CHAPTER-2

Combined synchronization of time-delayed chaotic systems with uncertain parameters

Chapter 2

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2.1 Introduction

Synchronization and chaos theory have useful applications in the many areas of engineering such as secure communication, digital communication, chemical reactions analysis and design, information processing etc. The synchronization phenomenon of chaotic systems is interesting area of chaos theory (Fujisaka and Yamada (1983a), Pecora and Carroll (1990), Chen and Dong (1998), Elabbssy et al. (2006), Lu and Chen (2002), Chen and Lu (2003), Li and Xu (2004b), Wang (2003)). Several types of synchronization phenomenon have been identified and demonstrated such as complete synchronization, phase synchronization, anti-phase synchronization, generalized synchronization, lag synchronization, projective synchronization, dual synchronization (Fujisaka and Yamada (1983b), Mahmoud and Mahmoud (2010), Liu (2006), Rosenblum et al. (1997), Singh et al. (2017), Yadav et al. (2015)) and so on. Among these types of synchronization, the combined synchronization of chaotic systems is identified by the taking more than one drive systems and one response system. So far many approaches have been used to synchronize chaotic systems such as active control, adaptive control, backstepping control, feedback control, sliding mode control and so on.

The study of synchronization between low dim systems leads to further development of synchronization between higher dimensional systems theoretically as well as on applied aspects. This further gave ample scope for exploring work on synchronization of time delayed chaotic systems (Pyragas (1998), He and Vaidya (1999)) and time-delayed system pertaining to nonlinear dynamic. The multiple time-delayed systems have enough

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scope of application on chaos-based encryption system leading to enhancement of message security together with chaos synchronization between multiple transmitter and receiver system and decoding of such message. The time-delayed chaotic systems are naturally related to the systems with memory that prevails for most of the physical and scientific systems such as blood production in patients with leukaemia (Mackey-Glass model), dynamics of optical systems (e.g. Ikeda system), laser physics, population dynamics, physiological model, neural networks, control system (Mackey and Glass (1977), Ikeda et al. (1980), Bunner et al. (1998), Yongzhen et al. (2011), Liao et al. (2007), Kwon et al. (2011)) and so on.

The initial work of Runzi et al. (Runzi et al. (2011)) on combination synchronization of two drive systems and one response system (as three chaotic systems) having physical application in secure communication in transmitted signals in the form of splitted signals being loaded in different drive systems which increase their efficiency/strength for any kind of anti-attack and anti-translation capability than that via usual method of transmission of such signals. The synchronization between chaotic systems with uncertainties is not an easy task for researchers since there are always possibilities of destroying synchronization under the effects of those parameters. In 2012, Chen et al. (Chen et al. (2012)) have studied disturbance-observer-based robust synchronization control of uncertain chaotic systems. Jawaadaa et al. (2012) studied robust active sliding mode anti-synchronization of hyperchaotic systems with uncertainties and external disturbances.

Overall the above facts have motivated the author to study the combined synchronization among time-delayed chaotic systems with uncertain parameters using nonlinear control method. In this chapter the nonlinear control method is proposed for combined synchronization of time-delayed chaotic systems and synchronization among three and four systems have been shown as examples of combined synchronization time-delayed chaotic systems.

2.2 Problem formulation

Consider the drive systems of the following form:

$$\frac{dX_1(t)}{dt} = f_1(X_1(t)) + (A_1 + \Delta A_1)X_1(t) + B_1 X_1(t-\tau_1),\ t > 0$$

$$X_1(t) = \phi_1(t),\ -\tau_1 \le t \le 0, \tag{2.1}$$

$$\frac{dX_2(t)}{dt} = f_2(X_2(t)) + (A_2 + \Delta A_2)X_2(t) + B_2 X_2(t-\tau_2),\ t > 0$$

$$X_2(t) = \phi_2(t),\ -\tau_2 \le t \le 0, \tag{2.2}$$

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$$\frac{dX_{n-1}(t)}{dt} = f_{n-1}(X_{n-1}(t)) + (A_{n-1} + \Delta A_{n-1})X_{n-1}(t) + B_{n-1} X_{n-1}(t-\tau_{n-1}),\ t > 0$$

$$X_{n-1}(t) = \phi_{n-1}(t),\ -\tau_{n-1} \le t \le 0 \tag{2.3}$$

and the response system is taken as

$$\frac{dX_n(t)}{dt} = f_n(X_n(t)) + (A_n + \Delta A_n)X_n(t) + B_n X_n(t-\tau_n) + U(t),\ t > 0$$

$$X_n(t) = \phi_n(t),\ -\tau_n \le t \le 0, \tag{2.4}$$

where X_1, X_2,..., X_{n-1}, $X_n \in R^n$ are state vectors of the systems, A_1, B_1, A_2, B_2,....., A_{n-1}, B_{n-1}, A_n, $B_n \in R^{n\times n}$ are the constant parameters matrices, $f_1, f_2, ..., f_{n-1}, f_n : R^n \to R^n$ are nonlinear functions, ΔA_1, ΔA_2,....., ΔA_{n-1}, $\Delta A_n \in R^{n\times n}$ are parametric uncertainties such that $\|\Delta A_1\| \le \delta_1$, $\|\Delta A_2\| \le \delta_2$,....., $\|\Delta A_{n-1}\| \le \delta_{n-1}$, $\|\Delta A_n\| \le \delta_n$ with δ_1, δ_2,...., δ_{n-1}, δ_n positive constants, $\phi_1(t)$, $\phi_2(t)$,....., $\phi_{n-1}(t)$, $\phi_n(t)$, represent the trajectories of the solutions in the past, τ_i, $i = 1,2,...n$ are the time delays and $U(t) \in R^n$ is the active control function to be designed later.

Now define the error function for combined synchronization as

$e(t) = X_n - X_{n-1} - ..... - X_2 - X_1$.

According to definition of error function the error system will be

$$\frac{de}{dt} = f_n(X_n(t)) - f_{n-1}(X_{n-1}(t)) - .... - f_2(X_2(t)) - f_1(X_1(t)) + (A_n + \Delta A_n)X_n(t)$$

$$-(A_{n-1} + \Delta A_{n-1})X_{n-1}(t) - .... - (A_2 + \Delta A_2)X_2(t) - (A_1 + \Delta A_1)X_1(t) + B_n X_n(t - \tau_n) \quad (2.5)$$

$$-B_{n-1}X_{n-1}(t - \tau_{n-1}) - .... - B_2 X_2(t - \tau_2) - B_1 X_1(t - \tau_1) + U(t).$$

Definition: The (n-1) drive and one response delay systems follow Combined synchronization among n-chaotic systems if

$$\lim_{t \to \infty} \| e(t) \| = \lim_{t \to \infty} \| X_n - X_{n-1} - ..... - X_2 - X_1 \| = 0.$$

2.3 Nonlinear Control method

Theorem 2.1: If the nonlinear controller is designed as

$$U(t) = -f_n(X_n(t)) + f_{n-1}(X_{n-1}(t)) + ... + f_2(X_2(t)) + f_1(X_1(t)) - (A_n + \Delta A_n)X_n(t)$$

$$+(A_{n-1} + \Delta A_{n-1})X_{n-1}(t) + ... + (A_2 + \Delta A_2)X_2(t) + (A_1 + \Delta A_1)X_1(t) \quad (2.6)$$

$$-B_n X_n(t - \tau_n) + B_{n-1}X_{n-1}(t - \tau_{n-1}) + ... + B_2 X_2(t - \tau_2) + B_1 X_1(t - \tau_1) - (1/2 + k)e(t),$$

then the combined synchronization among one response system and (n-1) drive systems satisfy the condition $\lim_{t \to \infty} \| X_n - X_{n-1} - ..... - X_2 - X_1 \| = 0$ for any positive constant *k*.

