_{15}$ space.

3.3.3 Fractional order Complex Lu system

The fractional order complex Lu system (Singh et al. (2016a)) is described as

$$\frac{d^q z_1'}{dt^q} = c_{11}(z_2' - z_1')$$

$$\frac{d^q z_2'}{dt^q} = -z_1' z_3' + c_{12} z_2' \tag{3.14}$$

$$\frac{d^q z_3'}{dt^q} = \frac{1}{2}(\bar{z}_1' z_2' + z_1' \bar{z}_2') - c_{13} z_3',$$

where $z_1' = z_{11} + jz_{12}$ and $z_2' = z_{13} + jz_{14}$, $j = \sqrt{-1}$ are the complex state variables and $z_3' = z_{15}$ is a real state variable.

System (3.14) is separated into real and imaginary part as follows

$$\frac{d^q z_{11}}{dt^q} = c_{11}(z_{13} - z_{11})$$

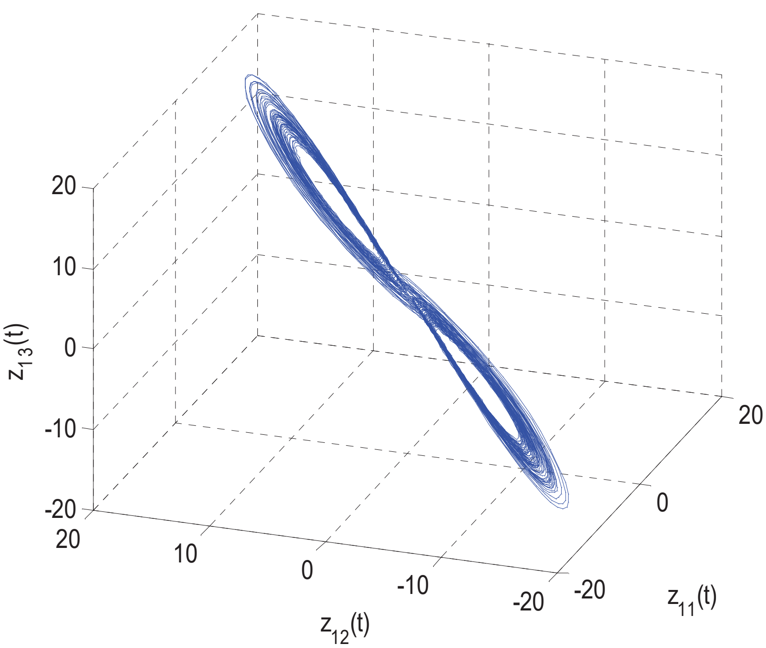
$$\frac{d^q z_{12}}{dt^q} = c_{11}(z_{14} - z_{12})$$

$$\frac{d^q z_{13}}{dt^q} = -z_{11}z_{15} + c_{12}z_{13} \tag{3.15}$$

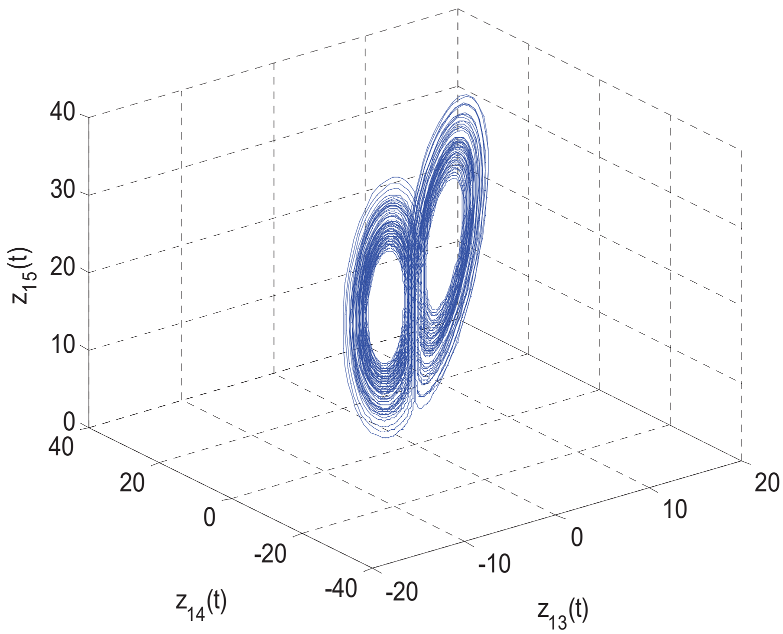
$$\frac{d^q z_{14}}{dt^q} = -z_{12}z_{15} + c_{12}z_{14}$$

$$\frac{d^q z_{15}}{dt^q} = z_{11}z_{13} + z_{12}z_{14} - c_{13}z_{15}.$$

The phase portraits of system (3.15) are depicted through Fig. 3.3 at $q = 0.95$ for the parameters' values $c_{11} = 40,\ c_{12} = 22,$ and $c_{13} = 5$ and initial conditions $z(0) = [2,\ 4,\ 6,\ 5,\ 3]^T$.



(a)



(b)

Fig. 3.3. Phase portraits of fractional order complex Lu system at $q = 0.95$: (a) in $z_{11} - z_{12} - z_{13}$ space, (b) in $z_{13} - z_{14} - z_{15}$ space.

3.3.4 Fractional order Complex Chen system

The fractional order complex Chen system (Luo and Wang (2014)) is considered as

$$\frac{d^q w_1'}{dt^q} = d_{11}(w_2' - w_1')$$

$$\frac{d^q w_2'}{dt^q} = (d_{13} - d_{11})w_1' - w_1'w_3' + d_{13}w_2' \tag{3.16}$$

$$\frac{d^q w_3'}{dt^q} = -d_{12}w_3' + \frac{1}{2}(\overline{w}_1'w_2' + w_1'\overline{w}_2'),$$

where $w' = [w_1', w_2', w_3']^T$ is the state variable vector of the system, $w_1' = w_{11} + iw_{12}$ and $w_2' = w_{13} + iw_{14}$ are complex variables, $w_3' = y_{15}$ is real variable and d_{11}, d_{12}, d_{13} are parameters.