Proof: According to Lyapunov stability theorem the error system (2.5) will be asymptotically stable, if the error system becomes zero, then the considered drive and response systems are regarded as combinationally synchronized. Let us construct the Lyapunov-Krasovskii functional V to succeed stability analysis as

$$V = \frac{1}{2}e^T(t)e(t) + \frac{1}{2}\int_{-\tau}^{0} e^T(t + \theta)e(t + \theta)d\theta.$$

The derivative of V along the trajectory of error dynamic system is

$$\frac{dV}{dt} = e^T(t)\frac{de(t)}{dt} + \frac{1}{2}(e^T(t)e(t) - e^T(t-\tau)e(t-\tau)). \tag{2.7}$$

After putting the value of $\frac{de(t)}{dt}$ and controller $U(t)$ from equations (2.5) & (2.6) in equation (2.7), then we obtain

$$\frac{dV}{dt} = -ke^T(t)e(t) - \frac{1}{2}(e^T(t-\tau)e(t-\tau)) < 0.$$

Thus, it may be concluded that if the control parameter $k > 0$, $V \in R$ is positive definite function and $\frac{dV}{dt} \in R$ is negative definite function, then the error system is globally and asymptotically stable according to Lyapunov-Krasovskii stability theory (Hale (1977), Krasovskii et al. (1963)). Consequently, the state trajectories of one response and (n-1) drive systems will be combinationally synchronized globally and asymptotically. It is also seen that the synchronization error $e(t)$ tend to zero as time becomes large. This completes the proof.

2.4 Systems' descriptions

2.4.1 Double delay Rossler system

A double delayed Rossler system (Ansari et al. (2015)) is given by

$$\frac{dx_1(t)}{dt} = -y_1(t) - z_1(t) + a_1\, x_1(t-\tau_1) + a_1'\, x_1(t-\tau_1')$$

$$\frac{dy_1(t)}{dt} = x_1(t) + b_1\, y_1(t) \tag{2.8}$$

$$\frac{dz_1(t)}{dt} = b_1 + x_1(t)\, z_1(t) - c_1\, z_1(t).$$

For the parameters' values $a_1 = 0.2$, $a_1' = 0.5$, $b_1 = 0.2$, $c = 5.7$ $\tau_1 = 1.0$, $\tau_1' = 2.0$ at the initial condition $(0.5, 1, 1.5)$, the chaotic attractor of system (2.8) is depicted through Fig. 2.1(a).

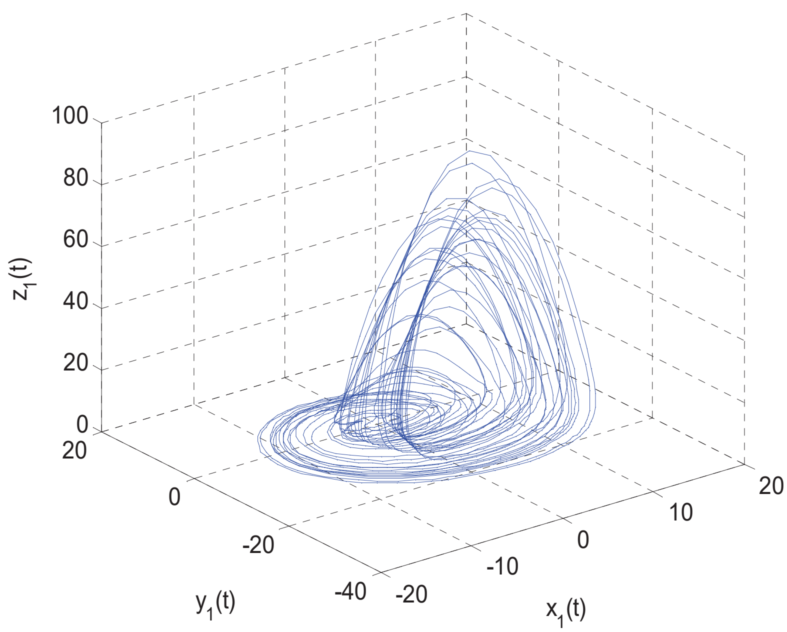
The double delayed Rossler system with uncertainties is described as

$$\frac{d\,x_1(t)}{dt} = -y_1(t) - z_1(t) + a_1\,x_1(t-\tau_1) + a_1'\,x_1(t-\tau_1') + 0.01\,x_1(t)$$

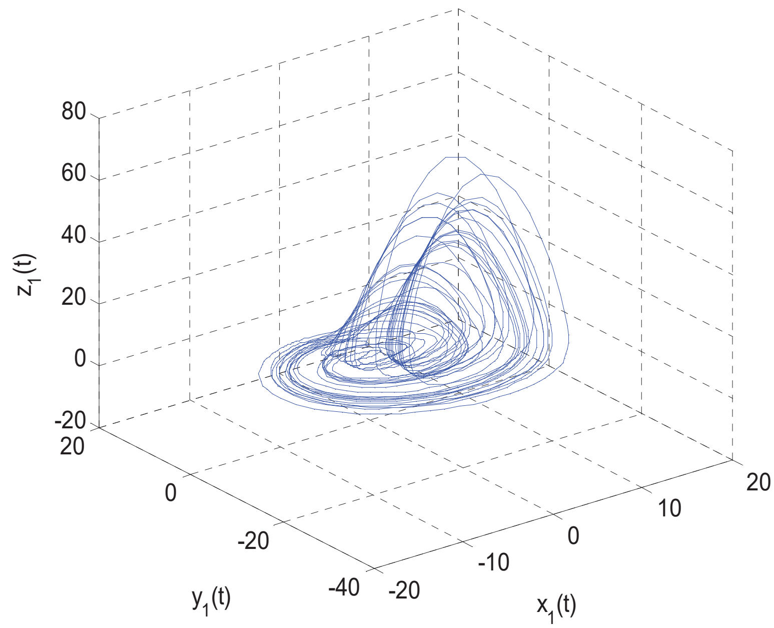
$$\frac{d\,y_1(t)}{dt} = x_1(t) + b_1\,y_1(t) + 0.05\,z_1(t) \qquad (2.9)$$

$$\frac{d\,z_1(t)}{dt} = b_1 + x_1(t)\,z_1(t) - c_1\,z_1(t) - 0.02\,y_1(t) + 0.1\,z_1(t),$$

where uncertain term $\Delta A_1 = \begin{bmatrix} 0.01 & 0 & 0 \\ 0 & 0 & 0.05 \\ 0 & -0.02 & 0.1 \end{bmatrix}$. The phase portrait of the system (2.9) in $x_1 - y_1 - z_1$ space is shown in Fig. 2.1(b).



(a)



(b)

Fig. 2.1. Phase portraits of Double delay Rossler system: (a) without uncertainties; (b) with uncertainties.