Separating complex variables into real and imaginary parts, the system (3.16) is reduced to

$$\frac{d^q w_{11}}{dt^q} = d_{11}(w_{13} - w_{11})$$

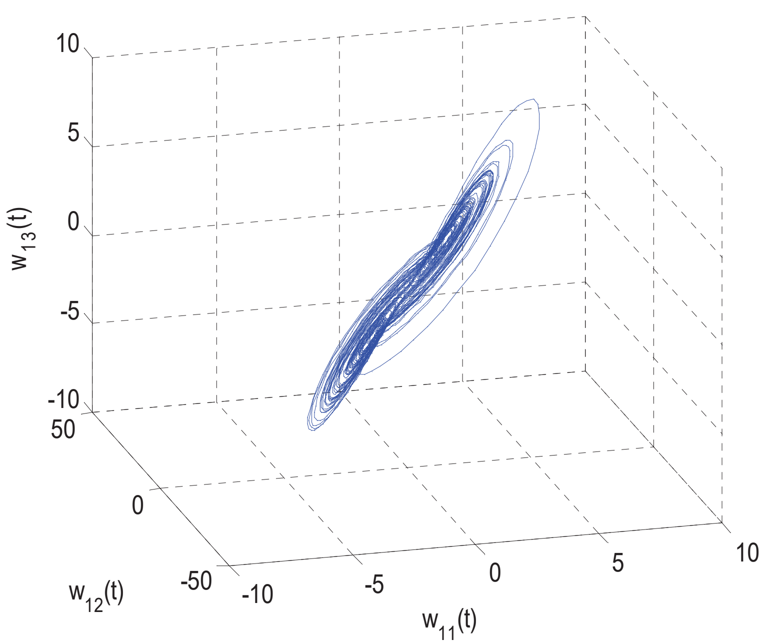
$$\frac{d^q w_{12}}{dt^q} = d_{11}(w_{14} - w_{12})$$

$$\frac{d^q w_{13}}{dt^q} = (d_{13} - d_{11})w_{11} - w_{11}w_{15} + d_{13}w_{13} \tag{3.17}$$

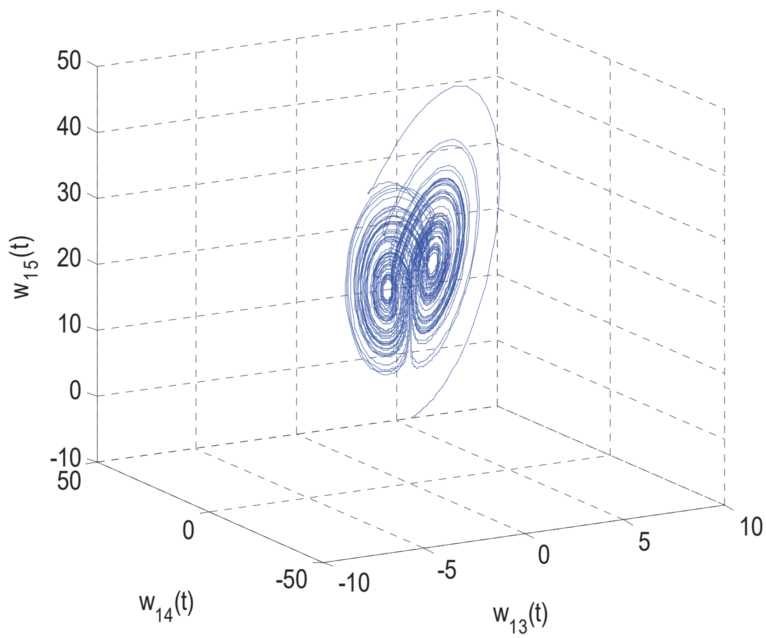
$$\frac{d^q w_{14}}{dt^q} = (d_{13} - d_{11})w_{12} - w_{12}w_{15} + d_{13}w_{14}$$

$$\frac{d^q w_{15}}{dt^q} = -d_{12}w_{15} + w_{11}w_{13} + w_{12}w_{14}.$$

The chaotic attractors of the system (3.17) are displayed through Fig. 3.4 for the parametric values $d_{11} = 35,\ d_{12} = 3,\ d_{13} = 28$, $q = 0.95$ and the initial condition $w(0) = [0.1,\ 0.2,\ 0.1,\ 0.3,\ -0.1]^T$.



(a)



(b)

Fig. 3.4. Phase portraits of fractional order complex Chen system at $q = 0.95$: (a) in $w_{11} - w_{12} - w_{13}$ space, (b) in $w_{13} - w_{14} - w_{15}$ space.

3.4 Combination-Combination phase synchronization between fractional order complex chaotic systems using nonlinear control method

The fractional-order complex Lorenz, T, Lu and Chen systems are considered as five dimensional real systems. Let us consider the fractional order complex Lorenz system as master systems-I and fractional order T-system as master system-II. The fractional order Lu and Chen systems are taken as response systems-I and II respectively.