2.4.2 Advanced Lorenz system

The advanced Lorenz system (Zhang et al. (2009)) is defined by equation (2.10)

$$\frac{dx_2(t)}{dt} = a_2(y_2(t) - x_2(t)) + p_2 x_2(t-\tau_2)$$

$$\frac{dy_2(t)}{dt} = -x_2(t)z_2(t) + b_2 x_2(t) + c_2 y_2(t) \tag{2.10}$$

$$\frac{dz_2(t)}{dt} = x_2^2(t) - d_2 z_2(t).$$

The system (2.10) shows chaotic behaviour for the control parameters $a_2 = 20$, $b_2 = 14$, $c_2 = 10.6$, $d_2 = 2.8$, $p_2 = 3$ and $\tau_2 = 0.001$ at initial condition $(-20, 8, 20)$, where $-\tau \le t \le 0$. The phase portrait is shown in the Fig. 2.2 (a).

Let us define system (2.10) in the presence of uncertain terms as

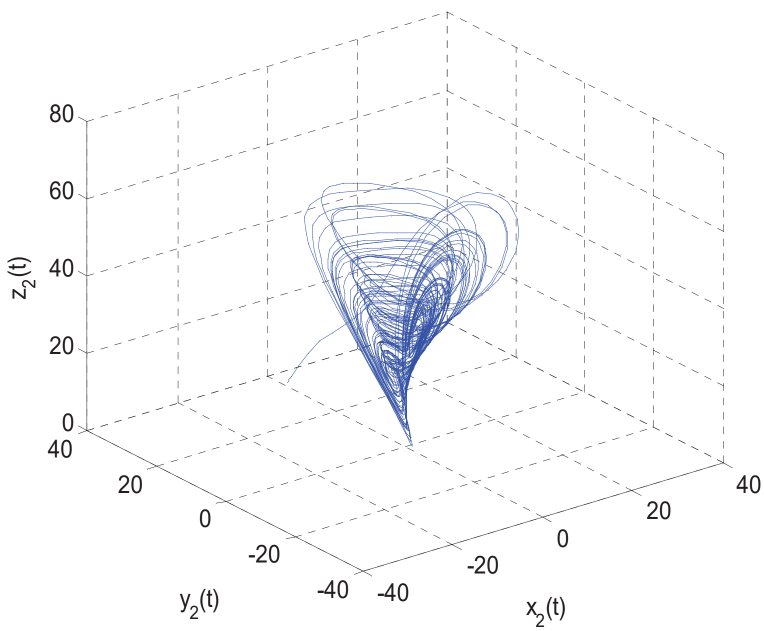
$$\frac{d\,x_2(t)}{dt} = a_2\,(y_2(t) - x_2(t)) + p_2 x_2(t-\tau_2) + 0.13 x_2(t) + 0.02 z_2(t)$$

$$\frac{d\,y_2(t)}{dt} = -x_2(t)\,z_2(t) + b_2 x_2(t) + c_2 y_2(t) + 0.08 x_2(t) \tag{2.11}$$

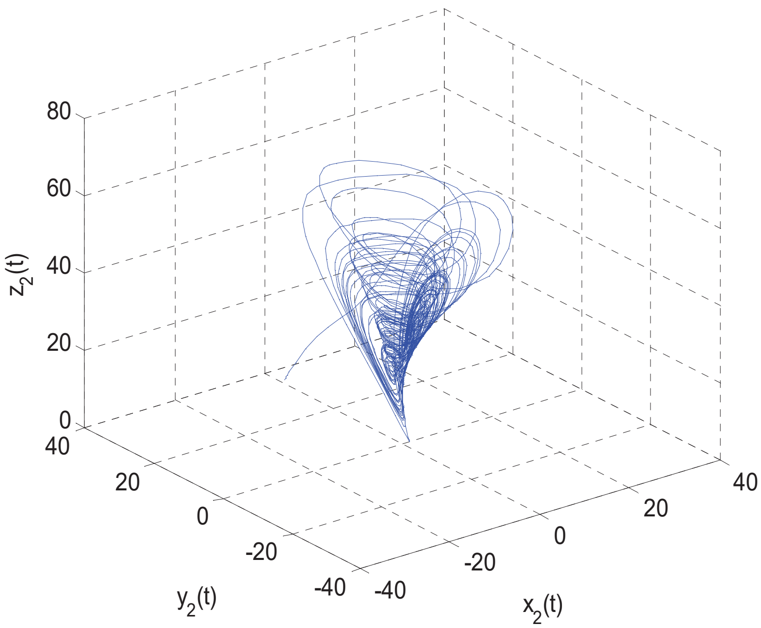
$$\frac{dz_2(t)}{dt} = x_2^2(t) - d_2 z_2(t) + 0.01 y_2(t) + 0.8\, z_2(t),$$

where the uncertain parameter $\Delta A_2 = \begin{bmatrix} 0.13 & 0 & 0.02 \\ 0.08 & 0 & 0 \\ 0 & 0.01 & 0.8 \end{bmatrix}$.

Fig. 2.2(b) shows the phase portrait of the system (2.11) in $x_2 - y_2 - z_2$ space.



(a)



(b)

Fig. 2.2. Phase portraits of advanced Lorenz system: (a) without uncertainties; (b) with uncertainties.

2.4.3 Time delay Liu system

The time delay Liu system (Bhalekar and Gejji (2010)) is given as

$$\frac{dx_3(t)}{dt} = a_3(y_3(t) - x_3(t-\tau_3))$$

$$\frac{dy_3(t)}{dt} = b_3 x_3(t-\tau_3) - k_3 x_3(t) z_3(t) \qquad (2.12)$$

$$\frac{dz_3(t)}{dt} = -c_3 z_3(t-\tau_3) + h_3\, x_3^2(t),\ t \in [-\tau,\ 0],$$

where $0 < \tau_3 \le 0.005$ represents the time delay term and $a_3 = 10$, $b_3 = 40$, $c_3 = 2.5$, $h_3 = 4$ and $k_3' = 1$ are the parameters and at initial condition (2.2, 2.4, 3.8) the system shows the chaotic behaviour in Fig. 2.3(a) in $x_3 - y_3 - z_3$ space.

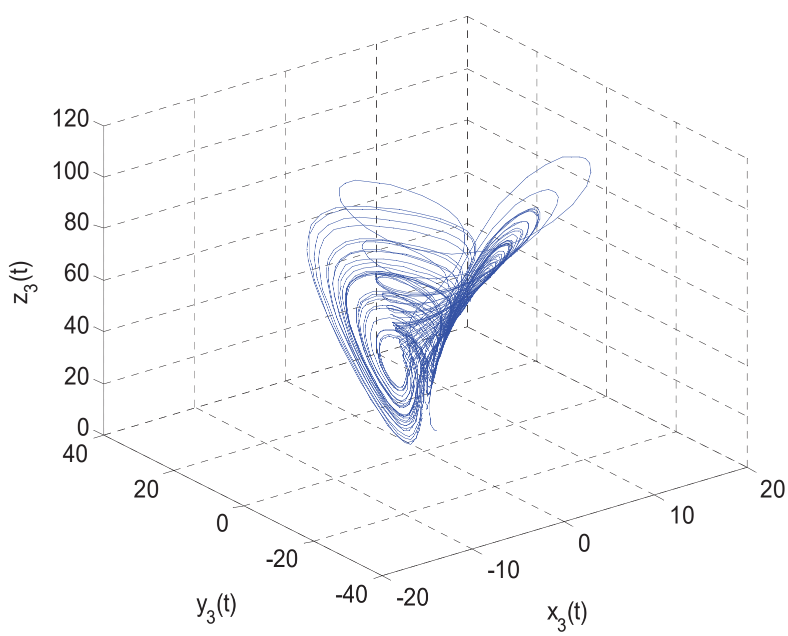
Now define the Liu system with uncertain parameters by equation (2.13) as

$$\frac{dx_3(t)}{dt} = a_3(y_3(t) - x_3(t-\tau_3)) + 0.05x_3(t) - 0.01z_3(t)$$

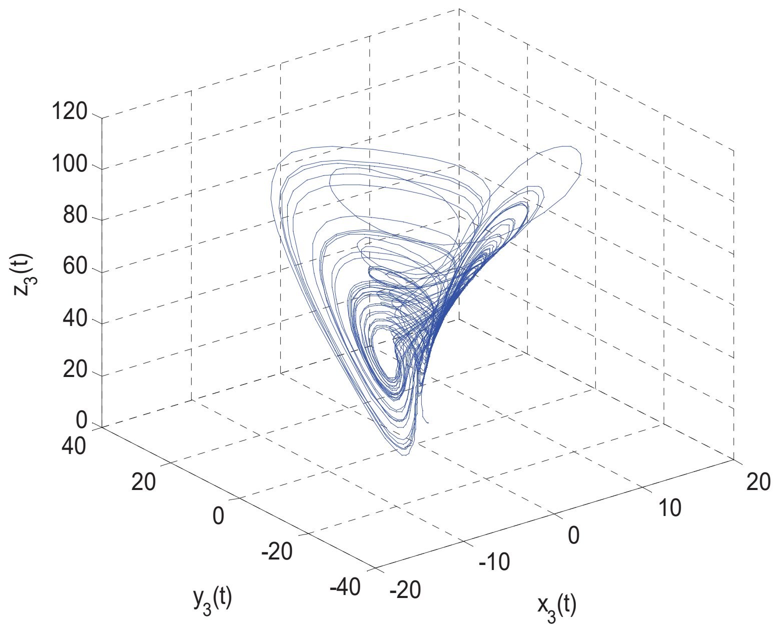
$$\frac{dy_3(t)}{dt} = b_3x_3(t-\tau_3) - k_3'x_3(t)z_3(t) + 0.2y_3(t) \tag{2.13}$$

$$\frac{dz_3(t)}{dt} = -c_3z_3(t-\tau_3) + h_3\, x_3^2(t) + 0.3x_3(t)\,,\ t \in [-\tau,\ 0]\,,$$

where the uncertain parameter $\Delta A_3 = \begin{bmatrix} 0.05 & 0 & -0.01 \\ 0 & 0.2 & 0 \\ 0.3 & 0 & 0 \end{bmatrix}$. The phase portrait of the Liu system in presence of uncertain parameters in $x_3 - y_3 - z_3$ space is demonstrate in Fig. 2.3(b).



(a)



(b)

Fig. 2.3. Phase portraits of time delay Liu system: (a) without uncertainties; (b) with uncertainties.

2.4.4 Time delay Chen system

The time delay Chen system (Gejji et al. (2012)) is defined by

$$\frac{dx_4(t)}{dt} = a_4(y_4(t) - x_4(t-\tau_4))$$

$$\frac{dy_4(t)}{dt} = (c_4 - a_4)x_4(t-\tau_4) - x_4(t)z_4(t) + c_4 y_4(t) \tag{2.14}$$

$$\frac{dz_4(t)}{dt} = x_4(t)y_4(t) - b_4 z_4(t-\tau_4)\,,\ t \in [-\tau,\ 0]\,,$$

where $0 \le \tau_4 < 0.010$ represent the time delay term, $a_4 = 35$, $b_4 = 3$, $c_4 = 27$ are the parameters and the initial condition is $(0.2,\ 0,\ 0.5)$. The phase portrait of system (2.14) is depicted in Fig. 2.4(a).

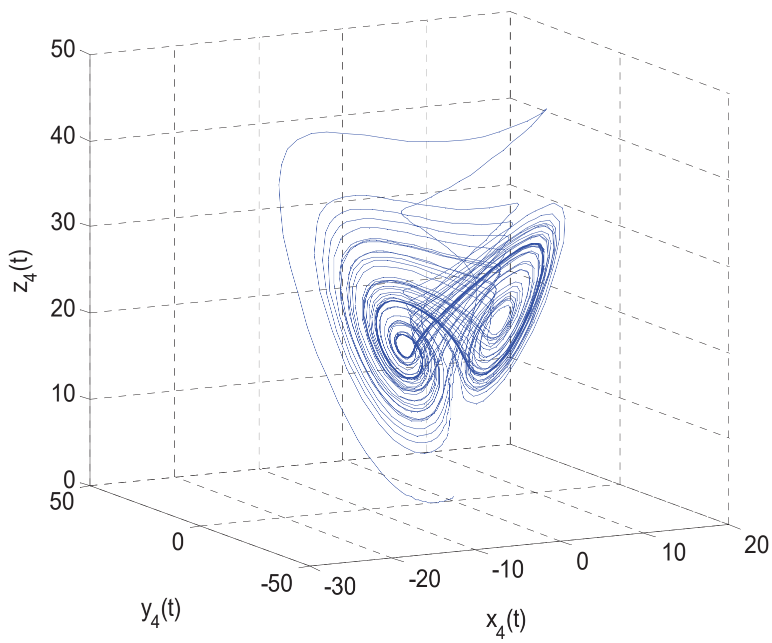
Now, we define the time delay Chen system in the presence of uncertain parameters as

$$\frac{dx_4(t)}{dt} = a_4(y_4(t) - x_4(t-\tau_4)) + 0.01x_4(t) - 0.02z_4(t)$$

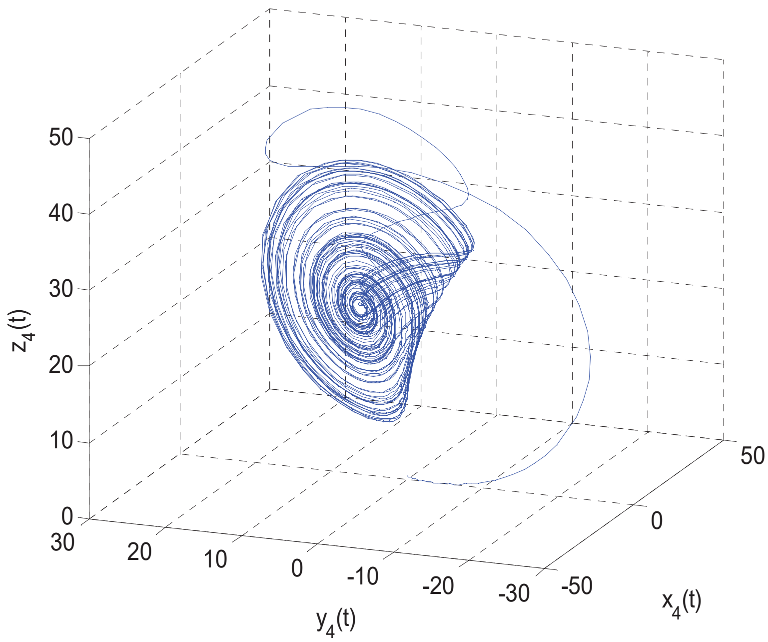
$$\frac{dy_4(t)}{dt} = (c_4 - a_4)x_4(t-\tau_4) - x_4(t)z_4(t) + c_4 y_4(t) + 0.1x_4(t) \tag{2.15}$$

$$\frac{dz_4(t)}{dt} = x_4(t)y_4(t) - b_4 z_4(t-\tau_4) + 0.04z_4(t)\,,\ t \in [-\tau,\ 0]\,,$$

where uncertain term $\Delta A_4 = \begin{bmatrix} 0.01 & 0 & -0.02 \\ 0.1 & 0 & 0 \\ 0 & 0 & 0.04 \end{bmatrix}$. The phase portrait of time delay Chen system with uncertain parameters (2.15) is shown in Fig. 2.4(b).