The response systems-I and response system-II with control functions are defined as

$$\frac{d^q z_{11}}{dt^q} = c_{11}(z_{13} - z_{11}) + u_{11}$$

$$\frac{d^q z_{12}}{dt^q} = c_{11}(z_{14} - z_{12}) + u_{12}$$

$$\frac{d^q z_{13}}{dt^q} = -z_{11}z_{15} + c_{12}z_{13} + u_{13} \tag{3.18}$$

$$\frac{d^q z_{14}}{dt^q} = -z_{12}z_{15} + c_{12}z_{14} + u_{14}$$

$$\frac{d^q z_{15}}{dt^q} = z_{11}z_{13} + z_{12}z_{14} - c_{13}z_{15} + u_{15}.$$

and

$$\frac{d^q w_{11}}{dt^q} = d_{11}(w_{13} - w_{11}) + v_{11}$$

$$\frac{d^q w_{12}}{dt^q} = d_{11}(w_{14} - w_{12}) + v_{12}$$

$$\frac{d^q w_{13}}{dt^q} = (d_{13} - d_{11})w_{11} - w_{11}w_{15} + d_{13}w_{13} + v_{13} \tag{3.19}$$

$$\frac{d^q w_{14}}{dt^q} = (d_{13} - d_{11})w_{12} - w_{12}w_{15} + d_{13}w_{14} + v_{14}$$

$$\frac{d^q w_{15}}{dt^q} = -d_{12}w_{15} + w_{11}w_{13} + w_{12}w_{14} + v_{15},$$

where u_{1i} and v_{1i} $(i = 1, 2, 3,4, 5)$ are control functions.

Now define the error functions are functions as

$e_{1i} = w_{1i} + z_{1i} - y_{1i} - x_{1i}$, $i = 1, 2, 3, 4, 5$.

The error systems obtained as

$$\frac{d^q e_{11}}{dt^q} = d_{11}(e_{13} - e_{11}) - d_{11}(z_{13} - y_{13} - x_{13} - z_{11} + y_{11} + x_{11}) + c_{11}(z_{13} - z_{11})$$
$$- b_{11}(y_{13} - y_{11}) - a_{11}(x_{13} - x_{11}) + u_1^*$$

$$\frac{d^q e_{12}}{dt^q} = d_{11}(e_{14} - e_{12}) - d_{11}(z_{14} - y_{14} - x_{14} - z_{12} + y_{12} + x_{12}) + c_{11}(z_{14} - z_{12})$$
$$- b_{11}(y_{14} - y_{12}) - a_{11}(x_{14} - x_{12}) + u_2^* \tag{3.20}$$

$$\frac{d^q e_{13}}{dt^q} = (d_{13} - d_{11})e_{11} - (d_{13} - d_{11})(z_{11} - y_{11} - x_{11}) - w_{11}w_{15} + d_{13}w_{13}$$
$$- z_{11}z_{15} + c_{12}z_{13} - (b_{12} - b_{11})y_{11} + b_{11}y_{11}y_{15} - a_{12}x_{11} + x_{13} + x_{11}x_{15} + u_3^*$$

$$\frac{d^q e_{14}}{dt^q} = (d_{13} - d_{11})e_{12} - (d_{13} - d_{11})(z_{12} - y_{12} - x_{12}) - w_{12}w_{15} + d_{13}w_{14}$$
$$- z_{12}z_{15} + c_{12}z_{14} - (b_{12} - b_{11})y_{12} + b_{11}y_{12}y_{15} - a_{12}x_{12} + x_{14} + x_{12}x_{15} + u_4^*$$

$$\frac{d^q e_{15}}{dt^q} = -d_{12}e_{15} + d_{12}(z_{15} - y_{15} - x_{15}) + w_{11}w_{13} + w_{12}w_{14} + z_{11}z_{13} + z_{12}z_{14}$$
$$- c_{13}z_{15} - y_{11}y_{13} - y_{12}y_{14} + b_{13}y_{15} - x_{11}x_{13} - x_{12}x_{14} + a_{13}x_{15} + u_5^*,$$

where $u_i^* = u_{1i} + v_{1i}$, $i = 1, 2, 3, 4, 5$.

Theorem 3.2: If the nonlinear control functions are designed as

$$u_1^* = -e_{11} - d_{11}(e_{13} - e_{11}) + d_{11}(z_{13} - y_{13} - x_{13} - z_{11} + y_{11} + x_{11}) - c_{11}(z_{13} - z_{11})$$
$$+ b_{11}(y_{13} - y_{11}) + a_{11}(x_{13} - x_{11})$$

$$u_2^* = -e_{12} - d_{11}(e_{14} - e_{12}) + d_{11}(z_{14} - y_{14} - x_{14} - z_{12} + y_{12} + x_{12}) - c_{11}(z_{14} - z_{12})$$
$$+ b_{11}(y_{14} - y_{12}) + a_{11}(x_{14} - x_{12})$$

$$u_3^* = -e_{13} - (d_{13} - d_{11})e_{11} + (d_{13} - d_{11})(z_{11} - y_{11} - x_{11}) + w_{11}w_{15} - d_{13}w_{13} + z_{11}z_{15}$$
$$- c_{12}z_{13} + (b_{12} - b_{11})y_{11} - b_{11}y_{11}y_{15} + a_{12}x_{11} - x_{13} - x_{11}x_{15} \tag{3.21}$$

$$u_4^* = -e_{14} - (d_{13} - d_{11})e_{12} + (d_{13} - d_{11})(z_{12} - y_{12} - x_{12}) + w_{12}w_{15} - d_{13}w_{14} + z_{12}z_{15}$$
$$- c_{12}z_{14} + (b_{12} - b_{11})y_{12} - b_{11}y_{12}y_{15} + a_{12}x_{12} - x_{14} - x_{12}x_{15}$$

$$u_5^* = -e_{15} + d_{12}e_{15} - d_{12}(z_{15} - y_{15} - x_{15}) - w_{11}w_{13} - w_{12}w_{14} - z_{11}z_{13} - z_{12}z_{14} + c_{13}z_{15}$$
$$+ y_{11}y_{13} + y_{12}y_{14} - b_{13}y_{15} + x_{11}x_{13} + x_{12}x_{14} - a_{13}x_{15},$$

then the Combination-Combination synchronization among considered systems are achieved since it satisfies the condition $\lim_{t\to\infty}\| e_i(t) \| = 0$, for $i = 1, 2, 3, 4, 5$.

Proof: Let us construct the Lyapunov function V to stabilize the error systems (3.20) as

$$V = \frac{1}{2}(e_{11}^2 + e_{12}^2 + e_{13}^2 + e_{14}^2 + e_{15}^2). \tag{3.22}$$

Taking the q-th order fractional derivative of equation (3.22) w. r. to t, we have

$$\frac{d^qV}{dt^q} \le e_{11}\frac{d^qe_{11}}{dt^q} + e_{12}\frac{d^qe_{12}}{dt^q} + e_{13}\frac{d^qe_{13}}{dt^q} + e_{14}\frac{d^qe_{14}}{dt^q} + e_{15}\frac{d^qe_{15}}{dt^q}. \text{ (Using Lemma 1.4)} \tag{3.23}$$

After putting the values of $\frac{d^qe_{11}}{dt^q}$, $\frac{d^qe_{12}}{dt^q}$, $\frac{d^qe_{13}}{dt^q}$, $\frac{d^qe_{14}}{dt^q}$, $\frac{d^qe_{15}}{dt^q}$ and control functions from equation (3.20) and (3.21) in equation (3.23), then we obtain

$$\frac{d^qV(e)}{dt^q} \le -(e_{11}^2 + e_{12}^2 + e_{13}^2 + e_{14}^2 + e_{15}^2) < 0.$$

Thus we may conclude that $\frac{d^qV(e)}{dt^q} < 0$ is negative definite, and thus master systems I, II and response system I, II are combination-combination synchronized.

After substituting the values of the control functions, the error system are obtained as

$$\frac{d^qe_{1i}}{dt^q} = -e_{1i}, \quad i = 1, 2, 3, 4, 5. \tag{3.24}$$

The eigen values of the error system (3.24) are negative and satisfy the condition $|\arg(\lambda_i)| > 0.5\,\pi\,q$ which will also lead the system (3.20) asymptotically converge to zero as $t \to \infty$ and hence the combination-combination synchronization is achieved.

3.5 Numerical simulation and results

In this section for numerical simulation, the earlier considered values of parameters of the fractional order complex chaotic systems are taken. For combination-combination phase synchronization the initial conditions of the master I, II and slave systems I, II are taken as

$(x_{11}(0),\ x_{12}(0),\ x_{13}(0),\ x_{14}(0),\ x_{15}(0)) = (2,\ 3,\ 5,\ 6,\ 9)$,

$(y_{11}(0),\ y_{12}(0),\ y_{13}(0),\ y_{14}(0),\ y_{15}(0)) = (8,\ 7,\ 6,\ 8,\ 7)$ and

$(z_{11}(0),\ z_{12}(0),\ z_{13}(0),\ z_{14}(0),\ z_{15}(0)) = (2,\ 4,\ 6,\ 5,\ 3)$,

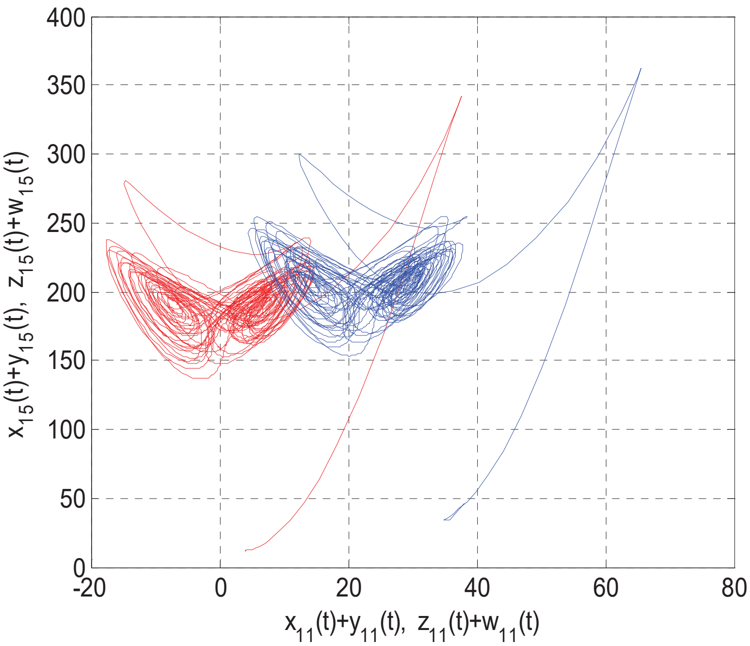
$(w_{11}(0),\ w_{12}(0),\ w_{13}(0),\ w_{14}(0),\ w_{15}(0)) = (0.1,\ 0.2,\ 0.1,\ 0.3,\ -0.1)$ respectively.