(a)



(b)

Fig. 2.4. Phase portraits of time delay Chen system: (a) without uncertainties; (b) with uncertainties.

2.5 Combined synchronization among double delay Rossler, advanced Lorenz and time delay Liu systems with uncertain parameters

For the study of combined synchronization among three time delay systems in the presence of uncertain parameters, two systems double delay Rossler (2.9) and advanced Lorenz (2.11) are considered as drive systems-I and drive system-II and third time delay Liu system (2.13) is considered as response system. The response system with control functions $u_1(t)$, $u_2(t)$, $u_3(t)$ is defined as

$$\frac{dx_3(t)}{dt} = a_3(y_3(t) - x_3(t-\tau_3)) + 0.05x_3(t) - 0.01z_3(t) + u_1(t)$$

$$\frac{dy_3(t)}{dt}=b_3x_3(t-\tau_3)-k_3x_3(t)z_3(t)+0.2y_3(t)+u_2(t) \tag{2.16}$$

$$\frac{dz_3(t)}{dt}=-c_3z_3(t-\tau_3)+h_3\,x_3^2(t)+0.3x_3(t)+u_3(t).$$

Defining the error functions as $e_1=x_3-x_2-x_1$, $e_2=y_3-y_2-y_1$, $e_3=z_3-z_2-z_1$, we will obtain the following error system

$$\frac{de_1(t)}{dt}=a_3(e_2(t)-x_3(t-\tau_3))+0.05e_1(t)-0.01e_3(t)+(a_2-0.08)x_2(t)+0.04x_1(t)$$

$$-p_2x_2(t-\tau_2)-a_1\,x_1(t-\tau_1)-a_1'\,x_1(t-\tau_1')+(a_3-a_2)\,y_2(t)$$

$$+(a_3+1)\,y_1(t)-0.03z_2(t)+0.99z_1(t)+u_1(t)$$

$$\frac{de_2(t)}{dt}=b_3x_3(t-\tau_3)-k_3'x_3(t)z_3(t)+0.2e_2(t)+x_2(t)\,z_2(t)-(b_2+0.008)x_2(t)$$

$$-x_1(t)+(0.2-c_2)\,y_2(t)+(0.2-b_1)\,y_1(t)-0.05\,z_1(t)+u_2(t) \tag{2.17}$$

$$\frac{de_3(t)}{dt}=-c_3z_3(t-\tau_3)+h_3\,x_3^2(t)+0.3e_1(t)-x_2^2(t)-x_1(t)\,z_1(t)+0.3x_2(t)+0.3x_1(t)$$

$$-0.01y_2(t)+0.02y_1(t)+(d_2-0.08)z_2(t)+(c_1-0.1)\,z_1(t)-b_1+u_3(t).$$

Theorem 2.2: If the nonlinear control functions are designed as

$$u_1(t)=-a_3(e_2(t)-x_3(t-\tau_3))-0.05e_1(t)+0.01e_3(t)-(a_2-0.08)x_2(t)-0.04x_1(t)$$

$$+p_2x_2(t-\tau_2)+a_1\,x_1(t-\tau_1)+a_1'\,x_1(t-\tau_1')-(a_3-a_2)\,y_2(t)-(a_3+1)\,y_1(t)$$

$$+0.03z_2(t)-0.99z_1(t)-\left(\frac{1}{2}+k_1\right)e_1(t)$$

$$u_2(t)=-b_3x_3(t-\tau_3)+k_3'x_3(t)z_3(t)-0.2e_2(t)-x_2(t)\,z_2(t)+(b_2+0.008)x_2(t)$$

$$+x_1(t)-(0.2-c_2)y_2(t)-(0.2-b_1)y_1(t)+0.05z_1(t)-\left(\frac{1}{2}+k_2\right)e_2(t) \tag{2.18}$$

$$u_3(t)=c_3z_3(t-\tau_3)-h_3x_3^2(t)-0.3e_1(t)+x_2^2(t)+x_1(t)z_1(t)-0.3x_2(t)-0.3x_1(t)$$

$$+0.01y_2(t)-0.02y_1(t)-(d_2-0.08)z_2(t)-(c_1-0.1)z_1(t)+b_1-(\frac{1}{2}+k_3)e_3(t),$$

then the combined synchronization among double delay Rossler (2.9), advanced Lorenz (2.11) and time delay Liu system (2.16) is achieved and satisfy the condition

$\lim_{t\to\infty}\| e_i(t)\|=0,\ i=1,2,3$ for any positive constants $k_i,\ i=1,2,3$.

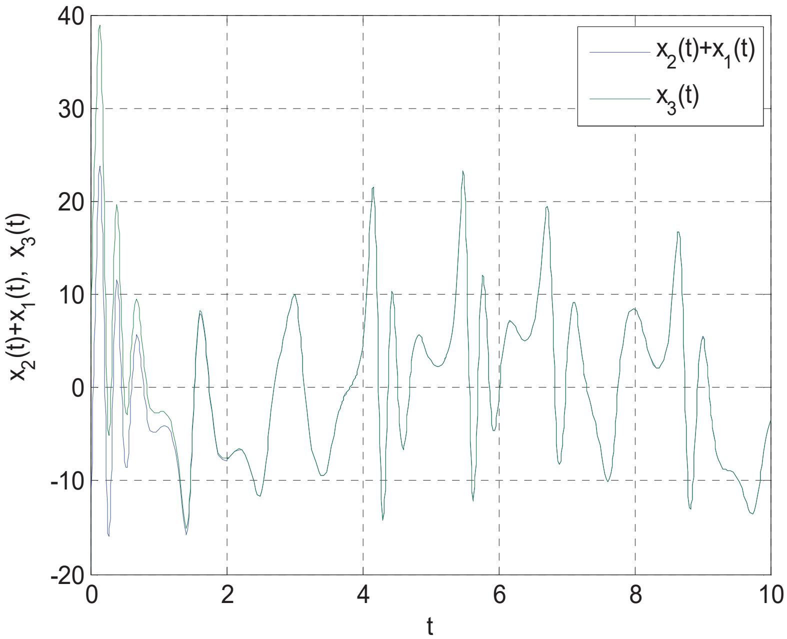
Proof: Let us construct the Lyapunov-Krasovskii functional V to succeed stability analysis as

$$V=\frac{1}{2}(e_1^2(t)+e_2^2(t)+e_3^2(t))+\frac{1}{2}\int_{-\tau}^{0}(e_1^2(t+\theta)+e_2^2(t+\theta)+e_3^2(t+\theta))d\theta.$$