Hence the initial conditions of error system for combination-combination phase synchronization will be

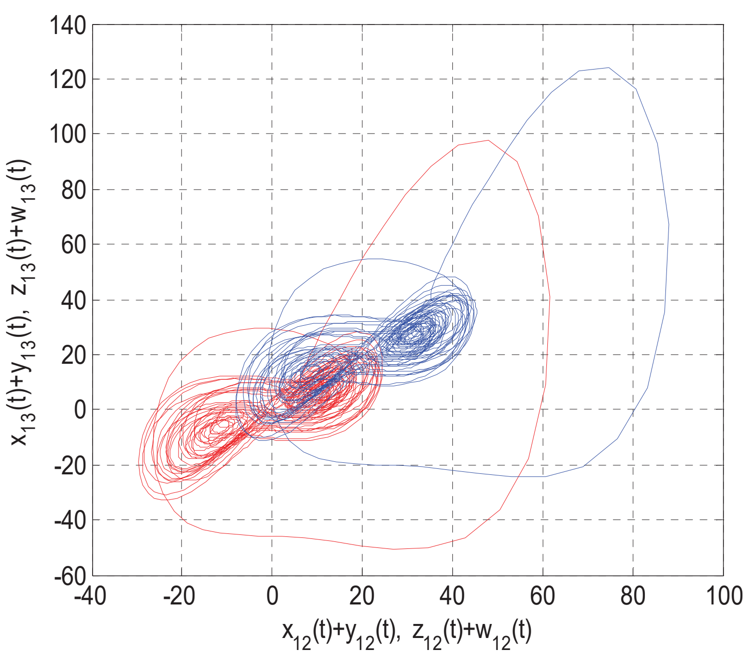
$(e_1(0),\ e_2(0),\ e_3(0),\ e_4(0),\ e_5(0),\ e_6(0)) = (-7.9,\ -5.8,\ -4.9,\ -8.7,\ -13.1)$.

During combination-combination phase synchronization of the systems the time step size is taken as 0.005. Now choosing $\lambda_1 = 0$, $\lambda_2 = -1$, $\lambda_3 = -1$, $\lambda_4 = -1$, $\lambda_5 = -1$, $\lambda_6 = -1$, the phase synchronization between signals $x_{11}(t) + y_{11}(t)$ and $z_{11}(t) + w_{11}(t)$ is achieved. It should be noted that, when $\lambda_1 = 0$, $\lambda_2 = -1$, $\lambda_3 = -1$, $\lambda_4 = -1$, $\lambda_5 = -1$, signals $x_{12}(t) + y_{12}(t)$ and $z_{12}(t) + w_{12}(t)$, $x_{13}(t) + y_{13}(t)$ and $z_{13}(t) + w_{13}(t)$, $x_{14}(t) + y_{14}(t)$ and $z_{14}(t) + w_{14}(t)$, $x_{15}(t) + y_{15}(t)$ and $z_{15}(t) + w_{15}(t)$ become synchronized. If $\lambda_1 = -1$, $\lambda_2 = 0$, $\lambda_3 = -1$, $\lambda_4 = -1$, $\lambda_5 = -1$; $\lambda_1 = -1$, $\lambda_2 = -1$, $\lambda_3 = 0$, $\lambda_4 = -1$, $\lambda_5 = -1$; $\lambda_1 = -1$, $\lambda_2 = -1$, $\lambda_3 = -1$, $\lambda_4 = 0$, $\lambda_5 = -1$ and $\lambda_1 = -1$, $\lambda_2 = -1$, $\lambda_3 = -1$, $\lambda_4 = -1$, $\lambda_5 = 0$ are taken, phase synchronizations between signals $x_{12}(t) + y_{12}(t)$ and $z_{12}(t) + w_{12}(t)$, $x_{13}(t) + y_{13}(t)$ and $z_{13}(t) + w_{13}(t)$, $x_{14}(t) + y_{14}(t)$ and $z_{14}(t) + w_{14}(t)$, $x_{15}(t) + y_{15}(t)$ and $z_{15}(t) + w_{15}(t)$ are obtained respectively. State trajectories of the combination-combination phase synchronization of complex chaotic systems are depicted through Fig. 3.7 for the order of the derivative $q = 0.95$. The plots of the systems combination-

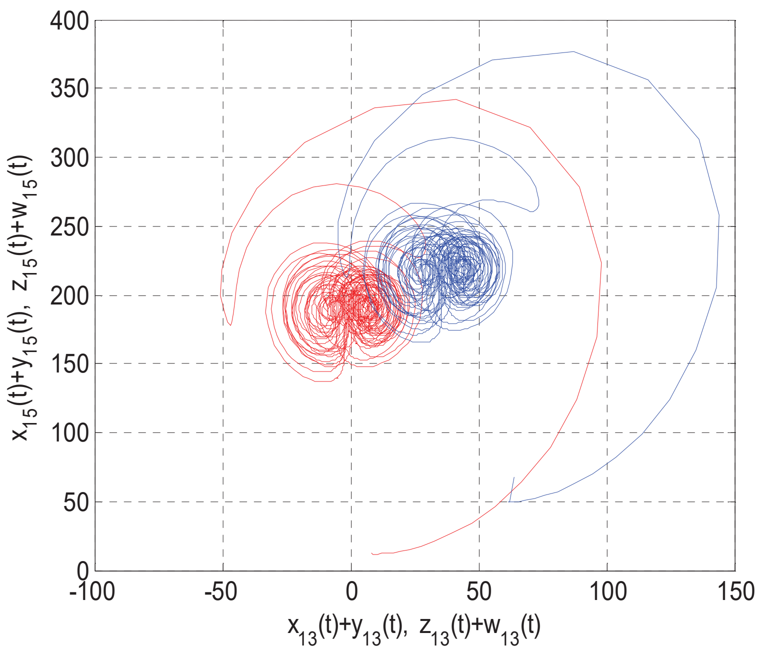
combination phase synchronization are also shown in Fig. 3.5 and Fig. 3.6 at the order $q = 0.95$.



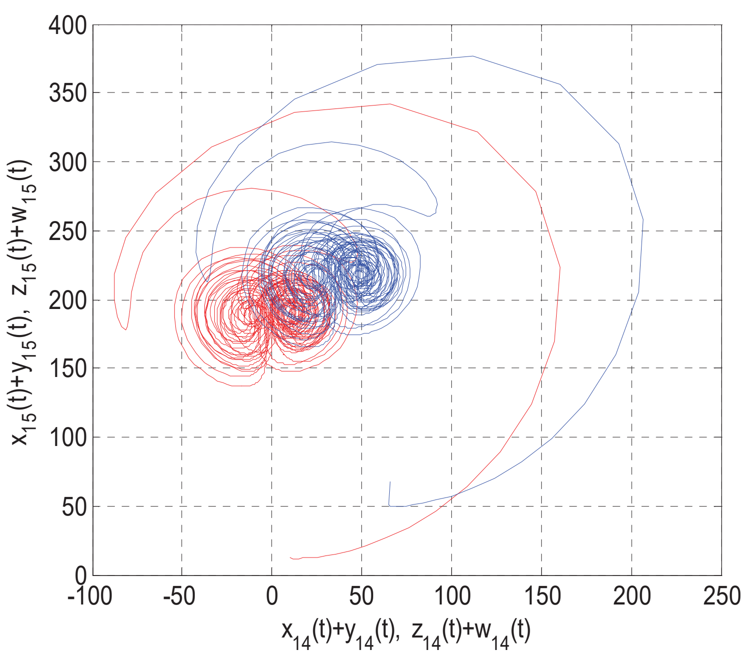
(a)



(b)

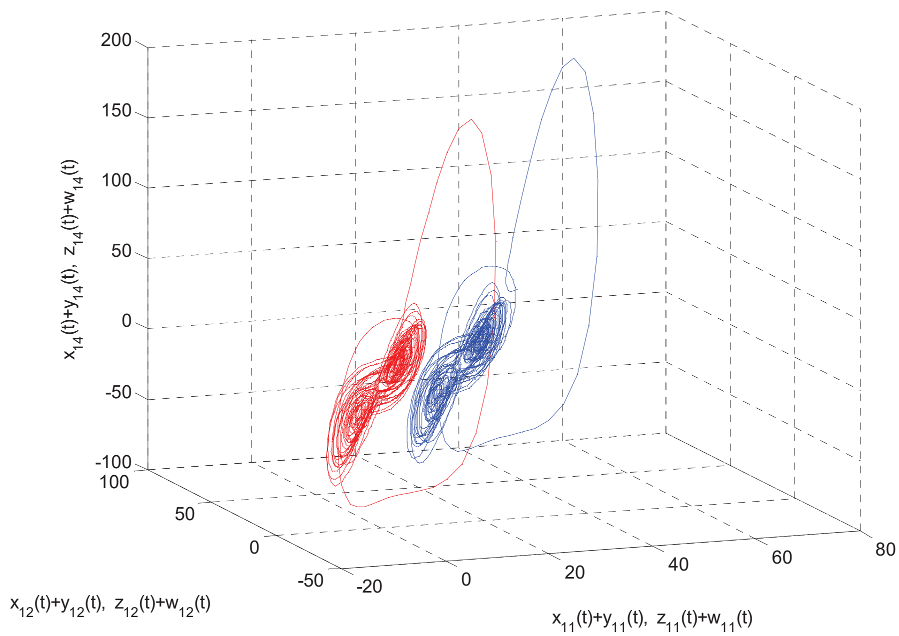


(c)

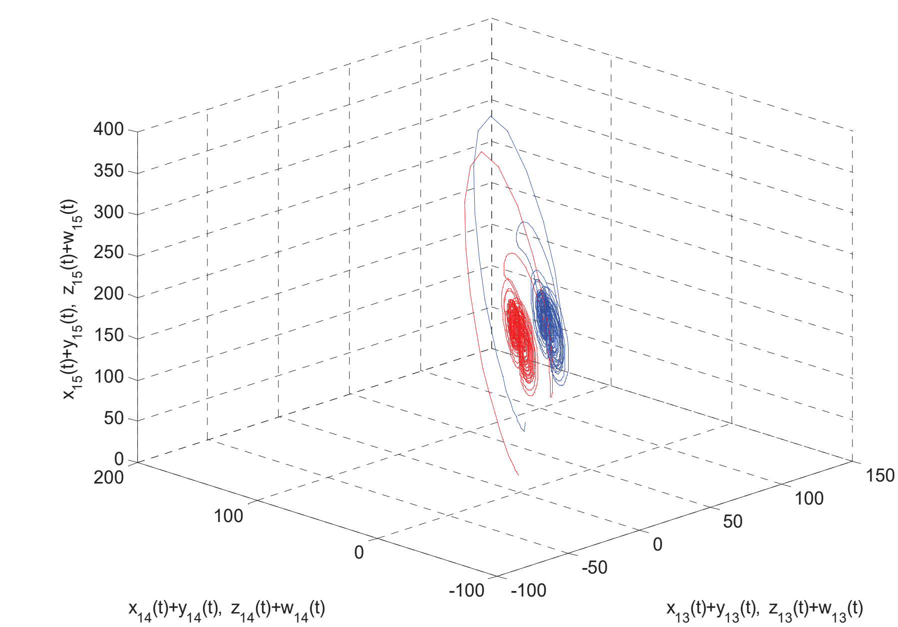


(d)

Fig. 3.5. 2D plots of combination-combination phase synchronization of fractional order complex Lorenz, T, Lu and Chen chaotic systems at $q = 0.95$.