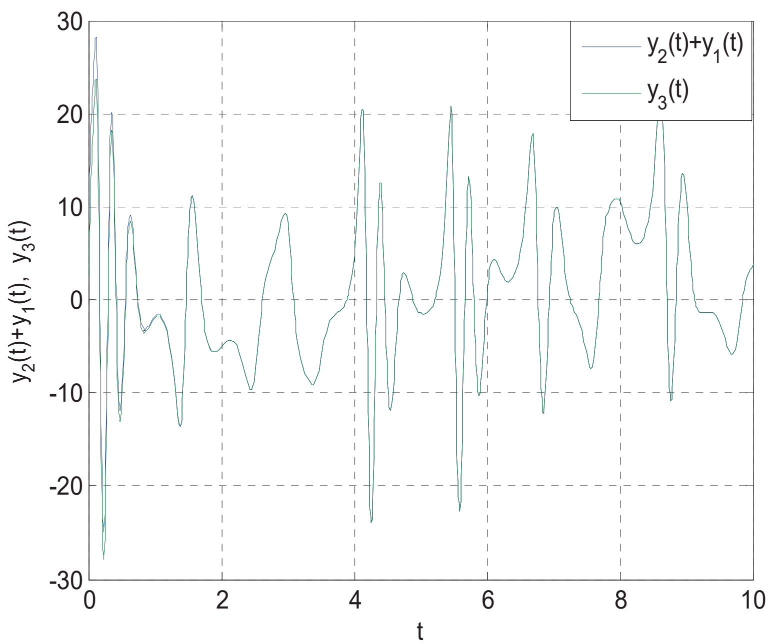
Now taking the derivative of V along the trajectory of error dynamic system and putting the value of $\frac{de_i(t)}{dt}$, $i=1,2,3$ and controller $u_i(t),\ i=1,2,3$ from equations (2.17) & (2.18), then we get

$$\frac{dV}{dt}=-k_1e_1^2(t)-k_2e_2^2(t)-k_3e_3^2-\frac{1}{2}(e_1^2(t-\tau)+e_2^2(t-\tau)+e_3^2(t-\tau))<0.$$

Thus, we may conclude that if the control parameters $k_1,\ k_2,\ k_3>0$, then the error system is globally and asymptotically stable according to Lyapunov-Krasovskii stability theory. Consequently, the state trajectories of two drive systems will be combinationally synchronized with one response system.



(a)



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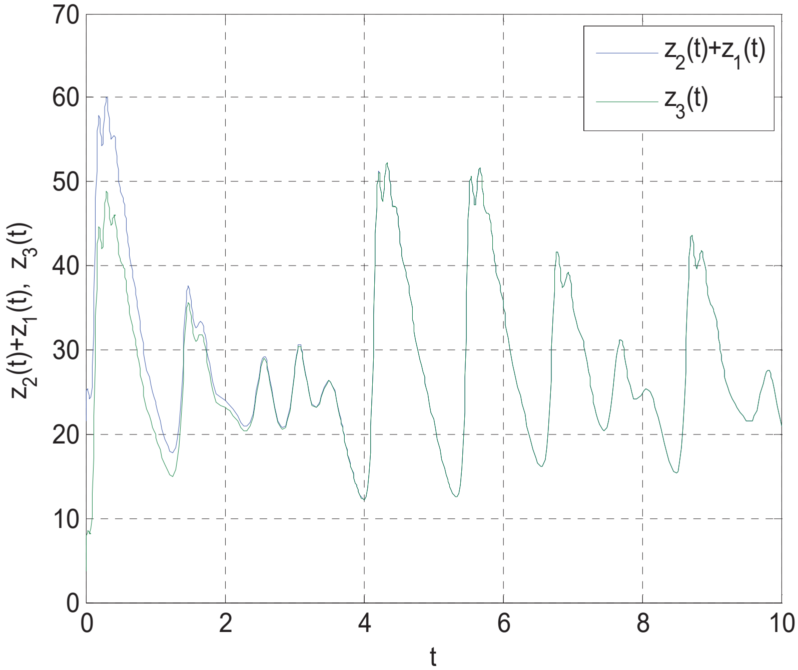

70
60
50
40
30
20
10
0
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2
4
6
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t
$z_2(t)+z_1(t)$
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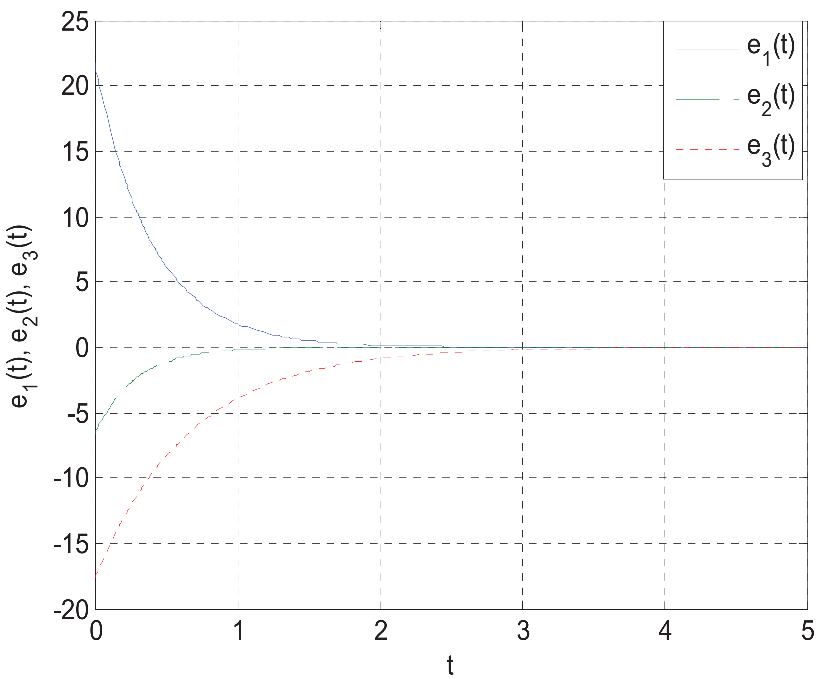
$z_2(t)+z_1(t)$, $z_3(t)$

(c)



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20
15
10
5
0
-5
-10
-15
-20
0
1
2
3
4
5
t
$e_1(t)$
$e_2(t)$
$e_3(t)$
$e_1(t)$, $e_2(t)$, $e_3(t)$

(d)

Fig. 2.5. Combined synchronization among three time delay systems (2.9), (2.11) and (2.13) : (a) between $x_2(t)+x_1(t)$ and $x_3(t)$; (b) between $y_2(t)+y_1(t)$ and $y_3(t)$; (c) between $z_2(t)+z_1(t)$ and $z_3(t)$; (d) the evolution of error functions $e_1(t)$, $e_2(t)$ and $e_3(t)$.