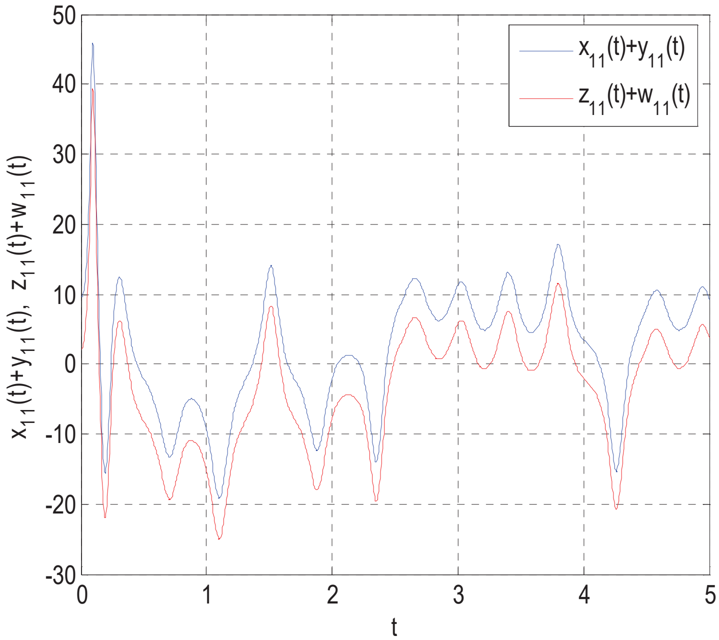


(a)

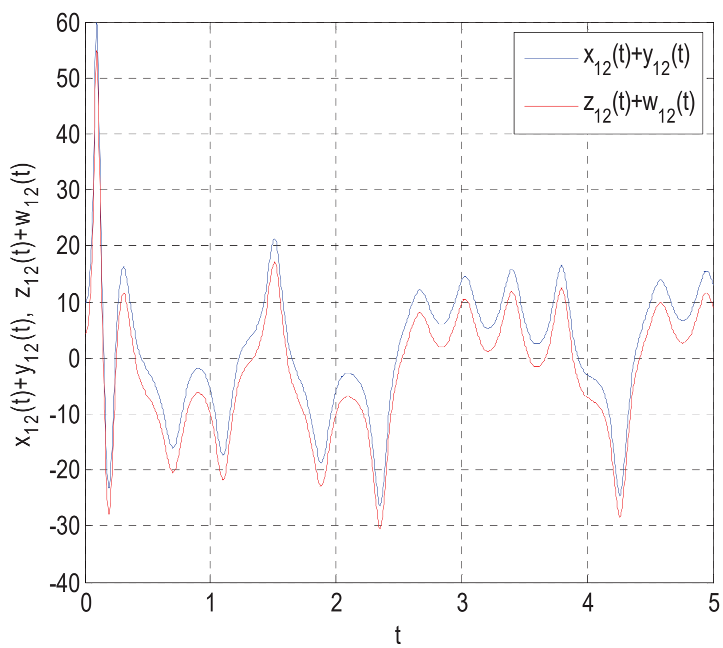


(b)

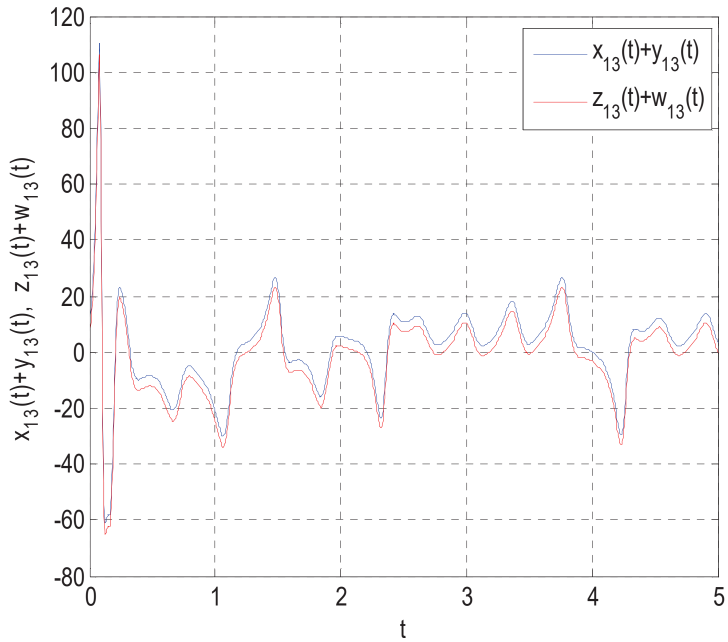
Fig. 3.6. 3D plots of combination-combination phase synchronization of fractional order complex Lorenz, T, Lu and Chen chaotic systems at $q = 0.95$.


50
40
30
20
10
0
-10
-20
-30
0
1
2
3
4
5
t
x11(t)+y11(t), z11(t)+w11(t)
x11(t)+y11(t)
z11(t)+w11(t)

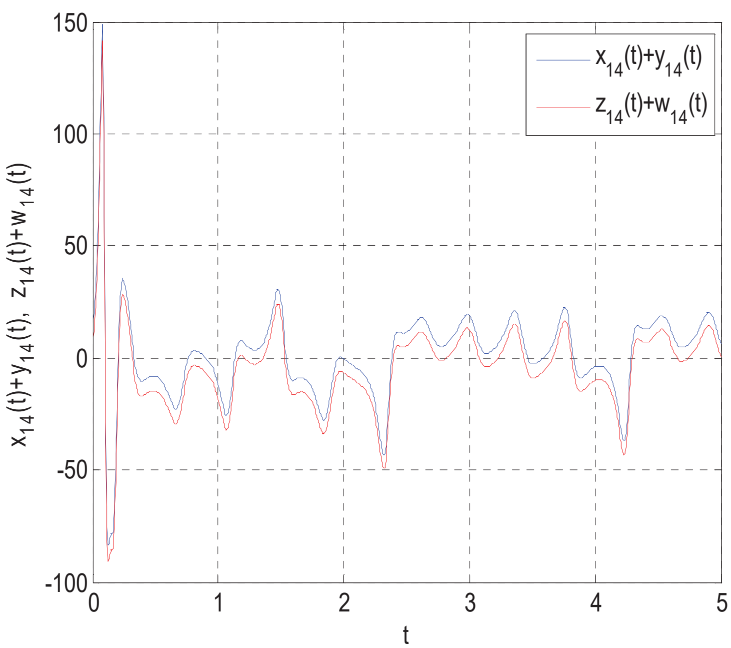
(a)


60
50
40
30
20
10
0
-10
-20
-30
-40
0
1
2
3
4
5
t
x12(t)+y12(t), z12(t)+w12(t)
x12(t)+y12(t)
z12(t)+w12(t)

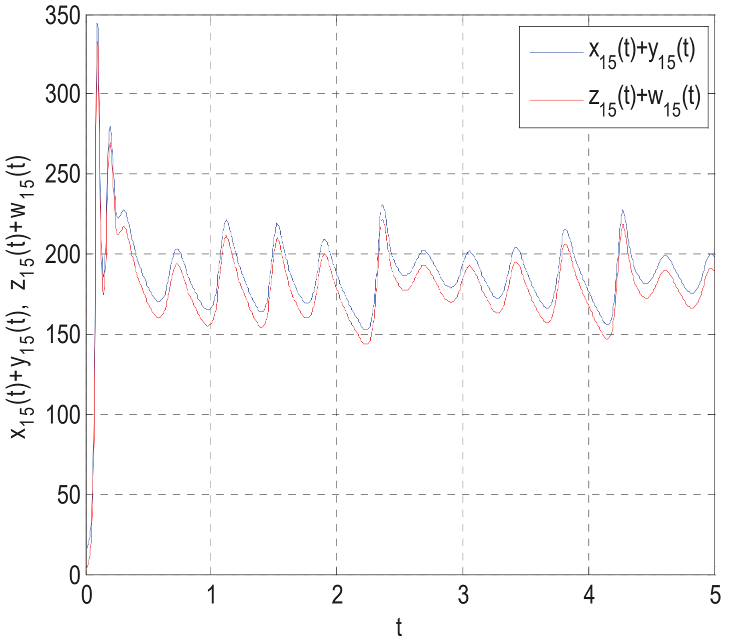
(b)


$x_{13}(t)+y_{13}(t)$
$z_{13}(t)+w_{13}(t)$
$x_{13}(t)+y_{13}(t)$, $z_{13}(t)+w_{13}(t)$
t

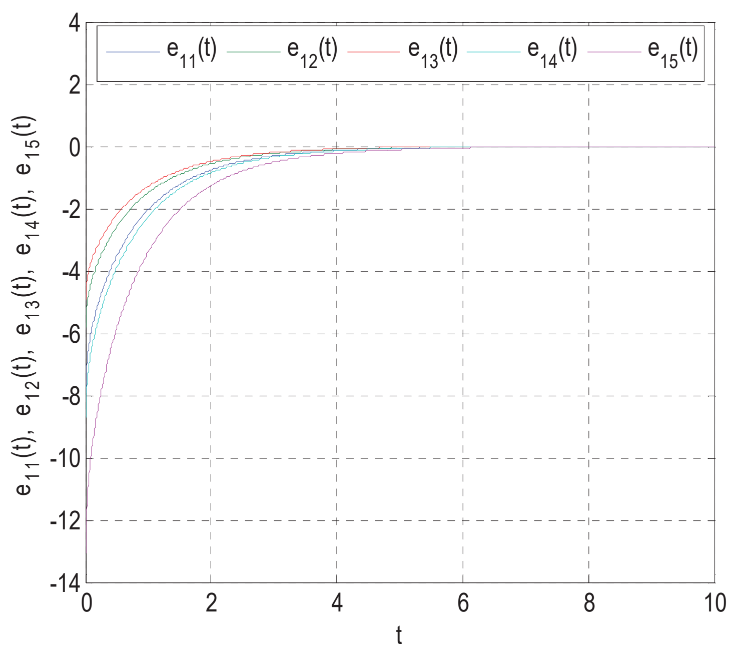
(c)


$x_{14}(t)+y_{14}(t)$
$z_{14}(t)+w_{14}(t)$
$x_{14}(t)+y_{14}(t)$, $z_{14}(t)+w_{14}(t)$
t

(d)



(e)



(f)

Fig. 3.7. Combination-Combination phase synchronization for signals at $q = 0.95$: (a) between $x_{11}(t) + y_{11}(t)$ and $z_{11}(t) + w_{11}(t)$, (b) between $x_{12}(t) + y_{12}(t)$ and

$z_{12}(t)+w_{12}(t)$, (c) between $x_{13}(t)+y_{13}(t)$ and $z_{13}(t)+w_{13}(t)$, (d) between $x_{14}(t)+y_{14}(t)$ and $z_{14}(t)+w_{14}(t)$, (e) between $x_{15}(t)+y_{15}(t)$ and $z_{15}(t)+w_{15}(t)$, (f) The evolution of the error functions of complex chaotic systems.

3.6 Conclusion

In the present chapter, the combination-combination phase synchronization between fractional order non-identical complex chaotic systems has been achieved, based on the stability theory of fractional-order systems. Nonlinear control laws are proposed to stabilize the fractional order complex chaotic systems using proper feedback control method. Graphical presentations of combination-combination phase synchronization of different fractional order complex chaotic systems have successfully demonstrated the reliability and effectiveness of the method. The author is optimistic about the simulation results of the present research work will be appreciated and utilized by the researchers involved in the field of fractional order nonlinear dynamical systems, and also it will work for the strong security of the secure communication.

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