2.6 Combined synchronization among double delay Rossler, advanced Lorenz, time delay Liu and time delay Chen systems with uncertain parameters

In this section to study the combined synchronization among the four time delay systems in the presence of uncertain parameters, we consider double delay Rossler system (2.9), advanced Lorenz system (2.11), time delay Liu system (2.13) as the drive systems I, II and III respectively. The time delay Chen system (2.15) is taken as response system with control inputs $u_1'(t)$, $u_2'(t)$, $u_3'(t)$ as

$$\frac{dx_4(t)}{dt} = a_4(y_4(t)-x_4(t-\tau_4))+0.01x_4(t)-0.02z_4(t)+u_1'(t)$$

$$\frac{dy_4(t)}{dt} = (c_4-a_4)x_4(t-\tau_4)-x_4(t)z_4(t)+c_4y_4(t)+0.1x_4(t)+u_2'(t) \tag{2.19}$$

$$\frac{dz_4(t)}{dt} = x_4(t)y_4(t)-b_4z_4(t-\tau_4)+0.04z_4(t)+u_3'(t).$$

Now define the error functions as $e_1 = x_4-x_3-x_2-x_1$, $e_2 = y_4-y_3-y_2-y_1$, $e_3 = z_4-z_3-z_2-z_1$, we obtain the error system as

$$\frac{de_1(t)}{dt} = 0.01e_1(t)+a_4e_2(t)-0.02e_3(t)-0.04x_3(t)+(a_2-0.12)x_2(t)+(a_4-a_3)y_3(t)$$

$$+(a_4-a_2)y_2(t)+(a_4+1)y_1(t)-0.01z_3(t)-0.04z_2(t)+0.98z_1(t)-a_4x_4(t-\tau_4)$$

$$+a_3x_3(t-\tau_3)-p_2x_2(t-\tau_2)-a_1\,x_1(t-\tau_1)-a_1'\,x_1(t-\tau_1')+u_1'(t)$$

$$\frac{de_2(t)}{dt} = 0.1e_1(t) + c_4 e_2(t) + 0.1x_3(t) + (0.02 - b_2)x_2(t) - 0.90x_1(t) + (c_4 - 0.2)y_3(t)$$

$$+(c_4 - c_2)y_2(t) + (c_4 - b_1)y_1(t) - 0.05z_1(t) + (c_4 - a_4)x_4(t - \tau_4) \qquad (2.20)$$

$$-b_3 x_3(t - \tau_3) - x_4(t)z_4(t) + k_3' x_3(t)z_3(t) + x_2(t)z_2(t) + u_2'(t)$$

$$\frac{de_3(t)}{dt} = 0.04e_3(t) + 0.04z_3(t) + (d_2 - 0.76)z_2(t) + (c_1 - 0.06)z_1(t) - 0.3x_3(t)$$

$$-0.01y_2(t) + 0.02y_1(t) - b_4 z_4(t - \tau_4) + c_3 z_3(t - \tau_3) + x_4(t)y_4(t)$$

$$-h_3 x_3^2(t) - x_2^2(t) - x_1(t)z_1(t) - b_1 + u_3'(t)$$

Theorem 2.3: If the nonlinear control functions are taken as

$$u_1'(t) = -0.01e_1(t) - a_4 e_2(t) + 0.02e_3(t) + 0.04x_3(t) - (a_2 - 0.12)x_2(t) - (a_4 - a_3)y_3(t)$$

$$-(a_4 - a_2)y_2(t) - (a_4 + 1)y_1(t) + 0.01z_3(t) + 0.04z_2(t) - 0.98z_1(t) + a_4 x_4(t - \tau_4)$$

$$-a_3 x_3(t - \tau_3) + p_2 x_2(t - \tau_2) + a_1 x_1(t - \tau_1) + a_1' x_1(t - \tau_1') - \left(\frac{1}{2} + k_1\right)e_1(t)$$

$$u_2'(t) = -0.1e_1(t) - c_4 e_2(t) - 0.1x_3(t) - (0.02 - b_2)x_2(t) + 0.90x_1(t) - (c_4 - 0.2)y_3(t)$$

$$-(c_4 - c_2)y_2(t) - (c_4 - b_1)y_1(t) + 0.05z_1(t) - (c_4 - a_4)x_4(t - \tau_4) \qquad (2.21)$$

$$+b_3 x_3(t - \tau_3) + x_4(t)z_4(t) - k_3' x_3(t)z_3(t) - x_2(t)z_2(t) - \left(\frac{1}{2} + k_2\right)e_2(t)$$

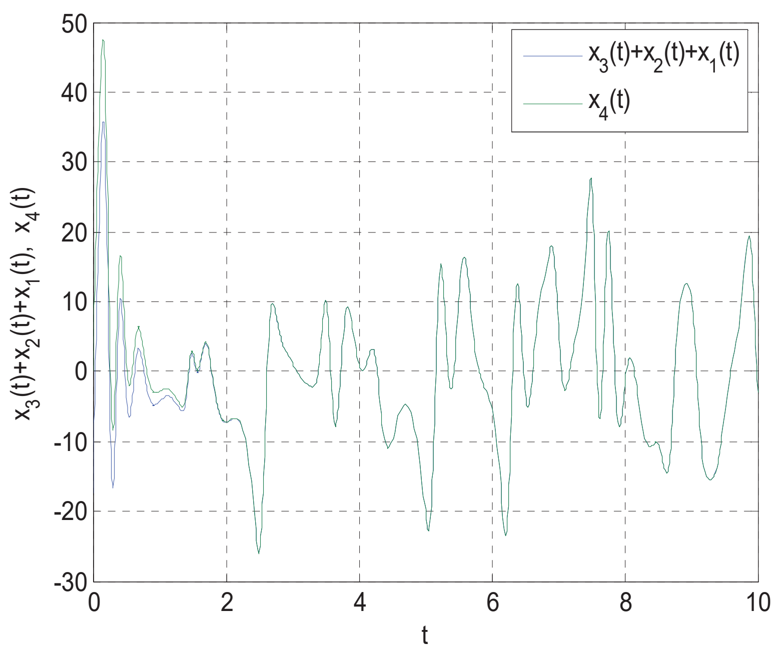
$$u_3'(t) = -0.04e_3(t) - 0.04z_3(t) - (d_2 - 0.76)z_2(t) - (c_1 - 0.06)z_1(t) + 0.3x_3(t) + 0.01y_2(t)$$

$$-0.02y_1(t) + b_4 z_4(t - \tau_4) - c_3 z_3(t - \tau_3) - x_4(t)y_4(t) + h_3 x_3^2(t)$$

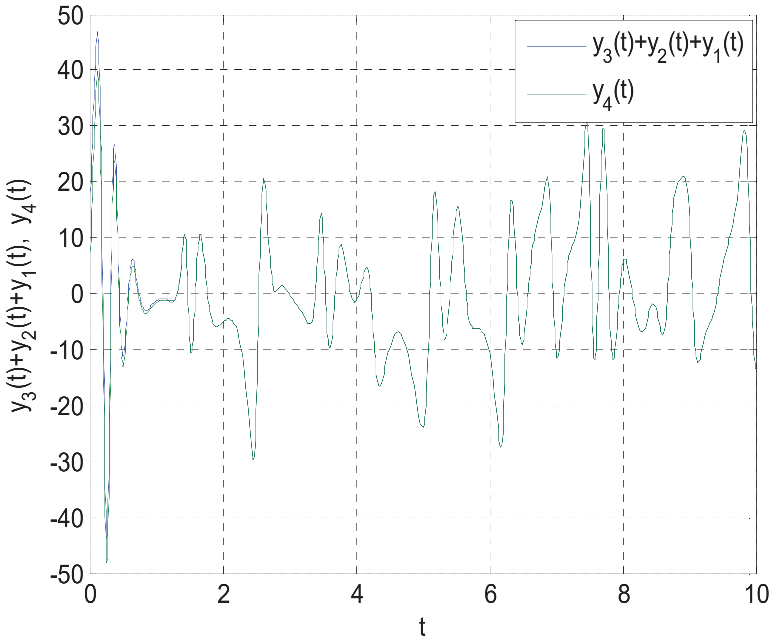
$$+x_2^2(t) + x_1(t)z_1(t) + b_1 - \left(\frac{1}{2} + k_3\right)e_3(t)$$

then the combined synchronization among double delay Rossler (2.9), advanced Lorenz (2.11), time delay Liu system (2.16) and time delay Chen system (2.19) are achieved and satisfy the condition $\lim_{t\to\infty}\| e_i(t) \| = 0,\ i = 1,2,3$ for any positive constants $k_i,\ i = 1,2,3$.

Proof: The proof of this theorem is similar to the Theorem 2.2. Therefore it is omitted.

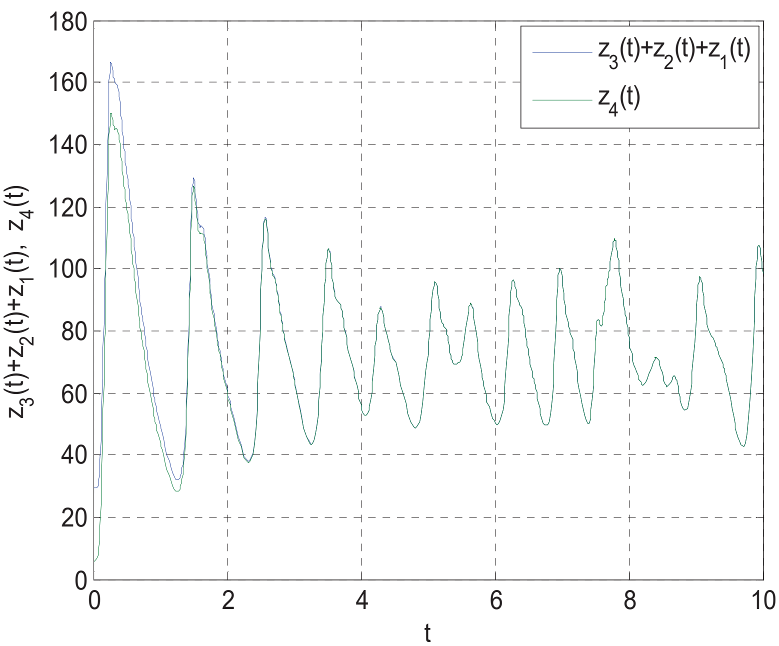


(a)



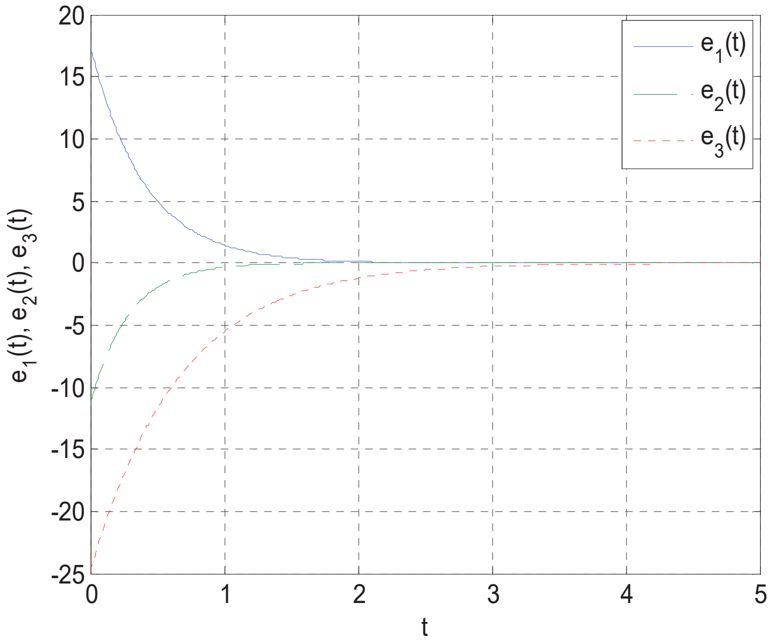
y3(t)+y2(t)+y1(t)
y4(t)
y3(t)+y2(t)+y1(t), y4(t)
t

(b)



z3(t)+z2(t)+z1(t)
z4(t)
z3(t)+z2(t)+z1(t), z4(t)
t

(c)



(d)

Fig. 2.6. Combined synchronization among four time delay systems (2.9), (2.11), (2.13) and (2.15) : (a) between $x_3(t)+x_2(t)+x_1(t)$ and $x_4(t)$; (b) between $y_3(t)+y_2(t)+y_1(t)$ and $y_4(t)$; (c) between $z_3(t)+z_2(t)+z_1(t)$ and $z_4(t)$; (d) the evolution of error functions $e_1(t)$, $e_2(t)$ and $e_3(t)$.

2.7 Numerical simulation and results

In this section the earlier values of the parameters and initial conditions of systems are considered. Fig. 2.5 shows that the synchronization among three fractional order systems is achieved through nonlinear control method. The initial values of error systems are taken as $(21.7, -6.6, -17.7)$ and $(17.5, -11.4, -24.8)$ during the synchronization among three and four systems respectively. Figs. 2.5(a), 2.5(b) and 2.5(c) depict the time response of the state trajectories $x_2(t)+x_1(t)$ and $x_3(t)$, $y_2(t)+y_1(t)$ and $y_3(t)$, $z_2(t)+z_1(t)$ and $z_3(t)$ respectively. The error states are displayed through Fig. 2.5(d). The synchronization among four time delayed chaotic systems is achieved using the same method. Fig. 2.6 displayed the combined synchronization among five time delayed chaotic systems and Figs. 2.6(a), 2.6(b) and 2.6(c) shows the time response of the states $x_3(t)+x_2(t)+x_1(t)$ and $x_4(t)$, $y_3(t)+y_2(t)+y_1(t)$ and $y_4(t)$, $z_3(t)+z_2(t)+z_1(t)$ and $z_4(t)$. The error states for this case are described through Fig 2.6(d). The time delayed terms and control parameters are taken as $\tau_1 = 1.0$, $\tau_1' = 2.0$, $\tau_2 = 0.001$, $\tau_3 = 0.002$, $\tau_4 = 0.005$ and $k_1 = 2$, $k_2 = 3$, $k_3 = 1$ respectively during the combined synchronization of three and four chaotic systems.

2.8 Conclusion

In this chapter, the combined synchronization has been successfully demonstrated using nonlinear control method among three and four time delayed chaotic systems in the presence of uncertain parameters. The combined synchronization of three and four systems are considered taking two systems and three systems as drive system respectively, while one system as response system. The combined synchronization controllers are developed using Lyapunov-Krasovskii stability theory for stabilizing the delay-differential equations. The graphical presentation of the numerical results with error states tending to zero as time becomes large, clearly exhibit that the applied nonlinear control method is effective and convenient to achieve global synchronization among non-identical time delayed chaotic systems with uncertain parameters.

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